GAUSSIAN PREPIVOTING FOR FINITE POPULATION CAUSAL INFERENCE

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ABSTRACT

In finite population causal inference exact randomization tests can be constructed for sharp null hypotheses, i.e. hypotheses which fully impute the missing potential outcomes. Oftentimes inference is instead desired for the weak null that the sample average of the treatment effects takes on a particular value while leaving the subject-specific treatment effects unspecified. Without proper care, tests valid for sharp null hypotheses may be anti-conservative should only the weak null hold, creating the risk of misinterpretation when randomization tests are deployed in practice. We develop a general framework for unifying modes of inference for sharp and weak nulls, wherein a single procedure simultaneously delivers exact inference for sharp nulls and asymptotically valid inference for weak nulls. To do this, we employ randomization tests based upon prepivoted test statistics, wherein a test statistic is first transformed by a suitably constructed cumulative distribution function and its randomization distribution assuming the sharp null is then enumerated. For a large class of commonly employed test statistics, we show that prepivoting may be accomplished by employing the push-forward of a sample-based Gaussian measure based upon a suitably constructed covariance estimator. In essence, the approach enumerates the randomization distribution (assuming the sharp null) of a P-value for a large-sample test known to be valid under the weak null, and uses the resulting randomization distribution to perform inference. The versatility of the method is demonstrated through a host of examples, including rerandomized designs and regression-adjusted estimators in completely randomized designs.

Keywords: Pivotal quantity, Stochastic dominance, Randomization tests, Sharp null, Weak null, Rerandomization

1 Introduction

In finite population causal inference two distinct hypotheses of “no treatment effect” are commonly tested: Fisher’s sharp null and Neyman’s weak null. Fisher’s sharp null of no effect refers to the null that the responses under treatment and under control are the same for all individuals in the study (Rosenbaum, 2002). The sharp null imputes the missing values of the potential outcomes for all individuals, in so doing facilitating the use of randomization tests to provide exact inference with randomization alone acting as the basis for inference (Fisher, 1935). Neyman’s weak null instead specifies only that the average of the treatment effects for the individuals in the experiment equals zero, while allowing for heterogeneity in the unit-specific effects. The missing potential outcomes are no longer imputed under the weak null, such that the randomization distribution under the weak null remains unknown. Consequently, inference for the weak null has historically proceeded using asymptotically conservative analytical approximations to the limiting distribution of the treated-minus-control difference in means.
While the exactness attained under the sharp null is appealing, randomization tests have been criticized for the seemingly restricted nature of the conclusions to which the researcher is entitled should the null be rejected (Caughey et al., 2017); while the researcher may suggest that the treatment effect is not zero for all individuals, generally one is not entitled to a statement of whether the treatment effect is positive or negative on average for the individuals in the study. To address this, a recent literature has emerged on how randomization tests may be made modified to maintain asymptotic validity under the weak null hypothesis. The resulting methods provide a single testing procedure that is asymptotically valid for the weak null hypothesis, while maintaining exactness should the sharp null also be true (Ding, 2017; Loh et al., 2017; Ding and Dasgupta, 2017; Wu and Ding, 2018; Fogarty, 2019).

The existing literature attains this unified mode of inference largely on a case-by-case basis: for a given experimental design, a specially catered test statistic is constructed so the corresponding randomization test under the sharp null maintains asymptotic validity under the weak null. In this work, we provide both general conditions under which the unification may be achieved and a general methodology for attaining it. The central idea is to leverage prepivoting, an idea introduced in Beran (1987, 1988). For most commonly employed experimental designs and test statistics, the reference distribution generated by the prepivoted statistic under the assumption of the sharp null asymptotically stochastically dominates the true, but unknowable, randomization distribution under the weak null, yielding asymptotically conservative inference for the weak null while maintaining exactness under the sharp null hypothesis. As we demonstrate, prepivoting succeeds in many scenarios where other common resolutions such as studentization prove inadequate.

At a high level, prepivoting takes a test statistic $T_0$ and composes it with a cumulative distribution function $\hat{F}$ constructed from the observed data, forming the new test statistic $T_1 = \hat{F}(T_0)$. If $\hat{F}$ were a consistent estimate of $T_0$’s limit distribution, $\hat{F}(T_0)$ would, through an asymptotic application of the probability integral transform, tend to a standard uniform. Under the weak null hypothesis, the true distribution function for common test statistics $T_0$ cannot generally be consistently estimated. Fortunately, as developed in §5 a distribution function for a random variable that asymptotically stochastically dominates $T_0$ may be constructed. For most common test statistics for the weak null hypothesis, under conditions outlined in §5 this dominating distribution function amounts to a suitable pushforward of a multivariate Gaussian measure constructed using conservative covariance estimator. Using this estimated distribution function, $T_1 = \hat{F}(T_0)$ is instead stochastically dominated by a standard uniform in the limit. Observe that through this construction, the prepivoted test $T_1 = \hat{F}(T_0)$ is precisely one minus the large sample $p$-value for the test statistic $T_0$ leveraging the central limit theorem. Rather than using this $p$-value to reach a conclusion by comparing its value to the desired $\alpha$, we instead use the reference distribution of this large-sample $p$-value enumerated over all possible randomizations assuming the sharp null holds. This reference distribution generally converges pointwise to the standard uniform distribution function for commonly used covariance estimators underpinning $\hat{F}$. As a result, inference is guaranteed to be asymptotically conservative under the weak null, while maintaining exactness under the sharp null. The general takeaway is that rather than looking at the randomization distribution of a test statistic itself under the sharp null, one should instead enumerate the randomization distribution of one minus an asymptotically valid $p$-value to restore validity of randomization tests when only the weak null holds.

In §2 we introduce notation for finite population causal inference and detail some standard assumptions. Section 3 defines the reference distribution assuming the truth of Fisher’s sharp null used in randomization tests and juxtaposes this with its true though unknowable randomization distribution under Neyman’s weak null of no effect on average. After an overview of useful asymptotic results on completley randomized designs in §4, §5 introduces Gaussian prepivoting in the context of suitably constructed functions of treated-minus control difference in means. Section 6 provides examples of and insight into prepivoting using Gaussian measure. Section 7 extends these results to other asymptotically linear estimators, including regression-adjusted estimators, while §8 provides simulation studies highlighting the benefits of Gaussian prepivoting.

### 2 Notation and Review

#### 2.1 Notation for finite population causal inference

While the developments in this work apply quite generally across experimental designs and with two or more levels of the treatment, in this work we focus on completely randomized experiments and rerandomized experiments with two treatments. Consider a collection of $N$ individuals, where $n_1$ receive treatment and $n_0 = N - n_1$ receive the control. For the $i$th individual, the random variable $Z_i$ is the treatment indicator, taking the value 1 if the $i$th individual receives treatment and 0 otherwise. We assume that SUTVA holds, such that there is no interference and that there are no hidden levels of the treatment (Rubin, 1980). The $i$th individual has two deterministic potential outcomes: $y_i(1)$, the $d$-dimensional outcome under treatment, and $y_i(0)$ the $d$-dimensional outcome under control.
Furthermore, the $i$th unit has deterministic covariates $x_i \in \mathbb{R}^k$. The random vector $Z$ represents $(Z_1, \ldots, Z_N)^T$; likewise $y(1) = (y_1(1), \ldots, y_N(1))^T$ and $y(0) = (y_1(0), \ldots, y_N(0))^T$. Under the finite population model the potential outcomes are viewed as fixed across randomizations, and the only randomness enters through $Z$ the treatment allocation. The observed outcomes corresponding to a given treatment allocation $Z$ is $y_i(Z_i)$ and the collection of these is denoted $y(Z)$.

The vector of treatment effects for the $i$th individual is $\tau_i = y_i(1) - y_i(0)$. The average treatment effect for the individuals in the experiment is $\bar{\tau} = N^{-1} \sum_{i=1}^N \tau_i$. As the two potential outcomes are not jointly observable, $\tau_i$ is unknown for all individuals. Neyman’s weak null of no treatment effect on average is $H_N: \bar{\tau} = 0$, while Fisher’s sharp null further stipulates $H_F: \tau_i = 0$ ($i = 1, \ldots, N$).

For any matrix $r \in \mathbb{R}^{N \times d}$ and any binary vector $W$ with $\sum_{i=1}^N W_i = n_1$, we define the function

$$\hat{\tau}(r, W) = \frac{1}{n_1} \sum_{i=1}^N W_i r_i - \frac{1}{n_0} \sum_{i=1}^N (1 - W_i) r_i.$$ 

Using this notation, the observed treated-minus-control difference in means for the outcome variables is $\hat{\tau}(y(Z), Z)$, and is often denoted by $\hat{\tau}$ as shorthand. Define $\bar{y}(0) = N^{-1} \sum_{i=1}^N y_i(0)$ and $\bar{y}(1) = N^{-1} \sum_{i=1}^N y_i(1)$ to be the average potential outcomes for the $N$ individuals in the study population. Likewise, we define the covariance matrices

$$\Sigma_y(z) = (N - 1)^{-1} \sum_{i=1}^N (y_i(z) - \bar{y}(z))(y_i(z) - \bar{y}(z))^T, \quad z \in \{0, 1\};$$
$$\Sigma_\tau = (N - 1)^{-1} \sum_{i=1}^N (\tau_i - \bar{\tau})(\tau_i - \bar{\tau})^T.$$ 

To emphasize the distinction between functions of observed outcomes and functions of covariates, we define the function $\hat{\delta}(x, W)$ with binary $W$ such that $\sum_{i=1}^N W_i = n_1$ as

$$\hat{\delta}(x, W) = \frac{1}{n_1} \sum_{i=1}^N W_i x_i - \frac{1}{n_0} \sum_{i=1}^N (1 - W_i) x_i.$$ 

The observed difference in means for covariates is $\hat{\delta}(x, Z)$, abbreviated as $\hat{\delta}$. The finite population mean of the covariates is $\bar{x} = N^{-1} \sum_{i=1}^N x_i$. The finite population covariance matrix for the covariates is $\Sigma_x$. The finite population covariance between potential outcomes and covariates is $\Sigma_y(x)$ for $z = 0, 1$, and the covariance between treatment effects and covariates is $\Sigma_{\tau x}$. Asymptotic arguments that follow will imagine a single sequence of finite populations of increasing size, with $N \to \infty$. As a result, quantities such as $\Sigma_x$ themselves vary as $N \to \infty$, and should be denoted by $\Sigma_{x,N}$ to reflect this. Generally the dependence is suppressed to reduce notational clutter; however, we do employ the notation $\Sigma_{\tau,\infty}$ to denote the limiting value of $\Sigma_{\tau,N}$ as $N \to \infty$, and likewise for other finite population quantities. For more on the finite population model for causal inference, see Imbens and Rubin (2015) and Ding et al. (2017) among many.

### 2.2 Rerandomized designs and balance criterion

The set of all possible treatment assignments $Z$ is denoted by $\Omega$, and is determined by the experimental design. In completely randomized experiments, covariates are not used to inform the chosen treatment assignment and $\Omega_{CRE} = \{z : \sum_{i=1}^N z_i = n_1\}$. To mitigate the risk of significant covariate imbalance, Morgan and Rubin (2012) suggest instead building covariate balance into the treatment allocation process through rerandomization. The study is conducted by collecting covariate data for the study participants, determining a measure of imbalance and a threshold for deciding what imbalances are acceptable, and selecting a treatment allocation uniformly over the set of allocations satisfying the balance criterion (Li et al., 2018). Stringent balance criterion reduce the cardinality of $\Omega$ by eliminating undesirable assignments, with the hopes of improving precision as a consequence. Naturally, randomization inference must take into account that the allowable realizations of $Z$ depend upon the condition that covariate balance is met.

A balance criterion is a Boolean-valued function $\phi(\cdot)$, where $\phi(\sqrt{N}\hat{\delta}) = 1$ is taken to mean that the treatment allocation $Z$ which results in the particular realization of $\hat{\delta}$ under consideration satisfies appropriate covariate balance. We impose the following restriction on $\phi$:

**Condition 1.** $\phi: \mathbb{R}^k \to \{0, 1\}$ is an indicator function such that the set $M = \{b : \phi(b) = 1\}$ is closed, convex, mirror-symmetric about the origin (i.e. $b \in M \iff -b \in M$) with non-empty interior.
2.3 Regularity conditions

We make the following assumptions about the structure of the finite populations and experimental designs as \( N \) goes to infinity. These assumptions are for the most part standard in the literature; see, for instance, Wu and Ding (2018).

**Assumption 1.** The proportion \( n_j/N \) limits to \( p \in (0, 1) \) as \( N \to \infty \).

**Assumption 2.** All finite population means and covariances are Cesàro summable for both the potential outcomes and the covariates. For instance, \( \lim_{N \to \infty} \bar{Y}(z) = \bar{Y}_\infty(z) \) for \( z \in \{0, 1\} \) and \( \lim_{N \to \infty} \Sigma_{y(1)} = \Sigma_{y(1), \infty} \).

**Assumption 3(a).** The worst-case squared distance from the average potential outcome is \( o(N) \); i.e.

\[
\lim_{N \to \infty} \max_{z \in \{0, 1\}} \max_{i \in \{1, \ldots, N\}} \frac{(y_{ij}(z) - \bar{y}_j(z))^2}{N} = 0.
\]

Further, the above holds for the covariates with \( x_{ij} \) replacing \( y_{ij}(z) \) above for \( j = 1, \ldots, k \).

At times, we will strengthen Assumption 3(a) to the following:

**Assumption 3(b).** There exists some \( C < \infty \) for which, for all \( z \in \{0, 1\} \), all \( j = 1, \ldots, d \) and all \( N \),

\[
\frac{\sum_{i=1}^{N} (y_{ij}(z) - \bar{y}_j(z))^4}{N} < C
\]

Further, the above holds for the covariates with \( x_{ij} \) replacing \( y_{ij}(z) \) above for \( j = 1, \ldots, k \).

Assumption 3(b) implies Assumption 3(a) (Wu and Ding, 2018, Proposition 1) and is used to obtain finite population-inference strong laws of large numbers for mean and variance estimators. Such an assumption is made at times for mathematical convenience to simplify the analysis of certain random distributions; such results may hold under weaker assumptions. Assumption 3(b) is commonplace in the literature on finite population causal inference; see, for instance, Wu and Ding (2018); Lin (2013); Freedman (2008a,b).

3 Randomization distributions and tests

3.1 Randomization distributions

Consider a scalar test statistic \( T(y(Z), Z) \), a function of the observed responses and the treatment assignment received. The randomization distribution for the test statistic \( T \) is

\[
\mathcal{R}_T(t) = \frac{1}{|\Omega|} \sum_{w \in \Omega} \mathbb{1} \{ T(y(w), w) \leq t \}.
\]

\( \mathcal{R}_T \) is the true cumulative distribution function of \( T(y(Z), Z) \) with respect to the randomness in treatment allocation \( Z \) distributed uniformly over \( \Omega \). If we had access to \( \mathcal{R}_T \) under the null hypothesis in question, we could make direct use of it to provide inference that is exact in finite samples, proceeding without dependence on asymptotics. Under Fisher’s sharp null hypothesis, \( \mathcal{R}_T \) is specified by the observed outcomes as \( y(Z) = y(w) \) for any \( w \in \Omega \). Unfortunately, the distribution is generally unknown under the weak null, as the weak null merely constrains the missing potential outcomes without determining them.

3.2 Randomization tests assuming the sharp null

In practice an experimenter draws a single realization of \( Z \), in so doing only revealing the values of the potential outcomes corresponding to the observed assignment. Suppose that regardless of whether or not Fisher’s sharp null hypothesis actually holds, the researcher considers use of the randomization distribution to which she or he would be entitled if the sharp null were true. This distribution takes the form

\[
\mathcal{P}_T(t) = \frac{1}{|\Omega|} \sum_{w \in \Omega} \mathbb{1} \{ T(y(Z), w) \leq t \}
\]

While \( \mathcal{R}_T = \mathcal{P}_T \) under the sharp null, under the weak null \( \mathcal{P}_T \) is a random distribution function as it varies with \( Z \). Inference using \( \mathcal{P}_T \) proceeds as though \( y(Z) \) would have been the observed treatment assignment for any \( w \in \Omega \). As the true response \( y(w) \) under assignment \( w \) need not align with \( y(Z) \), \( \mathcal{P}_T \) does not actually reflect the true
randomization distribution under the weak null. This gives rise to potentially anti-conservative inference should \( P_T \) be used to test the weak null hypothesis.

