This assignment will guide you through proofs of some important identities and evolution equations for curve shortening flow, which would apply verbatim if $\mathbb{R}^2$ were replaced with an arbitrary two-dimensional Riemannian manifold. (The proofs found in Chou-Zhu §1.3 do not easily generalize to the Riemannian setting.) No background in Riemannian geometry is necessary to complete this assignment.

**Problem 1.** Let $v, w : (\sigma_1, \sigma_2) \to \mathbb{R}^2$ be differentiable, and such that $\|v(\sigma)\| = \|w(\sigma)\| = 1$ and $v(\sigma) \cdot w(\sigma) = 0$ for all $\sigma \in [\sigma_1, \sigma_2]$. Show:

$$w_\sigma = -(v_\sigma \cdot w)v. \tag{\dagger_1}$$

**Problem 2.** Let $\gamma : (a, b) \times [0, T) \to \mathbb{R}^2$ be a $C^2$ portion of a solution of the curve shortening flow, $\gamma_t = \kappa$. If $\gamma = \gamma(p, t)$, then show

$$\frac{\partial}{\partial t} \|\gamma_p\| = -\|\kappa\|^2 \|\gamma_p\|. \tag{\dagger_2}$$

If differentiation with respect to arclength is $\frac{\partial}{\partial s} = \|\gamma_p\|^{-1} \frac{\partial}{\partial p}$, use $(\dagger_2)$ to show

$$\frac{\partial}{\partial t} \frac{\partial}{\partial s} - \frac{\partial}{\partial t} \frac{\partial}{\partial s} = \|\kappa\|^2 \frac{\partial}{\partial s}, \tag{\dagger_3}$$

then use $(\dagger_1)$ and $(\dagger_3)$ to show

$$\frac{\partial}{\partial t} \tau = \left[\frac{\partial}{\partial s}(\kappa \cdot \nu)\right] \nu, \tag{\dagger_4}$$

and then use $(\dagger_1)$ again to show

$$\frac{\partial}{\partial t} \nu = -\left[\frac{\partial}{\partial s}(\kappa \cdot \nu)\right] \tau. \tag{\dagger_5}$$

Finally, use $(\dagger_1)$-$(\dagger_5)$ to show

$$\frac{\partial}{\partial t}(\kappa \cdot \nu) = \frac{\partial^2}{\partial s^2}(\kappa \cdot \nu) + (\kappa \cdot \nu)^3. \tag{\dagger_6}$$

**Problem 3.** Let $\gamma : S^1 \times [0, T) \to \mathbb{R}^2$ denote a family of immersed $C^2$ curves moving by the curve shortening flow. If $\kappa \cdot \nu \geq 0$ at $t = 0$, then use $(\dagger_6)$ and the maximum principle to prove that $\kappa \cdot \nu > 0$ for all $t \in (0, T)$.

\[1\] In a Riemannian setting, we’d need to additionally assume that $\kappa \cdot \nu \neq 0$ at $t = 0$.  

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