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Swarm Stability Analysis of Nonlinear Dynamical Multi-Agent Systems via Relative Lyapunov Function

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Abstract Conditions for swarm stability of nonlinear high-order multi-agent systems are analyzed based on the idea of space transformation. Swarm stability can be assured by sufficient connectivity of graph topology and dissipative property regulated by relative Lyapunov function, with two independent variables. The problems addressed are general, since the models concerned can be time-varying or heterogeneous.

Keywords Swarm stability · Multi-agent system · Graph · Relative Lyapunov function

الخلاصة

تم تحليل الظروف اللازمة لاستقرار سرب من الأنظمة متعددة الوكيل ذات الترتيب العالي غير الخطية على أساس فكرة تحويل الفضاء. إن استقرار السرب يمكن تأكيده من الاتصال الكاف لطوبولوجيا الرسم البياني والممتلكات المبددة التي تنظمها وظيفة يابونوف النسبية، مع اثنين من المتغيرات المستقلة. والمشاكل المطروحة هي عامة حيث إن النماذج المعنية يمكنها أن تكون متغيرة مع الوقت أو غير متجانسة.

1 Introduction

During recent years, consensus of dynamical multi-agent systems has been paid extensive attention in control theory. Olfati-Saber et al. [1] first introduced the term "consensus" into control theory. Ren et al. [2] relaxed the condition in [1] and proved that to achieve consensus, including a spanning tree is a necessary and sufficient condition for the digraph of a linear system with first-order protocol. A set-valued Lyapunov function approach was developed by Moreau [3]. After 2007, scholars started to consider consensus problems for high-order models. For instance, Xiao et al. [4] proposed a criterion based on the structure of certain high-dimensional matrices. Wang et al. [5] attempted to determine whether an appropriate linear high-order consensus protocol exists under a given undirected graph topology. Cai et al. proved necessary and sufficient conditions for consensus of linear systems [6,7] and a class of nonlinear systems [8], respectively. Li et al. [9] paid attention to the robust stability problem of linear multi-agent systems with observer type interactive protocols. Xi et al. [10–12] devised a technique based on oblique decomposition of state space. Other relevant works include [13–17].

Only few papers in the literature involved consensus problems of high-order nonlinear time-varying heterogeneous systems. For these models, the normal Laplacian spectrum analysis is no longer applicable. The model in [4] is nonlinear and of high order, but the technical assumption that the motion of any agent is towards the convex hull formed by its neighbors is rather strict. Chopra et al. [16] discussed consensus problems of nonlinear systems; however, the model they concerned is essentially of first order. Kim et al. [17] dealt with heterogeneous models, whereas their models are linear.

The current paper endeavors to perform some preliminary research on the swarm stability problem of the systems that might be nonlinear, time-varying or even heterogeneous, which is a generalization of consensus. The main contribution of this paper is proposing an approach to deal with the

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swarm stability of multi-agent systems with high-order nonlinear models. The fundamental idea of the approach is based on relative Lyapunov function, which is a function with dual independent variables and can be regarded as a measure of distance between the trajectories of different agents.

The motivation of the current study arises out of an attempt to extend Lyapunov's second method for checking the stability of dynamical systems, from the case of single isolated system to the case of multiple interconnected systems. The idea behind Lyapunov's second method is powerful and convenient to handle nonlinear systems. Nonetheless, it is still very difficult to achieve the objective to perfectly extend Lyapunov's second method to multi-agent systems, because the situation is much more complicated.

By far, no approaches exist which are really effective in checking the swarm stability [8] of general nonlinear dynamical multi-agent systems. It is impossible to deal with most of the nonlinear swarm stability problems by the existing methods in the literature, whereas the relative Lyapunov function approach introduced in the current paper should be a direction of technical route towards solving many such problems. With an appropriate Lyapunov function, the stability of any equilibrium point of a nonlinear system can be easily checked. However, classical Lyapunov functions are no more suitable for the swarm stability problems of multi-agent systems. It is usually extremely difficult to construct a global Lyapunov function for large-scale systems composed of many interconnected subsystems [17–20]. Besides, the swarm stability of multi-agent systems is essentially a type of non-equilibrium stability [6]; sometimes a system might even have no equilibrium but still be swarm stable [6]. Classical Lyapunov function measures the distance from the trajectory of an isolated dynamic system to an equilibrium point, whereas relative Lyapunov function measures the distance of motions between dynamic systems. Thus, relative Lyapunov function should be used for analyzing the swarm stability of multi-agent systems instead of classical Lyapunov function.