For \( \alpha \in (0, 1) \) define the Fisher randomization test of nominal level \( \alpha \) by

\[
\varphi_T(\alpha) = 1 \{ T(y(Z), Z) \geq P_T^{-1}(1 - \alpha) \}. \tag{3}
\]

Under the sharp null, \( \mathbb{E}\{ \varphi_T(\alpha) \} \leq \alpha \) for any sample size as \( P_T = R_T \). Throughout this paper, we examine the extent to which certain choices of test statistics entitle us to genuine Type I error control at \( \alpha \) when \( \varphi_T(\alpha) \) is used to conduct inference but only the weak null holds. For a given test statistic \( T \), we will often proceed by juxtaposing its true limiting behavior under the randomization distribution \( R_T \) with the limiting behavior of \( P_T \), the randomization distribution if we (incorrectly) assumed that the sharp null held.

### 3.3 Towards a unified mode of inference

Suppose that for a test statistic \( T(y(Z), Z) \) based upon the observed outcomes \( y(Z) \) and the treatment allocation \( Z \),

1. \( P_T \) converges weakly in probability to a fixed distribution \( P_{T,\infty} \) as \( N \to \infty \); and
2. \( R_T \) converges pointwise to a fixed distribution \( R_{T,\infty} \) at all continuity points of \( R_{T,\infty} \).

The test statistic \( T(y(Z), Z) \) is called asymptotically sharp-dominant if, under \( H_N, P_{T,\infty}(t) \leq R_{T,\infty}(t) \) for any scalar \( t \). This implies that the \( (1 - \alpha)^{th} \) quantile of \( P_{T,\infty} \) is at or above the \( (1 - \alpha)^{th} \) quantile of \( R_{T,\infty} \). If \( T(y(Z), Z) \) is asymptotically sharp-dominant, then inference based upon the reference distribution \( P_T \) will be asymptotically conservative even if only \( H_N \) holds (Wu and Ding, 2018, Proposition 4), satisfying \( \lim \sup \mathbb{E}\{ \varphi_T(\alpha) \} \leq \alpha \) as \( N \to \infty \) while maintaining exactness should the sharp null be true.

Many common test statistics are not asymptotically sharp-dominant over all elements of the weak null. For instance, with univariate potential outcomes and under a completely randomized design with imbalanced treated and control groups, the absolute difference in means \( T(y(Z), Z) = |\sqrt{N}\tau| \) is not generally asymptotically sharp-dominant as there exist sequences of potential outcomes satisfying the weak null such that \( \lim \inf \mathbb{E}\{ \varphi_T(\alpha) \} > \alpha \); see Ding (2017), Wu and Ding (2018, Cor. 3), or Loh et al. (2017) for details. For this test statistic, simply studentizing by the usual standard error estimator ensures sharp dominance. However, studentization fails to generalize to other more complicated test statistics and complex experimental designs. Significant efforts have recovered appropriate studentization techniques for some test statistics, but each test statistic requires its own separate analysis (Wu and Ding, 2018). For some experimental designs, studentizing the difference in means is not sufficient to regain asymptotically valid inference even in the univariate case; we explore this topic in §5.2 and §8.1 in the context of rerandomization. In §5, we present a general method called Gaussian prepivoting which both recovers studentization when it alone would be sufficient, but also yields asymptotic sharp-dominance in circumstances where studentization would be insufficient. Before describing the method, we recall a few important results on the difference in means in completely randomized designs which underpin the success of Gaussian prepivoting.

### 4 Useful results for the difference-in-means in completely randomized designs

#### 4.1 Asymptotic normality and conservative covariance estimation for the randomization distribution

Consider the distribution of \( \sqrt{N}(\hat{\tau} - \bar{\tau}, \hat{\delta})^T \) in a completely randomized design. Under Assumptions 1, 2, and 3(a), a finite population central limit theorem applies (Li and Ding, 2017), and \( \sqrt{N}(\hat{\tau} - \bar{\tau}, \hat{\delta})^T \) converges in distribution to a mean-zero multivariate Gaussian with covariance matrix \( V \) of the form

\[
V = \begin{pmatrix}
V_{\tau\tau} & V_{\tau\delta} \\
V_{\delta\tau} & V_{\delta\delta}
\end{pmatrix}:
\]

\[
V_{\tau\tau} = p^{-1}\Sigma_{y(1),\infty} + (1 - p)^{-1}\Sigma_{y(0),\infty} - \Sigma_{\tau,\infty};
\]

\[
V_{\delta\delta} = \{p(1 - p)^{-1}\Sigma_{x,\infty};
\]

\[
V_{\tau\delta} = p^{-1}\Sigma_{y(1),\infty} + (1 - p)^{-1}\Sigma_{y(0),x,\infty} = V_{\delta\tau}^T.
\]

While \( V_{\delta\delta} \) and \( V_{\tau\delta} \) can be consistently estimated, \( V_{\tau\tau} \) cannot be in the presence of effect heterogeneity due to its dependence on \( \Sigma_{\tau} \), the covariance of the unobserved treatment effects. Consequently, one cannot consistently estimate the probability that \( \sqrt{N}(\hat{\tau} - \bar{\tau}) \) falls within a given region \( B \). While consistent variance estimates are not available,
there are several covariance estimators $\hat{V}_{\tau\tau}(y(Z), Z)$ for $V_{\tau\tau}$ satisfying $\hat{V}_{\tau\tau} - V_{\tau\tau} \overset{p}{\to} \Delta$ for some $\Delta \succeq 0$ under Assumptions 1 - 3(a) in completely randomized designs. These estimators typically have the property that $\Sigma_{\tau\tau} = 0$ implies consistency, rather than asymptotic conservativeness; see Ding et al. (2019) for more details. So while the matrix $V$ cannot generally be consistently estimated, one can construct an estimate converging in probability to a matrix $\hat{V}$ of the form

$$\hat{V} = \begin{pmatrix} \hat{V}_{\tau\tau} + \Delta & \hat{V}_{\tau\delta} \\ \hat{V}_{\delta\tau} & \hat{V}_{\delta\delta} \end{pmatrix}$$

with $\Delta \succeq 0$.

As an illustration, consider the conventional covariance estimator for the difference in means in a two-sample problem, $\hat{V}_{\tau\tau} = N \left( \frac{\hat{\Sigma}_{y(1)/n_1} + \hat{\Sigma}_{y(0)/n_0}}{n_1} \right)$ with

$$\hat{\Sigma}_{y(1)} = \frac{1}{n_1} \sum_{i=1}^{N} Z_i \left( y_i(1) - \frac{1}{n_1} \sum_{i=1}^{N} Z_i y_i(1) \right) \left( y_i(1) - \frac{1}{n_1} \sum_{i=1}^{N} Z_i y_i(1) \right)^T$$

and the analogous for $\hat{\Sigma}_{y(0)}$. Under both completely randomized experiments and rerandomized experiments with balance criterion satisfying Condition 1, this estimator satisfies $\hat{V}_{\tau\tau} - V_{\tau\tau} \overset{p}{\to} \Sigma_{\tau\tau, \infty} \succeq 0$ under Assumptions 1 - 3(a).

### 4.2 Limiting behavior of the reference distribution

Suppose we have a completely randomized design, and consider the random variable

$$\{ \sqrt{N} \hat{y}(Z), \sqrt{N} \hat{\delta}(x, W) \}^T$$

$$= \sqrt{N} \left\{ \frac{1}{n_1} \sum_{i=1}^{N} W_i \hat{y}_i(Z_i) - \frac{1}{n_0} \sum_{i=1}^{N} (1 - W_i) \hat{y}_i(Z_i), \frac{1}{n_1} \sum_{i=1}^{N} W_i x_i - \frac{1}{n_0} \sum_{i=1}^{N} (1 - W_i) x_i \right\}^T$$

where $Z$ and $W$ are independent, identically distributed, and drawn uniformly from $\Omega$ and $\hat{y}_i(Z_i) = y_i(Z_i) - Z_i \bar{\tau}$, such that

$$\hat{y}(Z) = y(Z) - Z \bar{\tau}^T.$$

**Proposition 1.** Subject to Assumptions 1, 2, and 3(b), under a completely randomized design the distribution of $\{ \sqrt{N} \hat{y}(Z), \sqrt{N} \hat{\delta}(x, W) \}^T | Z$ converges weakly in probability to a multivariate Gaussian measure, with mean zero and covariance $\hat{V}$ of the form

$$\hat{V} = \begin{pmatrix} \hat{V}_{\tau\tau} & \hat{V}_{\tau\delta} \\ \hat{V}_{\delta\tau} & \hat{V}_{\delta\delta} \end{pmatrix};$$

$$\hat{V}_{\tau\tau} = (1 - p)^{-1} \Sigma_{y(1), \infty} + p^{-1} \Sigma_{y(0), \infty};$$

$$\hat{V}_{\delta\delta} = \{ p(1 - p) \}^{-1} \Sigma_{x, \infty};$$

$$\hat{V}_{\tau\delta} = (1 - p)^{-1} \Sigma_{y(1), x, \infty} + p^{-1} \Sigma_{y(0), x, \infty} = \hat{V}_{\delta\tau}^T.$$

The proof of this statement is contained within the proof of Theorem 1 in Wu and Ding (2018) and is omitted. Under the sharp null, $\hat{V} = V$ as $y_i(1) = y_i(0)$ for all $i$. Under the weak null however, while $\hat{V}_{\delta\delta} = V_{\delta\delta}$ generally $\hat{V}_{\tau\tau} \neq V_{\tau\tau}$ and $\hat{V}_{\delta\tau} \neq V_{\delta\tau}$. The divergence between $V$ and $\hat{V}$ can render randomization tests for the weak null hypothesis anti-conservative; examples are given in §5.2. We now describe how prepivoting may be used to guarantee asymptotic correctness when inference for the weak null hypothesis is conducted using a reference distribution generated under the sharp null.

### 5 Gaussian Prepivoting

#### 5.1 Prepivoting with an estimated pushforward measure

We begin with statistics for $H_N$ of the form

$$T(y(Z), Z) = f_\xi(\sqrt{N} \hat{\tau}),$$

where $f_\eta(\cdot)$ and $\hat{\xi} = \hat{\xi}(\hat{y}(Z), Z)$ satisfy the following conditions for some set $\Xi$:
Condition 2. For any \( \eta \in \Xi \), \( f_\eta(\cdot) : \mathbb{R}^d \mapsto \mathbb{R}_+ \) is continuous, quasi-convex, and nonnegative with \( f_\eta(t) = f_\eta(-t) \) for all \( t \in \mathbb{R}^d \). Furthermore, \( f_\eta(t) \) is jointly continuous in \( \eta \) and \( t \).

Condition 3. With \( W, Z \) independent and each uniformly distributed over \( \Omega \),

\[
\hat{\gamma}(\hat{y}(Z), Z) \overset{D}{\rightarrow} \gamma; \quad \hat{\gamma}(\hat{y}(Z), W) \overset{D}{\rightarrow} \hat{\gamma},
\]

for some \( \gamma, \hat{\gamma} \in \Xi \).

As will be shown in §5.2, several commonly encountered statistics for Neyman’s null are of this form. Suppose further that one employs a covariance estimator \( \hat{\gamma}(\hat{y}(Z), Z) \) with the following property:

Condition 4. With \( W, Z \) independent, both uniformly distributed over \( \Omega \), and for some \( \Delta \geq 0, \Delta \in \mathbb{R}^{d \times d} \),

\[
\hat{\gamma}(\hat{y}(Z), Z) - V \overset{D}{\rightarrow} \begin{pmatrix} \Delta & 0_{d,k} \\ 0_{k,d} & 0_{k,k} \end{pmatrix}; \quad \hat{\gamma}(\hat{y}(Z), W) - \hat{V} \overset{D}{\rightarrow} 0_{(d+k),(d+k)}.
\]

Observe that when assuming the weak null for the purpose of testing, \( \hat{y}(Z) = y(Z) \) and \( \hat{\tau} - \bar{\tau} = \bar{\tau} \). Gaussian prepivoting transforms the test statistic \( T(y(Z), Z) = f_\xi(\sqrt{N} \bar{\tau}) \) into a new statistic of the form

\[
G(y(Z), Z) = \frac{\gamma_0^{(d+k)} \{ (a, b)^T : f_\xi(a) \leq T(y(Z), Z) \wedge \phi(b) = 1 \}}{\gamma_0^{(k)} \{ b : \phi(b) = 1 \}} \tag{5}
\]

where \( \gamma_{\mu, \Sigma}^{(p)}(B) \) is the \( p \)-dimensional Gaussian measure of a set \( B \) with mean parameter \( \mu \) and covariance \( \Sigma \), i.e.

\[
\gamma_{\mu, \Sigma}^{(p)}(B) = \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \int_{x \in B} \exp \left\{ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\} \, dx.
\]

For \( (A, B)^T \) jointly multivariate normal with mean zero and covariance \( \hat{V}, A \in \mathbb{R}^d, B \in \mathbb{R}^k \), \( G(y(Z), Z) \) represents the \( f_\xi \)-pushforward measure of \( A | \phi(B) = 1 \) evaluated on the set \( (-\infty, T(y(Z), Z)] \). That is, \( G(y(Z), Z) \) treats \( f_\xi \) and \( \hat{V} \) as fixed and computes the conditional probability that \( f_\xi(A) \) falls at or below the observed value for \( T(y(Z), Z) = f_\xi(\sqrt{N} \bar{\tau}) \) given that \( \phi(B) = 1 \). From the perspective of hypothesis testing, \( G(y(Z), Z) \) is 1 minus the large-sample \( p \)-value for \( T(y(Z), Z) \) levering the finite population central limit theorem and the estimated covariance \( \hat{V} \).
As a result, the randomization distribution for $G$.

**Algorithm 1:** Inference for the weak null through Gaussian prepivoting

**Input:** An observed treatment allocation $z$, with observed responses $y(z)$, test statistic $T(y(z), z) = f_\xi(\sqrt{N} \hat{\tau}_{obs})$ and covariance estimator $\hat{V}(y(z), z)$

**Result:** The $p$-value for the Gaussian prepivoted test statistic

Compute $f_{\hat{\xi}(y(z), z)}(\cdot); \hat{V}(y(z), z)$. Compute

$$g_z = \frac{\gamma_{(d+k)}_{0, \hat{V}(y(z), z)} \{ (a, b)^T : f_{\hat{\xi}(y(z), z)}(a) \leq T(y(z), z) \land \phi(b) = 1 \}}{\gamma_{(k)}_{0, \hat{V}_{kl}(y(z), z)} \{ b : \phi(b) = 1 \}}$$

for $w \in \Omega$

Compute $f_{\hat{\xi}(y(z), w)}(\cdot); \hat{V}(y(z), w)$.

Compute

$$g_w = \frac{\gamma_{(d+k)}_{0, \hat{V}(y(z), w)} \{ (a, b)^T : f_{\hat{\xi}(y(z), w)}(a) \leq T(y(z), w) \land \phi(b) = 1 \}}{\gamma_{(k)}_{0, \hat{V}_{kl}(y(z), w)} \{ b : \phi(b) = 1 \}}$$

end return

$$p_{cal} = \frac{1}{|\Omega|} \sum_{w \in \Omega} 1(g_w \geq g_z);$$

$$\varphi_G(\alpha) = 1(p_{cal} \leq \alpha).$$

Observe that $1 - g_n$ defined within Algorithm 1 is the usual large-sample $p$-value based upon a Gaussian approximation and using the covariance estimator $\hat{V}$. The large-sample test compares $1 - g_n$ to $\alpha$, the desired Type I error rate, and rejects if $1 - g_n \leq \alpha \Leftrightarrow g_n \geq 1 - \alpha$. The Gaussian prepivoted randomization test instead rejects if $g_n \geq \mathcal{P}_G^{-1}(1 - \alpha)$. The following Theorem, in concert with Lemma 11.2.1 of Lehmann and Romano (2005), show under our assumptions $\mathcal{P}_G^{-1}(1 - \alpha) \overset{P}{\rightarrow} 1 - \alpha$, such that the prepivoted randomization test is asymptotically equivalent to large sample test under the weak null. By using $\mathcal{P}_G^{-1}(1 - \alpha)$ instead of $1 - \alpha$, exactness under the sharp null is preserved.

