Problem 1. Consider the function \( \sigma : \mathbb{R} \rightarrow S^1 \) given by \( \sigma(t) := e^{it} \). Define, for \( k \in \mathbb{N}, \alpha \in [0,1] \),

\[
C^{k,\alpha}(S^1; \mathbb{R}^n) := \{ f : S^1 \rightarrow \mathbb{R}^n \text{ such that } \|f \circ \sigma\|_{C^{k,\alpha}(\mathbb{R}; \mathbb{R}^n)} < \infty \},
\]

and, for \( f \in C^{k,\alpha}(S^1; \mathbb{R}^n) \), set \( \|f\|_{C^{k,\alpha}(S^1; \mathbb{R}^n)} := \|f \circ \sigma\|_{C^{k,\alpha}(\mathbb{R}; \mathbb{R}^n)} \). We often omit \( \alpha \) from the notation when \( \alpha = 0 \). Note (but don’t prove) that \( f \in C^k(S^1; \mathbb{R}^n), k \in \mathbb{N} \setminus \{0\} \implies \|f\|_{C^{k-1,1}(S^1; \mathbb{R}^n)} = \|f\|_{C^k(S^1; \mathbb{R}^n)} \).

Suppose that \( \{f_i\}_{i=1,2,...} \subset C^{k,\alpha}(S^1; \mathbb{R}^n) \) is a sequence with

\[
\|f_i\|_{C^{k,\alpha}(S^1; \mathbb{R}^n)} \leq c \text{ for all } i = 1,2,\ldots
\]

and \( c > 0 \) fixed. If \( \alpha > 0 \), then prove that there exists a subsequence \( \{f_{i'}\}_{i'=1,2,...} \) and a \( f_\infty \in C^{k,\alpha}(S^1; \mathbb{R}^n) \) such that

\[
\lim_{i' \to \infty} \|f_{i'} - f_\infty\|_{C^{k',\alpha'}(S^1; \mathbb{R}^n)} = 0
\]

for every \( k' \in \mathbb{N}, \alpha' \in [0,1] \) with \( k' + \alpha' < k + \alpha \). What conclusions can you draw when \( \alpha = 0 \)? (i.e., what convergence can one expect if only assuming \( C^k \) bounds, but not Hölder bounds?)

Problem 2. Let \( \gamma_i : [0,1] \rightarrow \mathbb{R}^n, i = 1,2,\ldots, \) denote a sequence of regular \( C^2 \) curves such that, for every \( i = 1,2,\ldots: \)

1. \( \gamma_i \) is p.b.a.l.,
2. \( \text{img} \gamma_i \cap B_R(0) \neq \emptyset, \) and
3. \( \|\kappa_t(t)\| \leq c \) for all \( t \in [0,1]. \)

Here, \( c, R > 0 \) are fixed. Prove that there exists a subsequence that converges in \( C^1([0,1]) \) to a regular \( C^{1,1} \) curve \( \gamma_\infty : [0,1] \rightarrow \mathbb{R}^2 \). Construct examples to show that:

1. all three conditions above are necessary for the convergence to hold, and
2. the limit curve \( \gamma_\infty \) need not be \( C^2 \).