The rest of this paper is organized as follows. In Sect. 2, the multi-agent system model and the swarm stability problems are formulated. Section 3 discusses conditions for swarm stability of nonlinear systems, providing the main theoretical result. Section 4 analyzes the swarm stability for a class of nonlinear systems. Finally, Sect. 5 concludes the paper.

Notation: B_{r} denotes a closed hyper-sphere in any metric space with the origin its spherical center and r its radius.

2 Problem Formulation

In this paper, we consider dynamical multi-agent systems with $N > 1$ agents, no matter whether it is homogenous or heterogeneous. The state of each agent is $x_i \in R^d$, with $d \geq 1$ representing a common order. If agent j can influence the motion of agent i, then agent j is called a neighbor of agent i. The overall neighboring relationship forms the graph topology of a system. Suppose all interactions between agents are independent, the dynamics of agent i can be formulated as follows

$$\dot{x}_i = f_i(x_i, u_i, t) + \sum_{j \in N_i} g_{ij}(x_i, x_j, t) \quad (i, j \in \{1, 2, \ldots, N\}) \tag{1}$$

In (1), $f_i(\bullet)$ represents the self-governed component of the dynamics of agent i; and $g_{ij}(\bullet)$ represents the influence from agent j to i. Such a multi-agent model is a general extension of most models in the literature [1–6], complying with many typical natural/engineering systems.

It has been a common understanding that stability of a swarm system implies cohesion, which is formulated by the following definitions.

Definition 1 [6] (*Swarm stability*) For a dynamical multi-agent system (1) with $x_1, \ldots, x_N \in R^d$, the states of N agents, if for $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$, s.t. $\|x_i(t) - x_j(t)\| < \varepsilon$ $(t > 0)$ as $\|x_i(0) - x_j(0)\| < \delta(\varepsilon)$ $(\forall i, j \in \{1, 2, \ldots, N\})$, then the system is uniformly swarm stable. If $\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$, the system is globally uniformly swarm stable.

Definition 2 [6] (*Asymptotic swarm stability*) If a dynamical multi-agent system is globally uniformly swarm stable and for $\forall \varepsilon, c > 0$, $\exists T = T(\varepsilon, c) > 0$ s.t.

$$\|x_i(t) - x_j(t)\| < \varepsilon$$

as $t > T(\varepsilon, c)$ and $\|x_i(0) - x_j(0)\| < c$ $(\forall i, j \in \{1, 2, \ldots, N\})$, then the system is globally uniformly asymptotically swarm stable.

Remark 1 For a dynamical multi-agent system that may be nonlinear and time-varying, global uniform asymptotic swarm stability is equivalent to full state consensus.

Definition 3 (*Joint connectivity*) A time-varying undirected Nth order graph $G(t)$ being continuous in t is jointly connected if $\exists T > 0$ s.t. for $\forall t > 0$, $\int_t^{t+T} W(t)dt$ represents a connected graph, where $W(t) \in R^{N \times N}$ is the weighted adjacency matrix of $G(t)$.

3 Swarm Stability of General Nonlinear Dynamical Multi-Agent Systems

This section deals with the criteria for swarm stability of nonlinear time-varying multi-agent systems. Our approach is based on the idea of space transformation. The original system is mapped into an image system in an abstract space, with both systems being equal in the sense of swarm stability.

The approach will be effective if the swarm stability of the image system can be much more easily checked.

Although some of the theoretical results will be summarized as propositions and corollaries for convenience of reading, the discussions may not be limited to the literal statements of these propositions and corollaries because usually more results can be easily derived by trivial adjustments.