**Theorem 1.** Suppose we have either a completely randomized design or a rerandomized design with balance criterion $\phi$ satisfying Condition 11. Suppose $T(y(Z), Z)$ is of the form (4) for some $f_\xi$ and $\xi$ satisfying Conditions 2 and 3. Suppose further that we employ a covariance estimator $\hat{V}$ satisfying Condition 4 when forming the prepivoted test statistic $G(y(Z), Z)$. Then, under $H_N : \bar{\tau} = 0$ and under Assumptions 1 - 3(a), $G(y(Z), Z)$ converges in distribution to a random variable $\bar{U}$ taking values in $[0, 1]$ satisfying

$$P(\bar{U} \leq t) \geq t,$$

for all $t \in [0, 1]$. Furthermore, strengthening Assumption 3(a) to Assumption 3(b), the distribution $\mathcal{P}_G(t)$ satisfies

$$\mathcal{P}_G(t) \overset{L}{\rightarrow} t$$

for all $t \in [0, 1]$.

Theorem 1 states that under the weak null, $G(y(Z), Z)$ converges in distribution to a random variable which is stochastically dominated by the standard uniform. Meanwhile, the reference distribution for $G(y(Z), Z)$ constructed assuming (incorrectly) that the sharp null holds converges pointwise to the distribution function of a standard uniform. As a result, the randomization distribution for $G(y(Z), Z)$ is asymptotically sharp-dominant: the reference distribution generated in this manner yields asymptotically conservative inference for the weak null hypothesis, while maintaining exactness should the sharp null also hold.
Remark 1. Consider the function
\[ \hat{F}(t) = \gamma_{0, \hat{V}}^{(d+k)} \left\{ (a, b) : f_\hat{\xi}(a) \leq t \land \phi(b) = 1 \right\} \]
the estimated distribution function for \( f_\hat{\xi}(\sqrt{N} \hat{\tau}) | \phi(\sqrt{N} \hat{\delta}) = 1 \) based upon a finite population central limit theorem.
In special cases, the function \( \hat{F}(t) \) may have a known closed form. This is true of the test statistics which are sharp-dominated by a \( \chi^2_d \) distribution considered in Wu and Ding (2018), for example. Should this not be the case, one can approximate \( \hat{F}(\cdot) \) by way of Monte-Carlo approximation, replacing the measures \( \gamma_{0, \hat{V}} \) and \( \gamma_{0, \hat{V}_{is}} \) with estimates based upon a \( B \) draws from a multivariate normal with mean \( 0 \) and covariance \( \hat{V} \) when enumerating the reference distribution. Importantly, such Monte-Carlo approximation does not corrupt finite-sample correctness under Fisher’s sharp null.

5.2 Examples of Gaussian prepivoting

Through a series of examples, we now provide illustrations of the transformations achieved by (5). As will be demonstrated, the form recovers several randomization tests previously known to be valid for weak null hypotheses in the literature, while providing a basis for new randomization tests for weak nulls using other test statistics.

Example 1 (Absolute difference in means). Let \( \sqrt{N} \hat{\tau} \) be univariate, consider a completely randomized design with no rerandomization, and let \( T_{DiM}(y(Z), Z) = \sqrt{N} |\hat{\tau}| \), such that \( f_\eta(t) = |t| \) and \( \hat{\xi} = 1 \). The randomization distribution for \( T_{DiM}(y(Z), Z) \) is not asymptotically sharp-dominant, such that employing the reference distribution assuming that the sharp null holds may lead to anticonservative inference. The conventional fix is to studentize \( \sqrt{N}|\hat{\tau}| \) using a variance estimator estimator satisfying Condition 4, forming instead \( T_{Stu}(y(Z), Z) = \sqrt{N}|\hat{\tau}| / \sqrt{V_{\tau\tau}} \) (Loh et al., 2017).

As \( \phi(\cdot) = 1 \) deterministically in a completely randomized design, Gaussian prepivoting via (5) yields the test statistic
\[ G_{DiM}(y(Z), Z) = \gamma_{0, \hat{V}_{\tau\tau}}^{(1)} \{ a : |a| \leq \sqrt{N}|\hat{\tau}| \} = 1 - 2\Phi \left( -\frac{\sqrt{N}|\hat{\tau}|}{\sqrt{V_{\tau\tau}}} \right), \]
where \( \Phi(\cdot) \) is the standard normal distribution function. For any \( Z \), the pairs \( \{G_{DiM}(y(Z), w), T_{DiM}(y(Z), w)\} \) have rank correlation equal to 1 when computed for all \( w \in \Omega \). As a result, the reference distribution using the studentized difference in means assuming the sharp null will furnish identical \( p \)-values to those attained using Gaussian prepivoting. That is, in the univariate case Gaussian prepivoting is equivalent to studentization for completely randomized designs.

Example 2 (Multivariate studentization). Let \( \sqrt{N} \hat{\tau} \) now be multivariate and suppose we have a completely randomized design. Wu and Ding (2018) suggest the test statistic
\[ T_{\chi^2}(y(Z), Z) = \left( \sqrt{N} \hat{\tau} \right)^T \hat{V}_{\tau\tau}^{-1} \left( \sqrt{N} \hat{\tau} \right); \]
\[ \hat{V}_{\tau\tau} = \frac{N}{n_1} \hat{\Sigma}_{y(1)} + \frac{N}{n_0} \hat{\Sigma}_{y(0)}. \]
For this test statistic, \( f_\eta(t) = t^T \eta^{-1} t \) and \( \hat{\xi} = \hat{V}_{\tau\tau} \). Wu and Ding (2018) show that under our assumptions, under the weak null this test statistic converges in distribution to \( \sum_{i=1}^d w_i \zeta_i^2 \) where \( w_i \in [0, 1] \) are weights and \( \zeta_1, \ldots, \zeta_d \sim N(0, 1) \) while the reference distribution of \( T_{\chi^2}(y(Z), Z) \) attained assuming that the sharp null holds converges weakly in probability to the \( \chi^2_d \)-distribution (Wu and Ding, 2018). As a result, \( T_{\chi^2}(y(Z), Z) \) is asymptotically sharp-dominant, and its reference distribution assuming the sharp null may be used for inference for the weak null hypothesis. Here, Gaussian prepivoting produces
\[ G_{\chi^2}(y(Z), Z) = \gamma_{0, \hat{V}_{\tau\tau}}^{(d)} \{ a : a^T \hat{V}_{\tau\tau}^{-1} a \leq T_{\chi^2}(y(Z), Z) \} = F_{d}\{T_{\chi^2}(y(Z), Z)\}, \]
where \( F_{d}(\cdot) \) is the distribution function of a \( \chi^2_d \) random variable. For any \( Z \), the pairs \( \{G_{\chi^2}(y(Z), w), T_{\chi^2}(y(Z), w)\} \) have rank correlation equal to 1 when computed for all \( w \in \Omega \), such that Gaussian prepivoting yields equivalent inference to that attained using the distribution of \( T_{\chi^2}(y(Z), Z) \) under the sharp null.
Suppose instead that, erroneously, a practitioner proceeded with the more typical form of Hotelling’s $T$-squared statistic employing a pooled covariance estimator,

$$ T_{\text{Pool}}(y(Z), Z) = \left( \sqrt{N} \hat{\tau} \right)^{\top} \left( \hat{V}_{\text{Pool}} \right)^{-1} \left( \sqrt{N} \hat{\tau} \right); $$

$$ \hat{V}_{\text{Pool}} = N \left\{ \frac{(n_1 - 1)\hat{\Sigma}_y(1) + (n_0 - 1)\hat{\Sigma}_y(0)}{n_1 + n_0 - 2} \right\}. $$

For this test statistic, $f_\eta(t) = t^\top \eta^{-1} t$ as before, but $\hat{\xi} = \hat{V}_{\text{Pool}}$. In this case, $T_{\text{Pool}}(y(Z), Z)$ is not asymptotically sharp-dominant, such that the reference distribution using this statistic and assuming the sharp null may yield invalid inference. Gaussian prepivoting returns the test statistic

$$ G_{\text{Pool}}(y(Z), Z) = \gamma^{(d)}_{0, \hat{V}_{\text{Pool}}} \{ a : a^\top \hat{V}_{\text{Pool}}^{-1} a \leq T_{\text{Pool}}(y(Z), Z) \}. $$

Importantly, $T_{\text{Pool}}(y(Z), Z)$ continues to use the Gaussian measure computed with the covariance matrix $\hat{V}_{\text{Pool}}$ in forming the suitable transformation, despite the fact that the pooled covariance matrix is used in forming $T_{\text{Pool}}(y(Z), w)$. For fixed $Z$, $G_{\text{Pool}}(y(Z), w)$ generally will not have perfect rank correlation with $T_{\text{Pool}}(y(Z), w)$ when computed over $w \in \Omega$, such that the two randomization tests assuming the sharp null no longer furnish identical $p$-values. This divergence is necessary: while $T_{\text{Pool}}(y(Z), Z)$ is not asymptotically sharp-dominant, Theorem 1 asserts that $G_{\text{Pool}}(y(Z), Z)$ is, such that the reference distribution for $G_{\text{Pool}}(y(Z), Z)$ assuming the sharp null yields asymptotically conservative inference for the weak null. Gaussian prepivoting can thus restore asymptotic validity to a test statistic employing improper studentization.

**Example 3** (Max absolute $t$-statistic). Consider again multivariate $\sqrt{N} \hat{\tau}$ and a completely randomized design, and consider the test statistic

$$ T_{\text{max}1}(y(Z), Z) = \max_{1 \leq j \leq d} \frac{\sqrt{N} |\hat{\tau}_j|}{\sqrt{\hat{V}_{\tau, jj}}}, $$

where $\hat{V}_{\tau, jj}$ is the $j^{th}$ element of $\hat{V}_{\tau}$. For this statistic, $f_\eta(t) = \max_{1 \leq j \leq d} |t_j| / \eta_j$, and $\hat{\xi} = (\hat{V}_{\tau,11}, ..., \hat{V}_{\tau,d}^1)^\top$. For $d \geq 2$, $T_{\text{max}1}(y(Z), Z)$ is not asymptotically sharp-dominant under the weak null: the reference distribution generated under the sharp null depends upon the correlation matrix corresponding to $\hat{V}$, while the true randomization distribution is governed by the correlations encoded within $V$. The Gaussian prepivoted correction takes the form

$$ G_{\text{max}1}(y(Z), Z) = \gamma^{(d)}_{0, \hat{V}_{\tau}} \left\{ a : \max_{1 \leq j \leq d} \frac{|a_j|}{\sqrt{\hat{V}_{\tau, jj}}} \leq \max_{1 \leq j \leq d} \frac{\sqrt{N} |\hat{\tau}_j|}{\sqrt{\hat{V}_{\tau, jj}}} \right\}, $$

which composes $T_{\text{max}1}(y(Z), Z)$ with the distribution function for $\max |A_j| / \sqrt{\hat{V}_{\tau, jj}}$, $j = 1, ..., d$, when $A$ is multivariate Gaussian with mean zero and covariance $\hat{V}_{\tau}$.  

**Example 4** (Rerandomization). Let $\sqrt{N} \hat{\tau}$ be univariate and suppose we now consider a rerandomized design with balance criterion $\phi$ satisfying Condition 1. Consider the absolute difference in means, $f_\xi(\sqrt{N} \hat{\tau}) = \sqrt{N} |\hat{\tau}|$, such that $\hat{\xi} = 1$. Gaussian prepivoting yields the test statistic

$$ G_{\text{Re}}(y(Z), Z) = \gamma^{(1+k)}_{0, \hat{V}} \left\{ (a, b)^\top : |a| \leq \sqrt{N} |\hat{\tau}| \land \phi(b) = 1 \right\} $$

For completely randomized designs with $\phi(\cdot) = 1$ deterministically, Gaussian prepivoting is equivalent to studentizing as described in Example 1. In general rerandomized designs however, observe that the transformation depends upon the particular form of the balance criterion $\phi$, and that the reference distribution will depend upon the relationship between the potential outcomes and the covariates used in the balance criterion. As a result, it will generally not be the case that the reference distribution of $G_{\text{Re}}(y(Z), Z)$ under the sharp null yields equivalent inference to that attained using $\sqrt{N} |\hat{\tau}| / \sqrt{\hat{V}_{\tau}}$. This suggests that in rerandomized designs, studentization alone is insufficient for attaining an asymptotically sharp-dominant test statistic. In §8.1, we show this through an example in the case of Mahalanobis rerandomization. Lemmas A15 and A16 of Li et al. (2018) show that under our conditions, probability limits for estimators $\hat{V}$ derived under complete randomization are generally preserved under rerandomized designs. Once again, Theorem 1 ensures that $G_{\text{Re}}(y(Z), Z)$ will be asymptotically sharp-dominant, such that the randomization distribution assuming the sharp null may be employed for inference for the weak null.
6 Gaussian comparison, stochastic dominance, and the probability integral transform

6.1 Gaussian comparison and Anderson’s Theorem

We now highlight the essential technical ingredients underpinning the success of Gaussian prepivoting. Consider two mean-zero multivariate Gaussian vectors \((\mathbf{A}_1, \mathbf{B}_1)^T\) and \((\mathbf{A}_2, \mathbf{B}_2)^T\), with covariances

\[
M_1 = \begin{pmatrix} \Lambda^{(1)}_{aa} & \Lambda_{ab} \\ \Lambda_{ba} & \Lambda_{bb} \end{pmatrix}, \quad M_2 = \begin{pmatrix} \Lambda^{(2)}_{aa} & \Lambda_{ab} \\ \Lambda_{ba} & \Lambda_{bb} \end{pmatrix},
\]

satisfying \(\Lambda^{(2)}_{aa} - \Lambda^{(1)}_{aa} \geq 0\) and \(\Lambda_{bb} > 0\); the inequalities are stated with respect to the Loewner partial order on positive semidefinite matrices. Let the dimensions of \(\mathbf{A}_j\) and \(\mathbf{B}_j\) be \(d\) and \(k\) respectively for \(j = 1, 2\). Compare the tail probabilities for

\[
f(\mathbf{A}_1) \mid \phi(\mathbf{B}_1) = 1 \quad \text{and} \quad f(\mathbf{A}_2) \mid \phi(\mathbf{B}_2) = 1,
\]

where \(\phi\) and \(f\) satisfy Conditions 1 and Condition 2 respectively. The following result is a straightforward corollary of Anderson’s (1955) theorem for multivariate Gaussians; see also Theorem 4.2.5 of Tong (1990).

**Lemma 1.** Under the stated conditions, for any scalar \(v\),

\[
\mathbb{P}\{f(\mathbf{A}_1) \geq v \mid \phi(\mathbf{B}_1) = 1\} \leq \mathbb{P}\{f(\mathbf{A}_2) \geq v \mid \phi(\mathbf{B}_2) = 1\}.
\]

The result follows immediately from Anderson’s theorem after noting that the set \(\mathcal{B}_v = \{(a, b)^T : f(a) \leq v \wedge \phi(b) = 1\}\) is convex and mirror-symmetric for any \(v\). This can be seen through our assumption that \(f(\cdot)\) is quasi-convex and mirror-symmetric, such that its sublevel sets are convex and mirror symmetric. We further have that \(\mathbb{P}(\phi(\mathbf{B}_1) = 1) = \mathbb{P}(\phi(\mathbf{B}_2) = 1) > 0\) given the structure of the covariance matrices \(M_1\) and \(M_2\) and Condition 1, completing the proof.

6.2 Stochastic dominance and the probability integral transform

For two real valued random variables \(S\) and \(T\), \(S\) (first order) stochastically dominates \(T\) if \(F_S(a) \leq F_T(a)\) for all \(a \in \mathbb{R}\), where \(F_S\) and \(F_T\) are the distribution functions of \(S\) and \(T\) respectively.

Suppose now that \(S\) and \(T\) are continuous and that \(S\) stochastically dominates \(T\). By the probability integral transform, the distribution of \(F_T(T)\) would be standard uniform. The following proposition considers transforming the random variable \(T\) not by its own distribution function, but rather by the distribution function of \(S\), its stochastically dominating random variable.

**Lemma 2.** Suppose that \(S, T\) are continuous random variables and that \(S\) stochastically dominates \(T\). Then, \(F_S(T)\) is stochastically dominated by a standard uniform random variable.

**Proof.** For any \(t \in [0, 1]\), \(\mathbb{P}\{F_S(T) \leq t\} = \mathbb{P}\{T \leq F_S^{-1}(t)\} \geq \mathbb{P}\{S \leq F_S^{-1}(t)\} = t. \quad \square\)

In the setup of §6.1, under Conditions 1 and 2 we have by Proposition 1 that \(f(\mathbf{A}_2) \mid \phi(\mathbf{B}_2) = 1\) stochastically dominates \(f(\mathbf{A}_1) \mid \phi(\mathbf{B}_1) = 1\). Consequently, composing \(f(\mathbf{A}_1) \mid \phi(\mathbf{B}_1) = 1\) with the distribution function of \(f(\mathbf{A}_2) \mid \phi(\mathbf{B}_2) = 1\) would yield a random variable that is stochastically dominated by a standard uniform.

6.3 A sketch of Theorem 1

While a formal proof of Theorem 1 is deferred to the appendix, here we provide an informal sketch in light of Lemmas 1 and 2. Under Assumptions 1 - 3(a) and Condition 1, \(\sqrt{N}(\hat{\tau} - \bar{\tau})\) converges in distribution to \(\mathbf{A}_1 \mid \phi(\mathbf{B}_1) = 1\), where \((\mathbf{A}_1, \mathbf{B}_1)^T\) are jointly multivariate normal with covariance \(V\). Recall that \(T(\mathbf{y}(\mathbf{Z}), \mathbf{Z}) = f_\xi(\sqrt{N}\hat{\tau})\) for some \(f_\eta\) satisfying Condition 2 for all \(\eta \in \Xi\), some \(\xi\) satisfying Condition 3, and with a balance criterion \(\phi\) satisfying Condition 1. By Condition 3 and the assumption of the weak null, we have that \(\xi(\mathbf{y}(\mathbf{Z}), \mathbf{Z})\) converges in probability to \(\xi\). Therefore, under the weak null, by Lemma 1 the limiting distribution of \(T(\mathbf{y}(\mathbf{Z}), \mathbf{Z})\) would be stochastically dominated by that of \(f_\xi(\mathbf{A}_2) \mid \phi(\mathbf{B}_2) = 1\) for any \((\mathbf{A}_2, \mathbf{B}_2)^T\) multivariate Gaussian with covariance matrix

\[
\tilde{V} = V + \begin{pmatrix} \Delta & 0_{d,k} \\ 0_{k,d} & 0_{k,k} \end{pmatrix}.
\]
with $\Delta \geq 0$. The transformation
\[
\tilde{G}(y(Z), Z) = \gamma_{0, \tilde{V}}^{(d+k)} \left\{ (a, b)^T : f_\xi(a) \leq f_\xi(\sqrt{N}\tilde{r}) \land \phi(b) = 1 \right\}
\]
transforms $T(y(Z), Z)$ by the distribution function of a random variable which stochastically dominates its limiting distribution. By Lemma 2 and the continuous mapping theorem, $\tilde{G}(y(Z), Z)$ is stochastically dominated by a standard uniform. By Condition 4, the covariance estimator $\tilde{V}$ used in forming $G(y(Z), Z)$ has a probability limit of the required form for stochastic dominance. Therefore, another application of the continuous mapping theorem yields that $G(y(Z), Z) - \tilde{G}(y(Z), Z) = o_p(1)$, such that by Slutsky’s Theorem $G(y(Z), Z)$ is itself stochastically dominated by a standard uniform.

Meanwhile, Proposition 1 and Condition 1 yield that under the weak null the distribution of $\sqrt{N}\tilde{r}(y(Z), W) | Z$ converges weakly in probability to the distribution of $\tilde{A} | \phi(\tilde{B}) = 1$, where $(A, B)^T$ are jointly multivariate Gaussian with mean zero and covariance $\tilde{V}$. The distribution of $f_\xi(y(Z), W) \{ \sqrt{N}\tilde{r}(y(Z), W) | Z \}$ is precisely $\mathcal{P}_T$, the reference distribution assuming the sharp null holds for the test statistic $T(y(Z), Z) = f_\xi(\sqrt{N}\tilde{r})$. By Condition 4, $\tilde{V}(y(Z), W)$ converges in probability to $\tilde{V}$ itself. Further, by Condition 3 $\tilde{\xi}(y(Z), W)$ converges in probability to $\tilde{\xi}$. Applying the continuous mapping theorem and Slutsky’s Theorem for randomization distributions (Chung and Romano, 2016, Lemmas A5-A6), one sees that Gaussian preivoting furnishes a transformation that amounts to, asymptotically, an application of the probability integral transform. As a result, $\mathcal{P}_G(t)$ converges in probability to $t$, the distribution function of the standard uniform, for all $t \in [0, 1]$.

### 7 Extensions to asymptotically linear estimators

Theorem 1 may be extended to estimators other than the difference in means. Consider an estimator $\hat{\tau}(y(Z), Z)$ such that
\[
\sqrt{N}(\hat{r}(y(Z), Z) - \tau) = \sqrt{N} \left( \frac{1}{n_1} \sum_{i=1}^N Z_i r_i(Z_i) - \frac{1}{n_0} \sum_{i=1}^N (1 - Z_i) r_i(Z_i) \right) + o_p(1)
\]
for some constants $\{r_i(0), r_i(1)\}_{i=1}^N$ which may change with $N$ and that satisfy $(1/N) \sum_{i=1}^N (r_i(1) - r_i(0)) = 0$ along with Assumptions 2 and 3(a). Suppose further that $\hat{\tau}(\tilde{y}(Z), W)$, $W$ independent from $Z$ and drawn uniformly from $\Omega$, satisfies
\[
\sqrt{N}\hat{\tau}(\tilde{y}(Z), W) = \sqrt{N} \left( \frac{1}{n_1} \sum_{i=1}^N W_i \tilde{r}_i(Z_i) - \frac{1}{n_0} \sum_{i=1}^N (1 - W_i) \tilde{r}_i(Z_i) \right) + o_p(1)
\]
for potentially distinct constants $\{\tilde{r}_i(0), \tilde{r}_i(1)\}_{i=1}^N$, which may change with $N$ that satisfy $(1/N) \sum_{i=1}^N (\tilde{r}_i(1) - \tilde{r}_i(0)) = 0$ along with Assumptions 2 and 3(b). Observe that the difference in means estimator satisfies these conditions with $r_i(z) = \tilde{r}_i(z) = y_i(z) - z\tau$ for $z \in \{0, 1\}$. Let $\tau_{\tau_1} = r_i(1) - r_i(0)$. Let $\Sigma_{y(z)}$, $\Sigma_\tau$, $\Sigma_{y(z)x}$, $\Sigma_{\tau x}$ be the analogues of $\Sigma_{y(z)}$, $\Sigma_\tau$, $\Sigma_{y(z)x}$ and $\Sigma_{\tau x}$ for $z \in \{0, 1\}$, and let the same hold with $r$ replaced by $\tilde{r}$. Define $\tilde{V}(\tau)$ and $\tilde{V}(\tau)$ as the analogues of $V$ and $\tilde{V}$, computed now based upon $r(z)$ and $\tilde{r}(z)$ instead of $y(z)$ and $\tilde{y}(z)$ for $z \in \{0, 1\}$ and

Consider a test statistic for the weak null of the form $\tilde{T}(y(Z), Z) = f_\xi(\sqrt{N}\tilde{r})$ for some $f_\eta$ satisfying Condition 2 and $\tilde{\xi}$ satisfying Condition 3, and suppose that there exists a covariance estimator $\tilde{V}$ satisfying Condition 4 with $V$ and $\tilde{V}$ replaced by $V(\tau)$ and $\tilde{V}(\tau)$. The Gaussian preivoted test statistic is
\[
\tilde{G}(y(Z), Z) = \gamma_{0, \tilde{V}}^{(d+k)} \left\{ (a, b)^T : f_\xi(a) \leq \tilde{T}(y(Z), Z) \land \phi(b) = 1 \right\}
\]
\[
\gamma_{0, \tilde{V}}^{(d+k)} \{ b : \phi(b) = 1 \}
\]

**Theorem 2.** Suppose that Neyman’s null, $H_N : \tau = 0$, holds. Then, under the described restrictions on $\tilde{T}(y(Z), Z)$ and $\tilde{V}$ and under Assumption 1 and with Assumptions 2 and 3(a) applied to $r_i(z)$, $z = \{0, 1\}$, $\tilde{G}(y(Z), Z)$ converges in distribution to a random variable $\tilde{U}$ taking values in $[0, 1]$ satisfying
\[
\mathbb{P}(\tilde{U} \leq t) \geq t,
\]

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for all $t \in [0, 1]$. Furthermore, if Assumptions 2 and 3(b) hold for $\hat{r}(z), z = \{0, 1\}$, the permutation distribution $\mathcal{P}_G(t)$ satisfies

$$\mathcal{P}_G(t) \xrightarrow{P} t$$

for all $t \in [0, 1]$.

In the appendix we illustrate that the regression-adjusted average treatment effect estimator and its corresponding estimated variance presented in Lin (2013) can be viewed in this form. As a result, Theorem 2 provides justification for the use of the prepivoted randomization distribution of a regression-adjusted estimator.

8 Simulation studies

8.1 Studentization and prepivoting in rerandomized designs

In the $b$th of $B$ iterations, we draw, for $i = 1, \ldots, N$, covariates iid as

$$x_i \overset{iid}{\sim} \mathcal{N}\left(0, \begin{bmatrix} 1.0 & 0.8 & 0.2 \\ 0.8 & 1 & 0.3 \\ 0.2 & 0.3 & 1 \end{bmatrix}\right).$$

Given these covariates, we draw $r_i(0)$ and $r_i(1)$ as

$$r_i(0) = x_i^T \beta_0 + \epsilon_i(0); \quad r_i(1) = x_i^T \beta_1 + \epsilon_i(1),$$

where $\beta_0 = -(6.4, -4.0, -2.4)$, $\beta_1 = (0.2, 0.4, 0.6)^T$, $\epsilon_i(0) \overset{iid}{\sim} -\mathcal{E}(1) + 1$, $\epsilon_i(1) \overset{iid}{\sim} -\mathcal{E}(1/10) + 10$, $\epsilon_i(0)$ independent of $\epsilon_i(1)$, and $\mathcal{E}(\lambda)$ representing an exponential distribution with rate $\lambda$.

We form the potential outcomes under treatment and control in two distinct ways, one in which the sharp null holds and one in which only the weak null holds:

**Sharp Null**: $y_i(1) = y_i(0) = r_i(1)$

**Weak Null**: $y_i(1) = r_i(1); \quad y_i(0) = r_i(0) + \tilde{r}(1) - \tilde{r}(0)$

Of the $N$ individuals, $n_1 = 0.2N$ receive the treatment and $n_0 = 0.8N$ receive the control. We use a Mahalanobis-based rerandomized design, with criterion $\phi(\sqrt{N}\hat{\delta}) = 1 \left\{ (\sqrt{N}\hat{\delta})^T V_\delta^{-1} (\sqrt{N}\hat{\delta}) \leq 1 \right\}$. This balance criterion reduces the cardinality of $\Omega$ by roughly 80% relative to a completely randomized design. For each $b$, we draw a single $Z \in \Omega$, and proceed with inference using the reference distribution of the following test statistics under the incorrect assumption that the sharp null holds:

1. Absolute difference in means, unstudentized
2. Absolute difference in means, studentized
3. Gaussian prepivoting the absolute difference in means, studentized

The true reference distributions assuming the sharp null are replaced by Monte-Carlo estimates with 1000 draws from $\Omega$ for each $b$, and the desired Type I error rate is $\alpha = 0.1$. We also perform inference using the large-sample reference distribution for the absolute studentized difference in means in a rerandomized design; see Li et al. (2018) for more details. As a covariance estimator $\hat{V}$, we use the conventional unpooled covariance estimator for $(\sqrt{N}\hat{t}, \sqrt{N}\hat{\delta})^T$ in a two-sample design. For the generative models reflecting the sharp and weak nulls, we proceed with both $N = 50$ and $N = 1000$ to compare performance in small and large sample regimes. For each $N$, we conduct $B = 5000$ simulations.

Table 1 contains the results of the simulation study. Under the sharp null with $N = 50$, we see the benefits of using a randomization test: the randomization tests based upon the unstudentized, studentized, and prepivoted absolute difference in means all resulted in a Type I error rate of 0.1 (up to noise from the Monte-Carlo simulation) as desired. Contrast this with the large-sample test, which had an estimated Type I error rate of 0.165 under the sharp null hypothesis. Figure 1 explains the deficiency of the large-sample test by comparing the true distribution for the large-sample $p$-values to the standard uniform distribution. As is seen, at $N = 50$ small $p$-values are more likely to occur than what the standard uniform would predict at any point $t \in [0, 1]$, resulting in inflated Type I error rates. By $N = 1000$, the asymptotic approximation performs much better, as the true distribution of $p$-values lies on top of the standard uniform. Gaussian prepivoting uses 1 minus these large-sample $p$-values as the test statistic whose randomization distribution
Table 1: Inference after rerandomization. The rows describe the simulation settings, which vary between the sharp and weak nulls holding and between small and large sample sizes. The first three columns represent the performance of randomization tests assuming the sharp null hypothesis and using the unstudentized absolute difference in means, absolute studentized difference in means, and Gaussian prepivoted absolute difference in means respectively to perform inference. The last column is a large-sample test which is asymptotically valid for the weak null, based upon Li et al. (2018). The desired Type I error rate in all settings is $\alpha = 0.1$.

<table>
<thead>
<tr>
<th>Randomization Test</th>
<th>Large-Sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sharp, $N = 50$</td>
<td>0.104</td>
</tr>
<tr>
<td>Sharp, $N = 1000$</td>
<td>0.104</td>
</tr>
<tr>
<td>Weak, $N = 50$</td>
<td>0.128</td>
</tr>
<tr>
<td>Weak, $N = 1000$</td>
<td>0.117</td>
</tr>
</tbody>
</table>

is enumerated, such that the solid line in Figure 1 reflects 1 minus the randomization distribution of the Gaussian prepivoted test statistic. As Gaussian prepivoting uses a randomization test under the sharp null, the solid line also reflects the reference distribution employed for performing inference. That these coincide is a consequence of the sharp null holding, such that the randomization tests are exact tests for any sample size.

Under the weak null, we see in Table 1 that even at $N = 1000$, the unstudentized and studentized randomization tests erroneously assuming the sharp null have inflated Type I error rates. This pattern will persist even asymptotically, as in this simulation setup these test statistics are not asymptotically sharp-dominant. This may come as a surprise, as in completely randomized designs studentizing does furnish asymptotic sharp dominance. As evidenced here, the impact of covariates on the limiting distribution in rerandomized experiments invalidates studentization as a mechanism for attaining asymptotic sharp dominance. Figure 2 illustrates this in the case of the studentized test statistic. We see in the top panel that the true distribution function for the studentized test statistic lies below that of the reference distribution assuming the sharp null, such that the right-tail probabilities are larger for the true randomization distribution than they are for the reference distribution. This yields anti-conservative inference. We see in the bottom panel of Figure 2 that through use of Gaussian prepivoting, asymptotic conservativeness has been restored: the true randomization distribution of the prepivoted test statistic is stochastically dominated by the reference distribution assuming the sharp null, as predicted by Theorem 1. We further see that the cumulative distribution assuming the sharp null is converging to
To yield valid randomization tests under the weak null, the solid line needs to lie above the dotted line, such that the solid line attributes less mass in the right tail than the dotted line does.

the distribution function of the standard uniform (a straight line between 0 and 1), again reflecting Theorem 1. Table 1 further shows that the Gaussian prepivoted test and the large-sample test have very similar rejection rates at $N = 1000$, reflecting the asymptotic equivalence of the two methods under the weak null.