Proposition 1 *Suppose there is a continuously differentiable multi-agent system (1) in R^d with N agents, if there exists an injective linear operator $T : R^d \to H$, with H the image space, such that for any trajectory*

$$x_0(t) \in co(x_1(t), x_2(t), \ldots, x_N(t)) \in R^d$$

we have

$$T' x_i(t) = T(x_i - x_0) = \xi_i(t) \in H$$

and

$$T' x_0(t) \equiv 0$$

Let $k(t)$ denote the index of agent possessing the maximal value of $\|\xi_k\|$,

(1) if $\exists T > 0$ such that for $\forall t > 0$

$$\|\xi_k(t+T)\| - \|\xi_k(t)\| \le 0$$

then the system is globally uniformly swarm stable;

(2) if $\exists T > 0$ such that for $\forall t > 0$

$$\|\xi_k(t+T)\| - \|\xi_k(t)\| < 0$$

then the system is globally uniformly asymptotically swarm stable, and it achieves consensus as $t \to \infty$.

Proof The condition (1) implies that $B_{\|\xi_k\|}$ is an invariant set for the image trajectories of agents with

$$\xi_1, \xi_2, \ldots, \xi_N \in B_{\|\xi_k\|}$$

One knows that for any $i, j \in \{1, 2, \ldots, N\}$

$$\begin{aligned} T(x_i - x_j) &= T((x_i - x_0) - (x_j - x_0)) \\ &= T(x_i - x_0) - T(x_j - x_0) = \xi_i - \xi_j \end{aligned}$$

Because T is injective, it is invertible and

$$T^{-1}(\xi_i - \xi_j) = x_i - x_j$$

Therefore,

$$\begin{aligned} \left\| T^{-1}(\xi_i - \xi_j) \right\| &= \left\| x_i - x_j \right\| \le \left\| T^{-1} \right\| \left\| \xi_i - \xi_j \right\| \\ &\le 2 \left\| T^{-1} \right\| \|\xi_k(0)\| \end{aligned} \tag{2}$$

and

$$\left\| \xi_i - \xi_j \right\| = \left\| T(x_i - x_j) \right\| \le \|T\| \left\| x_i - x_j \right\|$$

Besides,

$$\begin{aligned} \|\xi_k(0)\| &\le \max_{i,j} \left\| \xi_i(0) - \xi_j(0) \right\| \\ &\le \|T\| \max_{i,j} \left\| x_i(0) - x_j(0) \right\| \end{aligned} \tag{3}$$

(2) and (3) naturally lead to

$$\left\| x_i - x_j \right\| \le 2 \left\| T^{-1} \right\| \|T\| \max_{i,j} \left\| x_i(0) - x_j(0) \right\|$$

Evidently, $\left\| x_i - x_j \right\| < \varepsilon$ so long as

$$\max_{i,j} \left\| x_i(0) - x_j(0) \right\| \le \frac{\varepsilon}{2 \left\| T^{-1} \right\| \|T\|}$$

and the multi-agent system is globally uniformly swarm stable.

On the other hand, if the condition (2) holds, because the value of $\|\xi_k\|$ is bounded from below,

$$\lim_{t \to \infty} \|\xi_k(t)\| = 0$$

Note that (2) can also yield

$$\left\| x_i(t) - x_j(t) \right\| \le 2 \left\| T^{-1} \right\| \|\xi_k(t)\|$$

Consequently,

$$\lim_{t \to \infty} \left\| x_i(t) - x_j(t) \right\| = 0 \quad (\forall i, j \in \{1, 2, \ldots, N\})$$

and the multi-agent system is globally uniformly asymptotically swarm stable and achieves consensus. □

Proposition 2 *Suppose that the image space in Proposition 1 is a Hilbert space,*

(1) if for $\forall t > 0$

$$\langle d\xi_k/dt, \xi_k \rangle \le 0$$

then the system is globally uniformly swarm stable;

(2) if for $\forall t > 0$

$$\langle d\xi_k/dt, \xi_k \rangle < 0$$

then the system is globally uniformly asymptotically swarm stable, and it achieves consensus as $t \to \infty$.