8.2 A comparison of multivariate tests

In each iteration $b = 1, ..., B$, we draw $\{r_i(1)\}_{i=1}^N$ and $\{r_i(0)\}_{i=1}^N$ independent from one another and iid from mean zero equicorrelated multivariate normals of dimension $k = 25$ with marginal variances one. The correlation coefficients governing $r_i(1)$ and $r_i(0)$ are 0 and 0.95 respectively. We will have two simulation settings, one each for the sharp and weak null:

- **Sharp Null**: $y_i(1) = y_i(0) = r_i(1)$.
- **Weak Null**: $y_i(1) = r_i(1); y_i(0) = r_i(0) + \bar{r}(1) - \bar{r}(0)$.

In both settings, $n_1 = 0.2N$ individuals receive the treatment and $n_0 = 0.8N$ receive the control. We consider a completely randomized design, and proceed with inference using the reference distribution of the following test statistics under the (erroneous) assumption that the sharp null holds:

1. Hotelling’s $T^2$-squared, unpooled covariance
2. Hotelling’s $T^2$-squared, pooled covariance
3. Max absolute $t$-statistic, unpooled standard error

For each candidate test, we proceed with the randomization distribution both of the untransformed test statistic and the Gaussian prepivoted test statistic. These tests are conducted using Monte-carlo simulation to generate the reference distributions, with 1000 draws from $\Omega$ for each iteration $b$. In addition to the two types of randomization tests, we also compute a large-sample $p$-value for each test which is asymptotically valid under the weak null hypothesis. As a covariance estimator $\hat{V}$, we use the conventional unpooled covariance estimator for $\sqrt{N}\hat{r}$. For each test, we seek to maintain the Type I error rate at or below $\alpha = 0.25$. The elevated value of $\alpha$ used here stems from the fundamental conservativeness of large-sample inference under the weak null in this context: more typical values for $\alpha$, say $\alpha = 0.05$,
The restrictions on the function $f_i$ outlined in Condition 2 require a quasi-convex, continuous function that is mirror-symmetric about the origin. This restriction results in convex, mirror-symmetric sublevel sets for $f_i$ symmetric about the origin. This restriction results in convex, mirror-symmetric sublevel sets for $f_i$, and facilitates the application of Anderson’s theorem, such that dominance in the Loewner order on covariance matrices translates to the stochastic dominance under the weak null. While the restrictions on $f_i$ are sensible with two-sided alternatives, they preclude testing directional alternatives because of the mirror symmetry condition. For instance, suppose one wanted to test the null hypothesis $\tau_i \leq 0$ for all $i = 1, \ldots, d$ versus the alternative that for at least one $i$ ($i = 1, \ldots, d$), $\tau_i > 0$. In the univariate case, choosing $T(y(Z), Z) = \hat{\tau}/\hat{V}_{\tau,\tau}^{1/2}$ does not provide a valid one-sided test for all $\alpha$. That said, it does provide a valid test for $\alpha \leq 0.5$, such that for any reasonable value for $\alpha$ to be deployed in practice a one-sided test is possible.

Suppose we have multivariate potential outcomes and consider the test statistic $T_{\text{max}}(y(Z), Z) = \max_{1 \leq i \leq d} \hat{\tau}_i/\hat{V}_{\tau,\tau}^{1/2}$, with $\hat{V}_{\tau,\tau}$ satisfying Condition 4. Consider the Gaussian prepivoted test statistic $G_{\text{max}}(y(Z), Z)$. The following is, to the best of our knowledge, an open question: is it the case that, for any $\alpha \leq 0.5$, $G_{\text{max}}$ is asymptotically sharp-dominant, in that $\lim \sup \mathbb{E}(G_{\text{max}}(\alpha)) \leq \alpha^*$? Under the assumptions imposed in this work, the answer would be true should the following conjecture on Gaussian comparisons hold:
Conjecture 1. Let $X = (X_1, ..., X_d)$, and $Y = (Y_1, ..., Y_d)$ be $d$-dimensional multivariate Gaussian vectors, with a common mean $\mu = (\mu_1, ..., \mu_d)$ but distinct covariances $\Sigma^X$ and $\Sigma^Y$, with $i,j$ entries $\sigma_{ij}^X$ and $\sigma_{ij}^Y$, respectively. Let $\gamma_{ij}^X = \mathbb{E}\{(X_i - X_j)^2\}$ and $\gamma_{ij}^Y = \mathbb{E}\{(Y_i - Y_j)^2\}$. Define $\text{med} (\max_{1 \leq i \leq d} Y_i)$ as the median of $\max_{1 \leq i \leq d} Y_i$, i.e. the value $c$ such that $\mathbb{P} \left( \max_{1 \leq i \leq d} Y_i \leq c \right) = 0.5$. Suppose that $\sigma_{ii}^Y \geq \sigma_{ii}^X$ for all $i$ and that $\gamma_{ij}^Y \geq \gamma_{ij}^X$ for all $i,j$. Consider any point $c \geq \text{med} (\max_i Y_i)$. Then,

$$\mathbb{P} \left( \max_{1 \leq i \leq d} X_i \geq c \right) \leq ? \mathbb{P} \left( \max_{1 \leq i \leq d} Y_i \geq c \right).$$

The conjecture is true in the univariate case. Under the assumptions of this conjecture, the Sudakov-Fernique inequality (Adler and Taylor, 2009, Theorem 2.2.5) asserts that $\mathbb{E}\{(X_i - X_j)^2\} \leq \mathbb{E}\{(Y_i - Y_j)^2\}$. Define $\text{med} (\max_{1 \leq i \leq d} Y_i)$ as the median of $\max_{1 \leq i \leq d} Y_i$, i.e. the value $c$ such that $\mathbb{P} \left( \max_{1 \leq i \leq d} Y_i \leq c \right) = 0.5$. Suppose that $\sigma_{ii}^Y \geq \sigma_{ii}^X$ for all $i$ and that $\gamma_{ij}^Y \geq \gamma_{ij}^X$ for all $i,j$. Consider any point $c \geq \text{med} (\max_i Y_i)$. Then,

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$$\mathbb{P} \left( \max_{1 \leq i \leq d} X_i \geq c \right) \leq \mathbb{P} \left( \max_{1 \leq i \leq d} Y_i \geq c \right).$$


9.2 Summary

In this work, we present a general framework for designing randomization tests that are both exact for Fisher’s sharp null and are asymptotically conservative for Neyman’s weak null in completely randomized experiments and rerandomized designs. Loosely stated, the approach may be summarized as follows: if one has access to a large-sample test that is asymptotically conservative under Neyman’s weak null, then a Fisher randomization test using the $p$-value produced by that large-sample test will maintain asymptotic correctness under the weak null while additionally restoring exactness. As the Fisher randomization distribution of these $p$-values converges weakly in probability to a uniform, the resulting randomization test assuming the sharp null will have the same large-sample performance under the weak null as large-sample test itself, and will further have the same asymptotic power under local alternatives as the large-sample test. We show that Gaussian pretesting exactly recovers several randomization tests known to be valid under the weak null, while providing a general approach to restore asymptotic correctness to randomization tests for a large class of test statistics. Importantly, our framework immediately provides valid randomization tests of the weak null hypothesis in rerandomized designs, absent from the literature until now.

Appendix

A Useful Lemmas

Lemma A. For any Borel measurable set $B \subseteq \mathbb{R}^\ell$, the centered Gaussian measure of $B$ is a continuous function in terms of the covariance parameter. In other words, $\gamma^\ell_{0,\Sigma}(B)$ is a continuous function of $\Sigma$ over the positive definite cone of $\ell \times \ell$ real matrices with metric induced by the Frobenius norm.

Proof. Denote the space of positive definite $\ell \times \ell$ matrices by $S^{\ell}_{++}$; this is a metric space under the metric induced by the Frobenius norm. Consider a sequence of matrices $\Sigma_N \in S^\ell_{++}$ for which $\Sigma_N \rightarrow \Sigma$. By definition for any Borel measurable set $B \subseteq \mathbb{R}^\ell$

$$\gamma^\ell_{0,\Sigma_N}(B) = \int_B \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\det(\Sigma_N)}} \exp \left( \frac{-x^T \Sigma_N^{-1} x}{2} \right) \, dx.$$

The function $f(M) = \det(M)^{-1/2}$ is continuous over the positive definite cone of $\ell \times \ell$ matrices. Thus, since $\Sigma_N \rightarrow \Sigma$ it follows that

$$\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\det(\Sigma_N)}} \rightarrow \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\det(\Sigma)}}.$$

(7)
All that remains to be examined is the limiting behavior of
\[ \int_B \exp \left( -\frac{x^T \Sigma_N^{-1} x}{2} \right) \, dx. \]

For \((M, x) \in S_{++}^\ell \times \mathbb{R}^\ell\) the function \(g(M, x) = \exp(-x^T M^{-1} x/2)\) is a jointly continuous of both \(x\) and \(M\). Consequently, for all \(x \in \mathbb{R}^\ell\)
\[ \exp \left( -\frac{x^T \Sigma_N^{-1} x}{2} \right) \rightarrow \exp \left( -\frac{x^T \Sigma_N^{-1} x}{2} \right). \]

Since all convergent sequences are bounded there exits a positive semidefinite matrix \(\Sigma_*\) that is greater than or equal to (in the Loewner partial order) all \(\Sigma_N\). Thus, \(\Sigma_N^{-1} \geq \Sigma_*^{-1}\) for all \(N \in \mathbb{N}\). Consequently \(g(\Sigma_N, x)\) is dominated by \(g(\Sigma_*, x)\) for all \(N\) and all \(x \in \mathbb{R}^\ell\). Thus, Lebesgue’s dominated convergence theorem implies that
\[ \int_B \exp \left( -\frac{x^T \Sigma_N^{-1} x}{2} \right) \, dx \rightarrow \int_B \exp \left( -\frac{x^T \Sigma_*^{-1} x}{2} \right) \, dx. \]

Combining (7) and (8) implies that for all sequences \(\Sigma_N \in S_{++}^\ell\) such that \(\Sigma_N \rightarrow \Sigma\)
\[ \gamma_{0, \Sigma_N}^{\ell}(B) \rightarrow \gamma_{0, \Sigma}^{\ell}(B). \] (9)

(9) establishes that \(\gamma_{0, \Sigma}^{\ell}(B)\) is a sequentially continuous function of the parameter \(\Sigma\) for all \(\Sigma \in S_{++}^\ell\). Sequential continuity in a metric space is equivalent to continuity (Giaquinta and Modica, 2007, Theorem 5.31); so \(\gamma_{0, \Sigma}^{\ell}(B)\) is a continuous function of the parameter \(\Sigma\) for all \(\Sigma \in S_{++}^\ell\).

\[\square\]

**Lemma B.** Let a function \(f_\eta(.)\) satisfy Condition 2 and let \(\phi(.)\) satisfy Condition 1. Let the matrix \(V \in S_{++}^{(d+k)}\) be defined blockwise as
\[ V = \begin{pmatrix} V_{\tau \tau} & V_{\tau \delta} \\ V_{\delta \tau} & V_{\delta \delta} \end{pmatrix}. \]

Consider
\[ h(V, \eta, x) = \frac{\gamma_{0, V}^{(d+k)} \{ (a, b)^T : f_\eta(a) \leq x \land \phi(b) = 1 \}}{\gamma_{0, V_{\delta \delta}}^{(k)} \{ b : \phi(b) = 1 \}}. \]

The function \(h(V, \eta, x)\) is a continuous function of \(V, \eta,\) and \(x\) jointly.

**Proof.** Because \(V\) is positive definite, \(V_{\delta \delta}\) must be as well. Thus, the centered Gaussian measure \(\gamma_{0, V_{\delta \delta}}^{(k)}(.)\) is non-singular. Furthermore, because \(\phi(.)\) satisfies Condition 1, the set \(\{ b : \phi(b) = 1 \}\) is Borel measurable with positive Lebesgue measure. Thus, \(\gamma_{0, V_{\delta \delta}}^{(k)} \{ b : \phi(b) = 1 \}\) is positive. Moreover, Lemma A establishes that \(\gamma_{0, V_{\delta \delta}}^{(k)} \{ b : \phi(b) = 1 \}\) is a continuous function of \(V_{\delta \delta}\), and thus of \(V\).

Consider the function \(\kappa : (\eta, x) \rightarrow \{ (a, b)^T : f_\eta(a) \leq x \land \phi(b) = 1 \}\). The range of \(\kappa\) is the set of Borel measurable sets in \(\mathbb{R}^{(d+k)}\). This space can be imbued with the metric\(^1\)
\[ d(B, B') = \mu(B \mathbin{\Delta} B') \]

where \(B \mathbin{\Delta} B'\) is the symmetric difference of \(B\) and \(B'\) and \(\mu(.)\) is Lebesgue measure on \(\mathbb{R}^{(d+k)}\), this is sometimes called the *Fréchet–Nikodým–Aronszajn distance* (Conci and Kubrusly, 2017, Section 4). Consider sequences of \(\eta_N\)

\(^1\)Actually, \(d(B, B')\) is a pseudo-metric unless one considers two sets equal if their symmetric difference is of measure zero. We take this convention since – by absolute continuity – sets of Lebesgue measure zero are of Gaussian measure zero as well.
Further, the above holds for the covariates with \( \eta \) and \( x_N \) which converge to \( x \). Let \( B_N \) denote \( \kappa(\eta_N, x_N) \); the set-theoretic limit of \( B_N \) converges to \( B \) under \( d(B, B') \). This relies upon the continuity of \( f_{\eta}(a) \) in \( \eta \). Thus, \( \kappa \) is sequentially continuous in \( \eta \) and \( x \) jointly. Sequential continuity in a metric space is equivalent to continuity (Giaquinta and Modica, 2007, Theorem 5.31); so \( \kappa \) is jointly continuous in \( \eta \) and \( x \).

The numerator of \( h(V, \eta, x) \) is the composition of \( \gamma_{0, V}^{(d+k)}(B) \) with \( \kappa(\eta, x) \); the former is continuous in \( V \) by Lemma A and in \( B \) by the absolute continuity of Gaussian measure, and the latter is jointly continuous in \( \eta \) and \( x \). Thus, the numerator of \( h(V, \eta, x) \) is jointly continuous in \( V, \eta, \) and \( x \). Since the denominator of \( h(V, \eta, x) \) is a continuous function of \( V \) that is always positive, the function \( h(V, \eta, x) \) itself is a jointly continuous function of \( V, \eta, \) and \( x \). □

### B Proof of main results

#### B.1 A reminder: assumptions and conditions

As in the main text, we rely on some regularity conditions which we restate below for convenience.

**Assumption 1.** The proportion \( n_1/N \) limits to \( p \in (0, 1) \) as \( N \to \infty \).

**Assumption 2.** All finite population means and covariances are Cesàro summable for both the potential outcomes and the covariates. For instance, \( \lim_{N \to \infty} \mathbb{E}(\bar{y}(z)) = \bar{y}(z) \) for \( z \in \{0, 1\} \) and \( \lim_{N \to \infty} \Sigma_y(1) = \Sigma_y(1, \infty) \).

**Assumption 3(a).** The worst-case squared distance from the average potential outcome is \( o(N) \); i.e.

\[
\lim_{N \to \infty} \max_{z \in \{0, 1\}} \max_{i \in \{1, \ldots, N\}} \frac{(y_{ij}(z) - \bar{y}_j(z))^2}{N} = 0.
\]

Further, the above holds for the covariates with \( x_{ij} \) replacing \( y_{ij}(z) \) above for \( j = 1, \ldots, k \).

At times, we will strengthen Assumption 3(a) to the following:

**Assumption 3(b).** There exists some \( C < \infty \) for which, for all \( z \in \{0, 1\} \), all \( j = 1, \ldots, d \) and all \( N \),

\[
\sum_{i=1}^{N} \frac{(y_{ij}(z) - \bar{y}_j(z))^4}{N} < C
\]

Further, the above holds for the covariates with \( x_{ij} \) replacing \( y_{ij}(z) \) above for \( j = 1, \ldots, k \).

---

Recall the \( \bar{y}(Z_i) \) is defined as

\[
\bar{y}_i(Z_i) = y_i(Z_i) - Z_i \bar{\bar{y}},
\]

such that \( \bar{y}(Z) = y(Z) - Z \bar{\bar{y}} \). Further recall the following conditions from the main text.

**Condition 1.** \( \phi : \mathbb{R}^k \to \{0, 1\} \) is an indicator function such that the set \( M = \{ b : \phi(b) = 1 \} \) is closed, convex, and mirror-symmetric about the origin (i.e. \( b \in M \iff -b \in M \)) with non-empty interior.

**Condition 2.** For any \( \eta \in \Xi \), \( f_{\eta}(\cdot) : \mathbb{R}^d \to \mathbb{R}_+ \) is continuous, quasi-convex, and nonnegative with \( f_{\eta}(t) = f_{\eta}(-t) \) for all \( t \in \mathbb{R}^d \). Furthermore, \( f_{\eta}(t) \) is jointly continuous in \( \eta \) and \( t \).

**Condition 3.** With \( W \), \( Z \) independent and each uniformly distributed over \( \Omega \),

\[
\xi(\bar{y}(Z), Z) \overset{p}{\to} \xi; \quad \bar{\eta}(\bar{y}(Z), W) \overset{p}{\to} \xi,
\]

for some \( \xi, \bar{\xi} \in \Xi \).

**Condition 4.** With \( W \), \( Z \) independent, both uniformly distributed over \( \Omega \), and for some \( \Delta \geq 0, \Delta \in \mathbb{R}^{d \times d} \),

\[
\hat{V}(\bar{y}(Z), Z) - V \overset{p}{\to} \Delta \begin{pmatrix} 0_{d,k} & 0_{d,k} \\ 0_{k,d} & 0_{k,d} \end{pmatrix} ; \quad \hat{V}(\bar{y}(Z), W) - V \overset{p}{\to} 0_{(d+k), (d+k)}.
\]
Oftentimes in the proofs it will implicitly be assumed that the weak null holds. For that reason, \( \hat{\xi} \) and \( \hat{V} \) may be written with \( y(Z) \) as inputs rather than \( y(Z) \). Let \( \Omega_{CRE} \) denote the set of allowable treatment allocation vectors \( z \) for a completely randomized experiment. Formally

\[
\Omega_{CRE} = \left\{ z \in \{0, 1\}^N \mid \sum_{i=1}^N z_i = n_1 \right\}.
\]

### B.2 A remark on limiting distributions for rerandomized designs

A completely randomized experiment can be considered a rerandomized experiment for which \( \phi(\cdot) \) is identically one. This trivial balance criterion satisfies Condition 1.\(^2\) When \( \phi(\cdot) \) is not vacuous, the interesting case for rerandomized designs, limiting distributions in completely randomized designs continue to provide corresponding limiting distributions after rerandomization under Condition 1.

By the finite population central limit theorem of Li and Ding (2017), \( \sqrt{N} (\hat{\tau} - \bar{\tau}, \hat{\delta}) \) is asymptotically distributed according to a mean-zero multivariate Gaussian distribution with covariance matrix \( V \), where

\[
\begin{bmatrix}
V_{\tau\tau} & V_{\tau\delta} \\
V_{\delta\tau} & V_{\delta\delta}
\end{bmatrix};
\]

\[
V_{\tau\tau} = p^{-1} \Sigma_{y(1), \infty} + (1 - p)^{-1} \Sigma_{y(0), \infty} - \Sigma_{\tau, \infty};
\]

\[
V_{\delta\delta} = \{p(p - 1)\}^{-1} \Sigma_{x, \infty}.
\]

\[
V_{\tau\delta} = p^{-1} \Sigma_{y(1), \infty} + (1 - p)^{-1} \Sigma_{y(0), \infty} = V_{\delta\tau}^T.
\]

Conditioning according to appropriate balance holding requires that \( V_{\delta\delta} > 0 \). In this case, the conditional probability of \( \sqrt{N} \hat{\tau}(y(Z), Z) \in B \) subject to \( \phi(\sqrt{N} \hat{\delta}(x, Z)) = 1 \) limits to

\[
\gamma_{0,V}^{(d+k)} \left\{ (a, b)^T : a \in B \land \phi(b) = 1 \right\}
\]

for any Borel measurable set \( B \).

Likewise, by Proposition 1 of the main text and Lemma 4.1 of Dümbgen and Del Conte-Zerial (2013), the conditional probability of \( \sqrt{N} \hat{\tau}(y(Z), W) \in B \) subject to \( \phi(\sqrt{N} \hat{\delta}(x, W)) = 1 \) limits to

\[
\gamma_{0,V}^{(d+k)} \left\{ (a, b)^T : a \in B \land \phi(b) = 1 \right\}
\]

for any Borel measurable set \( B \).

The finite population central limit theorem of Li and Ding (2017) and Proposition 1 are statements about joint convergence in distribution for the scaled differences in means for the observed outcomes and for the covariates. Passing to convergence in distribution conditional upon \( \phi(\sqrt{N} \hat{\delta}(x, Z)) = 1 \) or \( \phi(\sqrt{N} \hat{\delta}(x, W)) = 1 \) described in (10) and (11) rests upon the continuity-set argument used in the proof of Proposition A1 in Li et al. (2018). Condition 1 guarantees that such arguments remain valid: in particular the set \( M \) defined within Condition 1 is of positive Lebesgue measure. This allows results for completely randomized designs to provide asymptotics when \( \Omega_{CRE} \) is replaced with \( \Omega \) from a general rerandomized design.

### B.3 Proof of Theorem 1

**Theorem 1.** Suppose we have either a completely randomized design or a rerandomized design with balance criterion \( \phi \) satisfying Condition 1. Suppose \( T(y(Z), Z) \) is of the form \( f_\xi(\sqrt{N} \hat{\tau}) \) for some \( f_\xi \) and \( \xi \) satisfying Conditions 2 and 3. Suppose further that we employ a covariance estimator \( \hat{V} \) satisfying Condition 4 when forming the prepivoted test statistic \( G(y(Z), Z) \). Then, under Neyman’s null \( H_N : \tau = 0 \) and under Assumptions 1 - 3(a), \( G(y(Z), Z) \) converges in distribution to a random variable \( \hat{U} \) taking values in \([0, 1]\) satisfying

\[
P(\hat{U} \leq t) \geq t,
\]

\(^2\)When no covariate information is collected, this statement is then vacuous, but in such a context the comparison to a rerandomized experiment is also missing.
for all \( t \in [0, 1] \). Furthermore, strengthening Assumption 3(a) to Assumption 3(b), the permutation distribution \( \mathcal{P}_G(t) \) satisfies

\[
\mathcal{P}_G(t) \xrightarrow{p} t
\]

for all \( t \in [0, 1] \).

**Proof of Theorem 1.** A completely randomized experiment can be viewed as a rerandomized experiment for which \( \phi(b) = 1 \) for all \( b \in \mathbb{R}^k \); this \( \phi \) satisfies Condition 1. As such, the proof below proceeds with general \( \phi \) satisfying Condition 1 – making no distinction between rerandomized designs and completely randomized design.

First, we focus on the randomization distribution of the prepivoted test statistic; in other words, we examine the limiting distribution of \( G(y(Z), Z) \) under \( H_N \). By the finite population central limit theorem of Li and Ding (2017) in a completely randomized design or a rerandomized design with \( \theta \) satisfying Assumption 3(a), the permutation distribution \( \mathcal{P}_G(t) \) converges in distribution to a random variable \( N(0, V) \) with

\[
V = \begin{pmatrix} V_{\tau\tau} & V_{\tau\delta} \\ V_{\delta\tau} & V_{\delta\delta} \end{pmatrix};
V_{\tau\tau} = p^{-1}\Sigma_y(1, \infty) + (1 - p)^{-1}\Sigma_y(0, \infty) - \Sigma_{\tau, \infty};
V_{\delta\delta} = \{p(1 - p)\}^{-1}\Sigma_{x, \infty};
V_{\tau\delta} = p^{-1}\Sigma_y(1, \infty) + (1 - p)^{-1}\Sigma_y(0, x, \infty) = V^T_{\delta\tau}.
\]

Furthermore, by Condition 1 and Corollary A1 of Li et al. (2018), we have that for \( Z \) instead uniform over \( \Omega \) (accounting for the rerandomized design), \( \sqrt{N}(\bar{\tau} - \bar{x}) \xrightarrow{d} C \), where \( C \) follows the distribution of \( A | \phi(B) = 1 \) for \( A \in \mathbb{R}^d, B \in \mathbb{R}^k \), and \( (A, B)^T \) multivariate Gaussian with covariance \( V \) and mean zero.

By Condition 3 \( \hat{\xi}(y(Z), Z) \xrightarrow{p} \xi \) and by Condition 4

\[
\hat{V}(y(Z), Z) \xrightarrow{p} V + \begin{pmatrix} \Delta & 0_{d,k} \\ 0_{k,d} & 0_{k,k} \end{pmatrix} =: \hat{V}.
\]

Leveraging Lemma B and the continuous mapping theorem, under \( H_N \)

\[
h \left( \hat{V}(y(Z), Z), \hat{\xi}(y(Z), Z), \sqrt{N}\hat{\tau} \right) \xrightarrow{d} h \left( V + \begin{pmatrix} \Delta & 0_{d,k} \\ 0_{k,d} & 0_{k,k} \end{pmatrix}, \xi, C \right)
\]

where \( C \) distributed as before. Unwinding the notation of \( h(\cdot, \cdot, \cdot) \) gives that \( G(y(Z), Z) \) converges in distribution to

\[
\gamma_{0,\hat{V}}^{(d+k)} \{ (a, b)^T : f_{\xi}(a) \leq f_{\xi}(C) \wedge \phi(b) = 1 \}.
\]

(12)

If we had known to plug in \( V \) for \( \hat{V} \), (12) would exactly amount to applying the \( f_{\xi} \)-pushforward of the Gaussian measure \( \gamma_{0,V}^{(d+k)} \) conditional on \( \phi(b) = 1 \), which would result in a uniform random variable since this is just the asymptotic probability integral transform for \( T(y(Z), Z) \) given that \( \phi(\sqrt{N}\hat{\delta}) = 1 \). However, we do not know \( V \) and instead estimate it conservatively using a \( \hat{V} \) that satisfies Condition 4; this results in the discrepancy between the covariance of \( C \) versus the covariance used in the Gaussian measure \( \gamma_{0,\hat{V}}^{(d+k)} \) in (12). Consequently, (12) amounts to \( f_{\xi} \)-pushforward of the Gaussian measure \( \gamma_{0,\hat{V}}^{(d+k)} \) in the numerator (the denominator stays the same in both cases since the bottom right block of both \( \hat{V} \) and \( \hat{V} \) is \( V_{\delta\delta} \)). Since \( \hat{V} \succeq V \), it follows by Lemma 1 of the main text (and Anderson’s theorem more generally) that the numerator of (12) is no larger than the numerator of (12) with \( \hat{V} \) replaced by \( V \). Then, since applying the \( f_{\xi} \)-pushforward of the Gaussian measure \( \gamma_{0,V}^{(d+k)} \) conditional on \( \phi(b) = 1 \) results in a uniform random variable, it follows that (12) is stochastically dominated by a uniform random variable from Lemma 2 in the text. In other words, \( G(y(Z), Z) \) converges in distribution to a random variable \( \hat{U} \) taking values in \([0, 1]\) satisfying \( \mathbb{P}(\hat{U} \leq t) \geq t \) for all \( t \in [0, 1] \).

Now we turn our attention to the limiting value of \( \mathcal{P}_G(t) \) for any \( t \). Relying upon the result of Proposition 1 in the main text – which requires Assumptions 1, 2, and 3(b) – in a completely randomized design the distribution of
\{ \sqrt{N} \hat{\tau}(\hat{y}(Z), W), \sqrt{N} \delta(x, W) \}^T \mid Z \text{ converges weakly in probability to a multivariate Gaussian measure, with mean zero and covariance}

$$\hat{V} = \begin{pmatrix} \hat{V}_{\tau \tau} & \hat{V}_{\tau \delta} \\ \hat{V}_{\delta \tau} & \hat{V}_{\delta \delta} \end{pmatrix}.$$ 

By Dümbgen and Del Conte-Zerial (2013, Lemma 4.1), this is equivalent to

$$\begin{bmatrix} \{ \sqrt{N} \hat{\tau}(\hat{y}(Z), W), \sqrt{N} \delta(x, W) \}^T \\ \{ \sqrt{N} \hat{\tau}(\hat{y}(Z), W'), \sqrt{N} \delta(x, W') \}^T \end{bmatrix} \overset{d}{\rightarrow} \{ (\hat{A}, \hat{B}), (\hat{A}', \hat{B}') \}^T$$

(13)

where Z, W, and W' are independent and uniformly distributed over \(\Omega_{CRE}\) and (\(\hat{A}, \hat{B}\))^T and (\(\hat{A}', \hat{B}'\))^T are independent and identically distributed multivariate Gaussians with mean zero and covariance \(\hat{V}\). By the conditions on \(\phi\) outlined in Condition 1, we further have that for Z, W, and W' independently drawn from \(\Omega\) (now accounting for the restrictions imposed by rerandomization),

$$\begin{bmatrix} \sqrt{N} \hat{\tau}(\hat{y}(Z), W) \\ \sqrt{N} \hat{\tau}(\hat{y}(Z), W) \end{bmatrix} \overset{d}{\rightarrow} (D, D'),$$

(14)

where (D, D') are independent and identically distributed from the conditional distribution of \(\hat{A} \mid \phi(\hat{B}) = 1\).

By Conditions 3 and 4

$$\begin{bmatrix} \hat{\xi}(\hat{y}(Z), W) \\ \hat{V}(\hat{y}(Z), W) \\ \hat{\xi}(\hat{y}(Z), W') \\ \hat{V}(\hat{y}(Z), W') \end{bmatrix} \overset{d}{\rightarrow} \begin{bmatrix} \xi \\ \hat{V} \\ \hat{\xi} \\ \hat{V} \end{bmatrix}.$$  

(15)

Moreover, (14) and (15) hold jointly. Thus, the continuous mapping theorem implies that

$$\begin{bmatrix} h \left( \hat{V}(\hat{y}(Z), W), \hat{\xi}(\hat{y}(Z), W), \sqrt{N} \hat{\tau}(\hat{y}(Z), W) \right) \\ h \left( \hat{V}(\hat{y}(Z), W'), \hat{\xi}(\hat{y}(Z), W'), \sqrt{N} \hat{\tau}(\hat{y}(Z), W') \right) \end{bmatrix} \overset{d}{\rightarrow} 
\begin{bmatrix} h \left( \hat{V}, \hat{\xi}, D \right) \\ h \left( \hat{V}, \hat{\xi}, D' \right) \end{bmatrix}$$

(16)

where D and D' are distributed as before.