Proof Because T is linear, it is bounded. It is easy to know that the image system $\xi_1, \xi_2, \ldots, \xi_N$ is also continuously differentiable in H.

Consider the specific image agent ξ_k. Since the dynamics of ξ_k in H are continuous in time t,

$$\lim_{\Delta t \to 0} \langle \xi_k(t), \xi_k(t + \Delta t) \rangle = \|\xi_k(t)\| \|\xi_k(t + \Delta t)\| \tag{4}$$

Note that

$$\dot{\xi}_k = \lim_{\Delta t \to 0} (\xi_k(t + \Delta t) - \xi_k(t))/\Delta t \tag{5}$$

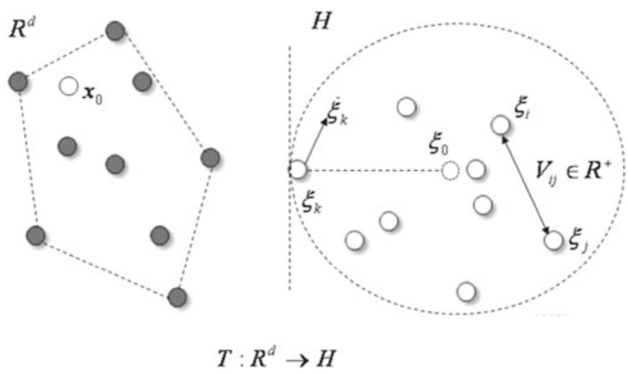

Fig. 1 Relative Lyapunov function

Thus, if it is true that

$$\langle \dot{\xi}_k, \xi_k \rangle = \lim_{\Delta t \to 0} \frac{(\langle \xi_k(t+\Delta t), \xi_k(t) \rangle - \langle \xi_k(t), \xi_k(t) \rangle)}{\Delta t} \le 0$$

with (2) and (3) taken into consideration, then it follows that

$$\lim_{\Delta t \to 0} (\|\xi_k(t+\Delta t)\| \, \|\xi_k(t)\| - \|\xi_k(t)\| \, \|\xi_k(t)\|)/\Delta t \le 0$$

and this implies that if $\Delta t > 0$

$$\lim_{\Delta t \to 0^+} \|\xi_k(t+\Delta t)\| \le \|\xi_k(t)\|$$

This means that $B_{\|\xi_k\|}$ is an invariant set for the image trajectories of agents with

$$\xi_1, \xi_2, \ldots, \xi_N \in B_{\|\xi_k\|}$$

According to the proof of Proposition 1, the mutli-agent system is globally uniformly swarm stable.

On the other hand, if at any time $\langle d\xi_k/dt, \xi_k \rangle < 0$, by the similar analysis above, we can conclude that

$$\lim_{\Delta t \to 0^+} \|\xi_k(t+\Delta t)\| < \|\xi_k(t)\|$$

According to the proof of Proposition 1, the multi-agent system is globally uniformly asymptotically swarm stable and achieves consensus. □

Definition 4 *(Relative Lyapunov function)* For any two agents i and j, a relative Lyapunov function represents the distance between the images in certain image space. Particularly, if the image space is Hilbert, a relative Lyapunov function can be formulated as follows:

$$V_{ij} = \sqrt{< \xi_i - \xi_j, \xi_i - \xi_j >} \in R^+$$

The idea is illustrated in Fig. 1.

Corollary 1 *Suppose there is a continuously differentiable multi-agent system of dth-order, including N agents. The dynamics of the agents are formulated by:*

$$\dot{x}_i = \sum_{j \in N_i} g_{ij}(x_i, x_j) \quad (i, j \in \{1, 2, \ldots, N\})$$

If there exists an injective linear operator $T : R^d \to H$, with the image space H being Hilbert, such that $Tx_i(t) = \xi_i(t) \in H$, and if

(1) At any time and for any two neighboring agents i and j,

$$\langle T g_{ij}(x_i, x_j), \xi_j - \xi_i \rangle < 0 \quad (i, j \in \{1, 2, \ldots, N\})$$

(2) The graph of the multi-agent system is jointly connected;

Then the multi-agent system is globally uniformly asymptotically swarm stable, and it achieves consensus as $t \to \infty$.