Recall that under the weak null, \(\hat{y}(Z) = y(Z)\) and \(h \left( \hat{V}(\hat{y}(Z), W), \hat{\xi}(\hat{y}(Z), W), \sqrt{N} \hat{\tau}(\hat{y}(Z), W) \right)\) is precisely \(G(y(Z), W)\) as previously defined. Observe that \(H(\hat{V}, \hat{\xi}, D)\) takes the form

$$h \left( \hat{V}, \hat{\xi}, D \right) = \frac{\gamma^{(d+k)}_{0, \hat{V}} \{ (a, b)^T : f_\xi(a) \leq f_\xi(D) \land \phi(b) = 1 \}}{\gamma^{(k)}_{0, \hat{V}}(b : \phi(b) = 1)}.$$  

(17)

The logic applied to (12) applies similarly to (17) except for the fact that the mismatch in the covariance of \(C\) and \(\gamma^{(d+k)}_{0, \hat{V}}\) of (12) no longer exists in (17) since D is derived from \((\hat{A}, \hat{B})^T \sim N(0, \hat{V})\) and the Gaussian measure \(\gamma^{(d+k)}_{0, \hat{V}}\) is applied. As remarked earlier, since the internal covariance matches the external covariance \(h \left( \hat{V}, \hat{\xi}, D \right)\) is uniformly distributed over \([0, 1]\). Applying Lemma 4.1 of Dümbgen and Del Conte-Zerial (2013) to (16) thus implies that \(\mathcal{S}_G\) converges weakly in probability to Unif[0, 1]. In other words, \(\mathcal{S}_G(t) \overset{d}{\rightarrow} t\) for all \(t \in [0, 1]\).

\[\Box\]

**B.4 Theorem 2**

Theorem 2 reduces to the proof of Theorem 1 by recognizing the \(r_i\) and \(\tilde{r}_i\) as potential outcomes satisfying the required assumptions. The asymptotically vanishing factor \(o_P(1)\) in the definitions of \(\sqrt{N} \{ \hat{\tau}(y(Z), Z) - \tau \}^T\) and \(\sqrt{N} \hat{\tau}(y(Z) - Z \tau^T, W)\) plays no role in the analysis of their limiting distributions, thereby allowing for application of the same proofs used to show Proposition 1 and Theorem 1.
C Gaussian prepotiving after regression adjustment

C.1 Regression adjustment in completely randomized experiments

In completely randomized experiments with covariate information, a common practice is to use regression-based estimators for treatment effects to improve efficiency. Assume that $k$ is fixed and smaller than $N$, and let the potential outcomes be univariate. Define $\hat{\tau}_{\text{reg}}(y(Z), Z)$ to be the estimated coefficient on $Z_i$ in an ordinary least squares regression of $y_i(Z_i)$ on $Z_i$, $(x_i - \bar{x})$, and $Z_i(x_i - \bar{x})$. Lin (2013) shows that under suitable regularity conditions, $\hat{\tau}_{\text{reg}}$ is $\sqrt{N}$-consistent for $\hat{\tau}$ and has an asymptotic variance that is no larger than that of $\hat{\tau}$. Importantly, this result holds true without assuming that the linear model inspiring $\hat{\tau}_{\text{reg}}$ is actually true.

Let

$$Q_1 = \lim_{N \to \infty} \left( \sum_{i=1}^{N} (x_i - \bar{x})(x_i - \bar{x})^T \right)^{-1} \left( \sum_{i=1}^{N} (x_i - \bar{x})^T (y_i(1) - \bar{y}(1)) \right)$$

be the limit of the OLS slopes for potential outcome under treatment regressed upon covariates, and define $Q_0$ analogously for the potential outcomes under control. The population level treatment residuals based upon the limiting slopes are then defined as

$$\varepsilon_i(1) = (y_i(1) - \bar{y}(1)) - (x_i - \bar{x})^T Q_1;$$

$$\varepsilon_i(0) = (y_i(0) - \bar{y}(0)) - (x_i - \bar{x})^T Q_0.$$

Let $\tilde{Q} = pQ_1 + (1-p)Q_0$ and further define

$$\tilde{\varepsilon}_i(1) = (y_i(1) - \bar{y}(1)) - (x_i - \bar{x})^T \tilde{Q};$$

$$\tilde{\varepsilon}_i(0) = (y_i(0) - \bar{y}(0)) - (x_i - \bar{x})^T \tilde{Q}.$$  \hfill (18)

**Proposition 2.** Suppose Assumption 1 holds, and suppose further that Assumptions 2 and 3(b) hold for the potential outcomes and covariates. Then,

$$\sqrt{N} \{ \hat{\tau}_{\text{reg}}(y(Z), Z) - \hat{\tau} \} = \sqrt{N} \left( \frac{1}{n_1} \sum_{i=1}^{N} Z_i \varepsilon_i(Z_i) - \frac{1}{n_0} \sum_{i=1}^{N} (1 - Z_i) \varepsilon_i(Z_i) \right) + o_p(1)$$

$$\sqrt{N} \{ \hat{\tau}_{\text{reg}}(y(Z) - Z \bar{\tau}^T, W) \} = \sqrt{N} \left( \frac{1}{n_1} \sum_{i=1}^{N} W_i \tilde{\varepsilon}_i(Z_i) - \frac{1}{n_0} \sum_{i=1}^{N} (1 - W_i) \tilde{\varepsilon}_i(Z_i) \right) + o_p(1)$$

Let $\tilde{\varepsilon}_i(\hat{y}(Z), W)$ be the $i$th sample residual from a regression of $\hat{y}(Z)$ on $W_i$, $(x_i - \bar{x})$, and $W_i(x_i - \bar{x})$. Using the sample residuals $\tilde{\varepsilon}_i(\hat{y}(Z), W)$ form the variance estimators

$$\hat{\sigma}^2_0(\hat{y}(Z), W) = \frac{1}{n_0 - 1} \sum_{i=1}^{N} (1 - W_i) \left\{ \tilde{\varepsilon}_i(\hat{y}(Z), W) - \frac{1}{n_0} \sum_{j=1}^{N} (1 - W_j) \tilde{\varepsilon}_j(\hat{y}(Z), W) \right\}^2$$

$$\hat{\sigma}^2_1(\hat{y}(Z), W) = \frac{1}{n_1 - 1} \sum_{i=1}^{N} W_i \left\{ \tilde{\varepsilon}_i(\hat{y}(Z), W) - \frac{1}{n_1} \sum_{j=1}^{N} W_j \tilde{\varepsilon}_j(\hat{y}(Z), W) \right\}^2$$

For the $\tilde{\varepsilon}_i(\hat{y}(Z), Z)$’s form $\hat{\sigma}^2_0(\hat{y}(Z), Z)$ and $\hat{\sigma}^2_1(\hat{y}(Z), Z)$ analogously but replace $W$ with $Z$.

Consider the variance estimators

$$\hat{V}_{\text{reg}}(\hat{y}(Z), Z) = \frac{N}{n_1} \hat{\sigma}^2_1(\hat{y}(Z), W) + \frac{N}{n_0} \hat{\sigma}^2_0(\hat{y}(Z), W)$$

$$\hat{V}_{\text{reg}}(\hat{y}(Z), W) = \frac{N}{n_1} \hat{\sigma}^2_1(\hat{y}(Z), W) + \frac{N}{n_0} \hat{\sigma}^2_0(\hat{y}(Z), W).$$

Observe that $\hat{\sigma}^2_j(y(Z), Z) = \hat{\sigma}^2_j(\hat{y}(Z), Z)$ for $j = 0, 1$ regardless of whether or not the weak null holds, but that $\hat{\sigma}^2_j(\hat{y}(Z), W) \neq \hat{\sigma}^2_j(y(Z), W)$ unless the weak null holds.
Proposition 3. \(\hat{V}_{\text{reg}}(\tilde{y}(Z), W)\) satisfies Condition 4 with \(V_{\tau_{\tau}}\) replaced by \(V_{\tau_{\tau}}(c)\) and \(\bar{V}_{\tau_{\tau}}\) replaced by \(\tilde{V}_{\tau_{\tau}}(c)\). The particular form of \(\Delta\), the degree to which \(\hat{V}_{\text{reg}}(\tilde{y}(Z), Z)\) is asymptotically conservative, is

\[
\Delta = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} (\tau_i - \bar{\tau} - (x_i - \bar{x})^T (Q_1 - Q_0))^2.
\]

By Theorem 2, one may apply Gaussian prepivoting to \(\sqrt{N} \hat{\tau}_{\text{reg}}\) using \(\hat{V}_{\text{reg}}\) and any function \(f_{\hat{\xi}}\) satisfying Condition 2 and 3; for instance, take \(f_{\hat{\xi}}(\sqrt{N} \hat{\tau}_{\text{reg}}) = \sqrt{N} |\hat{\tau}_{\text{reg}}|\). Note that other asymptotically equivalent forms for \(\hat{V}_{\text{reg}}\) to the one given here exist. For example, §5 of Lin (2013) suggests using the sandwich variance estimator corresponding to \(\hat{\tau}_{\text{reg}}\).

C.2 Proof of Proposition 2

We begin with the following Lemma:

Lemma C. If Assumptions 2 and 3(a) hold for the potential outcomes and covariates, then Assumptions 2 and 3(a) hold for the collection of \(\varepsilon_i(z)\). Likewise, if Assumptions 2 and 3(b) hold for the potential outcomes and covariates, then Assumptions 2 and 3(b) hold for the collection of \(\hat{\varepsilon}_i(z)\).

Proof. For each \(N\), expanding by the definition of \(\varepsilon_i(1)\) yields

\[
\bar{\varepsilon}(1) = N^{-1} \sum_{i=1}^{N} \left( (y_i(1) - \bar{y}(1)) - (x_i - \bar{x})^T Q_1 \right) = 0;
\]

\[
\varepsilon_{\varepsilon(1)} = (N-1)^{-1} \sum_{i=1}^{N} (y_i(1) - \bar{y}(1)) - (x_i - \bar{x})^T Q_1)^2.
\]

By inspection, Assumption 2 holds for the collection of \(\varepsilon_i(1)\) so long as the potential outcomes and covariates satisfy Assumption 2. Similar proofs establish Assumption 2 for \(\varepsilon_i(0), \bar{\varepsilon}_i(0)\), and \(\hat{\varepsilon}_i(1)\).

Suppose that Assumption 3(a) holds for the potential outcomes and covariates. Then

\[
\lim_{N \to \infty} \max_{\varepsilon \in \{0,1\}} \max_{i \in \{1, \ldots, N\}} \frac{(y_i(z) - \bar{y}^z(z))^2}{N} = 0
\]

and

\[
\lim_{N \to \infty} \max_{j \in \{1, \ldots, k\}} \max_{i \in \{1, \ldots, N\}} \frac{(x_{ij} - \bar{x}_{ij})^2}{N} = 0.
\]

As a consequence of the second statement,

\[
\lim_{N \to \infty} \max_{i \in \{1, \ldots, N\}} \sum_{j=1}^{d} \frac{(x_{ij} - \bar{x}_{ij})^2}{N} = 0
\]

and so, by the Cauchy-Schwarz inequality,

\[
\lim_{N \to \infty} \max_{i \in \{1, \ldots, N\}} \frac{(x_{ij}^T Q_1 - \bar{x}^T Q_1)^2}{N} \leq \lim_{N \to \infty} \max_{i \in \{1, \ldots, N\}} \frac{||Q_1||^2}{N} \sum_{j=1}^{d} (x_{ij} - \bar{x}_{ij})^2 = 0.
\]

Because \((y_i(1) - \bar{y}(1)) - (x_i - \bar{x})^T Q_1)^2 \leq 2(y_i(1) - \bar{y}(1))^2 + 2((x_i - \bar{x})^T Q_1)^2\) it follows from (19) and (20) that Assumption 3(a) holds for the collection of \(\varepsilon_i(z)\).

Now suppose that Assumption 3(b) holds for the potential outcomes and covariates: there exists some \(C < \infty\) for which, for all \(z \in \{0, 1\}\) and all \(N\),

\[
\sum_{i=1}^{N} \frac{(y_{ij}(z) - \bar{y}_{ij}(z))^4}{N} < C \quad \forall j = 1, \ldots, d
\]

and

\[
\sum_{i=1}^{N} \frac{(x_{ij} - \bar{x}_{ij})^4}{N} < C \quad \forall j = 1, \ldots, k.
\]
We split the proof of Proposition 2 into two: Proposition 2(a) and Proposition 2(b).

**Proposition 2(a).**

\[ \sqrt{N} \{ \hat{\tau}_{\text{reg}}(y(Z), Z) - \bar{\tau} \} = \sqrt{N} \left( \frac{1}{n_1} \sum_{i=1}^{N} Z_i \hat{\varepsilon}_i(Z_i) - \frac{1}{n_0} \sum_{i=1}^{N} (1 - Z_i) \hat{\varepsilon}_i(Z_i) \right) + o_p(1) \]

**Proof.** By Lemma A.3 of Lin (2013),

\[ \hat{\tau}_{\text{reg}}(y(Z), Z) - \bar{\tau} = \frac{1}{n_1} \sum_{i=1}^{N} Z_i \hat{\varepsilon}_i(y(Z), Z) - \frac{1}{n_0} \sum_{i=1}^{N} (1 - Z_i) \hat{\varepsilon}_i(y(Z), Z) \]

where the sample residuals \( \hat{\varepsilon}_i(y(Z), Z) \) are derived from the regression of \( y_i(Z_i) \) on \( Z_i, (x_i - \bar{x}) \), and \( Z_i(x_i - \bar{x}) \).

Let \( \hat{Q}_1(y(Z), Z) \) be the sample slope coefficient in the OLS regression of \( y_i(Z_i) \) on \( x_i \) in the group of individuals for which \( Z_i = 1 \); similarly, let \( \hat{Q}_0(y(Z), Z) \) be the sample slope coefficient in the population OLS regression of \( y_i(Z_i) \) on \( x_i \) in the group of individuals for which \( Z_i = 0 \) (Lin, 2013).

Define

\[ \hat{\varepsilon}_i(1) = (y_i(1) - \bar{y}(1)) - (x_i - \bar{x})^T \hat{Q}_1(y(Z), Z); \]
\[ \hat{\varepsilon}_i(0) = (y_i(0) - \bar{y}(0)) - (x_i - \bar{x})^T \hat{Q}_0(y(Z), Z); \]

these are random and depend upon \( Z \). The sample residual \( \hat{\varepsilon}_i(y(Z), Z) \) is \( \hat{\varepsilon}_i(Z_i) \).

By standard OLS theory the slope coefficient matrix \( \hat{Q}_1(y(Z), Z) \) is defined by

\[ \hat{Q}_1(y(Z), Z) = \left( \frac{1}{n_1 - 1} \sum_{i=1}^{N} Z_i \left( x_i - n_1^{-1} \sum_{j=1}^{N} Z_j x_j \right) \left( x_i - n_1^{-1} \sum_{j=1}^{N} Z_j x_j \right)^T \right)^{-1} \times \]
\[ \left( \frac{1}{n_1 - 1} \sum_{i=1}^{N} Z_i \left( x_i - n_1^{-1} \sum_{j=1}^{N} Z_j x_j \right) \left( y_i(Z_i) - n_1^{-1} \sum_{j=1}^{N} Z_j y(Z_j) \right) \right) \]

\( \hat{Q}_0(y(Z), Z) \) is defined analogously.

By weak laws of large numbers for covariance matrices in finite populations, \( \hat{Q}_0(y(Z), Z) \) and \( \hat{Q}_1(y(Z), Z) \) converge in probability to \( Q_0 \) and \( Q_1 \), respectively (Lin, 2013, Lemma A.5). Thus,

\[ \hat{\varepsilon}_i(1) - \varepsilon_i(1) = o_P(1) \]
\[ \hat{\varepsilon}_i(0) - \varepsilon_i(0) = o_P(1) \]

From this, it follows that

\[ \sqrt{N} \{ \hat{\tau}_{\text{reg}}(y(Z), Z) - \bar{\tau} \} = \sqrt{N} \left( \frac{1}{n_1} \sum_{i=1}^{N} Z_i \hat{\varepsilon}_i(Z_i) - \frac{1}{n_0} \sum_{i=1}^{N} (1 - Z_i) \hat{\varepsilon}_i(Z_i) \right) + o_P(1) \]

This proof closely parallels the logic used in the proof for Theorem 1 of Lin (2013). \qed
Before proving the remaining component of Proposition 2 we provide a convenient lemma.

Consider a function $g : \Omega \times \Omega \rightarrow \mathbb{R}$. Let $Z$ and $W$ independently distributed uniformly over $\Omega$. Define two properties:

**Property A.** The random variable $g(Z, W) \mid Z = z$ converges in probability to $c$ for all conditioning sets $\{z\}_{N \in \mathbb{N}}$ except for a set of measure zero.

**Property B.** The random variable $g(Z, W)$ converges in probability to $c$ with respect to randomness in both $Z$ and $W$.

**Lemma D.** Consider a function $g : \Omega \times \Omega \rightarrow \mathbb{R}$. For $Z$ and $W$ independently distributed uniformly over $\Omega$ Property A implies Property B.

**Proof.** Assume that Property A holds. Fix $\varepsilon > 0$; then

$$\mathbb{P}_{W \mid Z} (|g(Z, W) - c| \geq \varepsilon \mid Z) \xrightarrow{a.s.} 0.$$  \hfill (21)

Consider $\mathbb{P}_{W, Z} (|g(Z, W) - c| \geq \varepsilon)$; by the law of total probability

$$\mathbb{P}_{Z, W} (|g(Z, W) - c| \geq \varepsilon) = \sum_{z \in \Omega} \mathbb{P}_{W \mid Z} (|g(Z, W) - c| \geq \varepsilon \mid Z = z) \mathbb{P}_Z (Z = z)$$

$$= \mathbb{E}_Z [\mathbb{P}_{W \mid Z} (|g(Z, W) - c| \geq \varepsilon \mid Z)]$$

Since $\mathbb{P}_{W \mid Z} (|g(Z, W) - c| \geq \varepsilon \mid Z) \xrightarrow{a.s.} 0$ and $\mathbb{P}_{W \mid Z} (|g(Z, W) - c| \geq \varepsilon \mid Z) \in [0, 1]$ the bounded convergence theorem implies that

$$\lim_{N \rightarrow \infty} \mathbb{E}_Z [\mathbb{P}_{W \mid Z} (|g(Z, W) - c| \geq \varepsilon \mid Z)] = \mathbb{E}_Z [0] = 0$$

Thus, $g(Z, W)$ converges in probability to $c$ with respect to randomness in both $Z$ and $W$. \hfill $\Box$

**Proposition 2(b).**

$$\sqrt{N} \left\{ \hat{\tau}_{reg}(y(Z) - Z\bar{\tau}, W) \right\} = \sqrt{N} \left( \frac{1}{n_1} \sum_{i=1}^{N} W_i \hat{\epsilon}_i(Z_i) - \frac{1}{n_0} \sum_{i=1}^{N} (1 - W_i) \hat{\epsilon}_i(Z_i) \right) + o_p(1)$$

**Proof.** By definition $\hat{\tau}_{reg}(y(Z) - Z\bar{\tau}, W)$ is the estimated coefficient on $W_i$ in an ordinary least squares regression of $y_i(Z_i) - Z_i\bar{\tau}$ on $W_i, (x_i - \bar{x})$, and $W_i(x_i - \bar{x})$.

By the same logic that gave rise to Lemma A.3 of Lin (2013),

$$\hat{\tau}_{reg}(y(Z) - Z\bar{\tau}, W) = \frac{1}{n_1} \sum_{i=1}^{N} W_i \hat{\epsilon}_i(y(Z), W) - \frac{1}{n_0} \sum_{i=1}^{N} (1 - W_i) \hat{\epsilon}_i(y(Z), W)$$

where the sample residuals $\hat{\epsilon}_i(y(Z), W)$ are derived from the regression of $y_i(Z_i) - Z_i\bar{\tau}$ on $W_i, (x_i - \bar{x})$, and $W_i(x_i - \bar{x})$. Let $\hat{Q}_1(y(Z), W)$ be the sample slope coefficient in the OLS regression of $y_i(Z_i) - Z_i\bar{\tau}$ on $x_i$ in the group of individuals for which $W_i = 1$; similarly, let $\hat{Q}_0(y(Z), W)$ be the sample slope coefficient in the OLS regression of $y_i(Z_i) - Z_i\bar{\tau}$ on $x_i$ in the group of individuals for which $W_i = 0$. For convenience of notation, denote $N^{-1} \sum_{i=1}^{N} \hat{y}_i(Z_i)$ by $\bar{y}(Z)$.

Consequently

$$\hat{\epsilon}_i(y(Z), W) = \begin{cases} \hat{y}_i(Z_i) - \bar{y}(Z) - (x_i - \bar{x})^\top \hat{Q}_1(y(Z), W); & \text{if } W_i = 1 \\ \hat{y}_i(Z_i) - \bar{y}(Z) - (x_i - \bar{x})^\top \hat{Q}_0(y(Z), W); & \text{if } W_i = 0. \end{cases}$$

these are random and depend upon both $Z$ and $W$.  

[26]
By standard OLS theory the slope coefficient matrix \( \hat{Q}_1(\hat{y}(Z), W) \) is

\[
\hat{Q}_1(\hat{y}(Z), W) = \left( \frac{1}{n_1 - 1} \sum_{i=1}^{N} W_i \left( x_i - n_1^{-1} \sum_{j=1}^{N} W_j x_j \right) \left( x_i - n_1^{-1} \sum_{j=1}^{N} W_j x_j \right)^\top \right)^{-1} \times \\
\left( \frac{1}{n_1 - 1} \sum_{i=1}^{N} W_i \left( x_i - n_1^{-1} \sum_{j=1}^{N} W_j x_j \right) \left( (y_i(Z_i) - Z_i \bar{\tau}) - n_1^{-1} \sum_{j=1}^{N} W_j (y_j(Z_j) - Z_j \bar{\tau}) \right) \right)
\]

In Lemma A.5 of Lin (2013), it is shown that the first term of \( \hat{Q}_1(\hat{y}(Z), W) \) converges in probability to \( \Sigma_{x,\infty}^{-1} \). Now we turn our analysis to the second term of \( \hat{Q}_1(\hat{y}(Z), W) \); denote this term by \( M_1(\hat{y}(Z), W) \).

The centering of the potential outcomes under treatment that occurred when translating \( y_i(z) \) to \( \hat{y}_i(z) \) does not impact Assumptions 1, 2, 3(a), and 3(b). Thus, the finite population strong law for second moments (Wu and Ding, 2018, Lemma A.3, Part ii) applies to the sample covariances \( \hat{\Sigma} \).

\[
\hat{\Sigma}(1)x = \frac{1}{n_1 - 1} \sum_{i=1}^{N} Z_i \left( x_i - n_1^{-1} \sum_{j=1}^{N} Z_j x_j \right) \left( \hat{y}_i(1) - n_1^{-1} \sum_{j=1}^{N} Z_j \hat{y}_j(1) \right)^\top \\
\text{and} \\
\hat{\Sigma}(0)x = \frac{1}{n_0 - 1} \sum_{i=1}^{N} (1 - Z_i) \left( x_i - n_0^{-1} \sum_{j=1}^{N} (1 - Z_j) x_j \right) \left( \hat{y}_i(0) - n_0^{-1} \sum_{j=1}^{N} (1 - Z_j) \hat{y}_j(0) \right)^\top.
\]

Since the centering of the potential outcomes under treatment that occurred when translating \( y_i(z) \) to \( \hat{y}_i(z) \) does not impact the above covariance structure, it follows from Lemma A.3 of Wu and Ding (2018) that \( \hat{\Sigma}(1)x \xrightarrow{a.s.} \Sigma_{y(1)x,\infty} \) and \( \hat{\Sigma}(0)x \xrightarrow{a.s.} \Sigma_{y(0)x,\infty} \) (This statement relies upon Assumptions 1, 2, and 3(b)). Condition on a sequence of treatment allocations \( \{Z\}_{N \in \mathbb{N}} \) for the growing sequence of experiments such that \( \hat{\Sigma}(1)x \mid Z \xrightarrow{a.s.} \Sigma_{y(1)x,\infty} \) and \( \hat{\Sigma}(0)x \mid Z \xrightarrow{a.s.} \Sigma_{y(0)x,\infty} \); this requirement is met for all \( Z \) except for a set of measure zero.

Fix the treatment allocations \( \{Z\}_{N \in \mathbb{N}} \); after this conditioning we are left with fully determined “imputed potential outcomes”:

- \( \{\hat{y}_i(Z_i)\}_{i=1}^{N} \) for the “imputed treatment potential outcomes”
- \( \{\hat{y}_i(Z_i)\}_{i=1}^{N} \) for the “imputed control potential outcomes”

The imputed population can be envisioned as the population that an experiment would imagine to exist if she observed outcomes \( \hat{y}(Z) \) and believed that Fisher’s sharp null held. Consider \( W \) as a treatment allocation for this imputed population. Under this interpretation \( M_1(\hat{y}(Z), W) \) is the sample covariance between covariates and the imputed outcomes observed under “treatment” \( W_i = 1 \). Instead of working with \( M_1(\hat{y}(Z), W) \), we first focus attention to the underlying quantity that \( M_1(\hat{y}(Z), W) \) seeks to estimate: the covariance between covariates and the imputed potential outcomes \( \{\hat{y}_i(Z_i)\}_{i=1}^{N} \); we proceed with analysis based upon a fixed sequence of treatment allocations \( Z \). This quantity is

\[
\Sigma_{\text{imputed},1}(x) = \frac{1}{N} \sum_{i=1}^{N} \left( x_i - N^{-1} \sum_{j=1}^{N} x_j \right) \left( \hat{y}_i(Z_i) - N^{-1} \sum_{j=1}^{N} \hat{y}_j(Z_j) \right)^\top \\
= \frac{1}{N} \sum_{i | Z_i = 1} \left( x_i - N^{-1} \sum_{j=1}^{N} x_j \right) \left( \hat{y}_i(1) - N^{-1} \sum_{j=1}^{N} \hat{y}_j(Z_j) \right)^\top \\
+ \frac{1}{N} \sum_{i | Z_i = 0} \left( x_i - N^{-1} \sum_{j=1}^{N} x_j \right) \left( \hat{y}_i(0) - N^{-1} \sum_{j=1}^{N} \hat{y}_j(Z_j) \right)^\top.
\]
By the strong laws for the sample means, this shares the same limit as
\[
\frac{1}{N-1} \sum_{i=1}^{N} \left( x_i - n_1^{-1} \sum_{j=1}^{N} Z_j x_j \right) \left( \hat{y}_i(1) - n_1^{-1} \sum_{j=1}^{N} Z_j \hat{y}_j(1) \right)^T + \frac{1}{N-1} \sum_{i=0}^{N} \left( x_i - n_0^{-1} \sum_{j=1}^{N} (1 - Z_j) x_j \right) \left( \hat{y}_i(0) - n_0^{-1} \sum_{j=1}^{N} (1 - Z_j) \hat{y}_j(0) \right)^T.
\]
In turn, these two terms can be rewritten as
\[
\frac{n_1 - 1}{N-1} \bar{\Sigma} \bar{y}(1)x + \frac{n_0 - 1}{N-1} \bar{\Sigma} \bar{y}(0)x
\]
which limits to \( p\Sigma \bar{y}(1)x,\infty + (1 - p)\Sigma \bar{y}(0)x,\infty \) for all \( Z \) except for a set of measure zero. Since the centering of the potential outcomes under treatment that occurred when translating \( y_i(z) \) to \( \hat{y}_i(z) \) does not impact Assumptions 1, 2, 3(a), and 3(b) it follows from Lemma 1 of Lin (2013) that
\[
M_1(\hat{y}(Z), W) \mid Z \xrightarrow{p} p\Sigma \bar{y}(1)x,\infty + (1 - p)\Sigma \bar{y}(0)x,\infty
\]
almost surely in \( Z \); combining this with Lemma D implies that
\[
M_1(\hat{y}(Z), W) \xrightarrow{p} p\Sigma \bar{y}(1)x,\infty + (1 - p)\Sigma \bar{y}(0)x,\infty.
\]
Thus
\[
\hat{Q}_1(\bar{y}(Z), W) \xrightarrow{p} \Sigma^{-1}_{Z,\infty} \left( p\Sigma \bar{y}(1)x,\infty + (1 - p)\Sigma \bar{y}(0)x,\infty \right) = Q.
\]
The remainder of the proof proceeds in direct analogy with the proof used for Proposition 2(a).

**Remark 2.** The utility of Lemma D in the proof of Proposition 2(b) arose from our choice to analyze \( M_1(\hat{y}(Z), W) \) through conditioning upon treatment allocation \( Z \). With this conditioning argument, Assumption 3(b) is leveraged to attain strong laws with respect to randomness in \( Z \); these guarantee that arguments based upon conditioning on \( Z = z \) hold for all but a set of measure zero. An alternative approach to arrive at the statement \( M_1(\hat{y}(Z), W) \xrightarrow{p} p\Sigma \bar{y}(1)x,\infty + (1 - p)\Sigma \bar{y}(0)x,\infty \) may be to work unconditionally: appealing to a suitable weak law while allowing for randomness in both \( Z \) and \( W \). With an approach of this nature, Assumption 3(b) may be stronger than necessary.

### C.3 Proof of Proposition 3

First we show that
\[
\hat{V}_{reg}(\bar{y}(Z), Z) = \sum_{i=1}^{N} \left( \varepsilon_i(1) - \frac{1}{n_1} \sum_{j=1}^{N} Z_j \varepsilon_j(1) \right)^2 + \frac{1}{n_0} \sum_{i=1}^{N} (1 - Z_i) \left( \varepsilon_i(0) - \frac{1}{n_0} \sum_{j=1}^{N} (1 - Z_j) \varepsilon_j(0) \right)^2 + o_p(1).
\]
Since \( N/n_1 \to p^{-1} \) and \( N/n_0 \to (1 - p)^{-1} \) this has the same limit as \( N \to \infty \) as
\[
\frac{1}{p n_1 - 1} \sum_{i=1}^{N} Z_i \left( \varepsilon_i(1) - \frac{1}{n_1} \sum_{j=1}^{N} Z_j \varepsilon_j(1) \right)^2 + \frac{1}{1 - p n_0 - 1} \sum_{i=1}^{N} (1 - Z_i) \left( \varepsilon_i(0) - \frac{1}{n_0} \sum_{j=1}^{N} (1 - Z_j) \varepsilon_j(0) \right)^2.
\]
Thus, (22) holds by the weak law of large numbers for second moments (Lin, 2013, Lemma A.1) and second part of Theorem 2 from Lin (2013).

Next we show that

$$\hat{V}_{reg}(\mathbf{y}(\mathbf{Z}), \mathbf{W}) - \hat{V}_{\tau\tau}(\hat{\mathbf{\varepsilon}}) \overset{P}{\to} 0.$$  

(23)

By the proof of Proposition 2(b)

$$\hat{V}_{reg}(\mathbf{y}(\mathbf{Z}), \mathbf{W}) = \frac{N}{n_1} \sum_{i=1}^{N} W_i \left( \hat{\varepsilon}_i(Z_i) - \frac{1}{n_1} \sum_{j=1}^{N} W_j \hat{\varepsilon}_j(Z_j) \right)^2 + \frac{N}{n_0} \sum_{i=1}^{N} (1 - W_i) \left( \hat{\varepsilon}_i(Z_i) - \frac{1}{n_0} \sum_{j=1}^{N} (1 - W_j) \hat{\varepsilon}_j(Z_j) \right)^2 + o_P(1). \quad (24)$$

By conditioning upon $\mathbf{Z}$, an argument similar to that used to analyze $M_1(\hat{\mathbf{y}}(\mathbf{Z}), \mathbf{W})$ in the proof of Proposition 2(b) can then be applied to compute the probability limits of the first two terms in (24) almost surely with respect to the conditioning variable $\mathbf{Z}$. Then leveraging Lemma D yields that the probability limit is the same when considering randomness in both $\mathbf{Z}$ and $\mathbf{W}$. Finally, using $N/n_1 \to p^{-1}$ and $N/n_0 \to (1 - p)^{-1}$ yields that

$$\hat{V}_{reg}(\mathbf{y}(\mathbf{Z}), \mathbf{W}) \overset{P}{\to} \frac{1}{p} \Sigma_{\hat{\varepsilon}(1),\infty} + \frac{1}{1 - p} \Sigma_{\hat{\varepsilon}(1),\infty} = \hat{V}_{\tau\tau}(\hat{\mathbf{\varepsilon}}).$$

D Software

Code written in R that builds the figures of the paper is available via Github at https://github.com/PeterLCohen/PrepivotingCode.

References