This corollary is a direct derivative of the ideas behind Propositions 1 and 2.

Example 1 [21] Suppose the dynamic equation of agents in a nonlinear multi-agent system is:

$$\dot{x}_i = \sum_{j \in N_i} \phi_{ij}(x_j - x_i) \quad (i \in \{1, 2, \ldots, N\})$$

where $x_i \in R$. The interaction function $\phi(x)$ satisfies the following properties:

i) $\phi(x)$ is continuous and locally Lipschitz;
ii) $\phi(x) = 0 \Leftrightarrow x = 0$;
iii) $\phi(-x) = -\phi(x)$;
iv) $(x - y)(\phi(x) - \phi(y)) > 0, \forall x \ne y$.

In this example, the state space is one-dimensional Euclidean. Let the image space be the same space. The condition for Corollary 1 is satisfied and the system achieves consensus as $t \to \infty$.

Corollary 2 *Suppose there is a continuously differentiable multi-agent system in R^d composed of N agents. The dynamics of agents are formulated by:*

$$\dot{x}_i = \sum_{j \in N_i} g_{ij}(x_i, x_j) \quad i, j \in \{1, 2, \ldots, N\}$$

If there exists relative Lyapunov function $\phi : R^d \times R^d \to R^+$ between any two agents: $V_{ij} = \phi(x_i, x_j) = \phi(x_j, x_i) \in R^+$, with $\phi(x, y) = 0$ as $x = y$, such that the equation below always holds:

$$\langle g_{ij}(x_i, x_j), \nabla_{x_i} V_{ij} \rangle < 0 \quad (i, j \in \{1, 2, \ldots, N\})$$

and the graph topology is jointly connected, then the system is globally uniformly asymptotically swarm stable, and it achieves consensus as $t \to \infty$.

If there exists such a kind of relative Lyapunov function, then we can take $\phi(x_i, x_j)$ as the metric of image space H, and the inner product in H can be induced by this metric. In such an induced space H, the direction of image for vector field $g_{ij}(x_i, x_j)$ is always towards the direction of the shortest distance between the images of agents i and j, therefore, consensus must be achieved according to Corollary 1.

Remark 2 For Corollary 2, it is not necessary that $g_{ij}(x_i, x_j)$ be a conservative field. It only needs to have the identical direction to certain gradient field. Such a dynamical system may be called quasi-gradient system. If two scalar fields in a space share the same quasi-gradient field, that is, the directions of gradient vectors for both fields are identical everywhere, then the two scalar fields are co-quasi-gradient. For instance, all differentiable K^∞ functions are co-quasi-gradient fields in R^1, since the gradient of any K^∞ function always points rightward of the R^1 axis.

Remark 3 Both the image space and the original space should have the same dimension since the mapping is injective. Actually, from another viewpoint, the image space is just a space endued with a new metric, which is redefined by relative Lyapunov function. As a simple and common instance, sometimes the inner product between two different vectors x_1 and x_2 could be redefined by $x_1^T P x_2$ instead of $x_1^T x_2$, with P a positive definite matrix. Naturally, the distance between two different points x_1 and x_2 could be redefined by $\sqrt{(x_1 - x_2)^T P (x_1 - x_2)}$, which shares the same quasi-gradient field with $(x_1 - x_2)^T P (x_1 - x_2)$.

Example 2 Consider the problem about synchronization of the Kuramoto model of coupled nonlinear oscillators, which is well-known in nonlinear theory [22]. The dynamic equation of any agent is

$$\dot{\theta}_i = \omega_i + K_i \sum_{j \in N_i} \sin(\theta_j - \theta_i) \qquad (K_i > 0)$$

When all ω_i are identical: $\omega_i = \omega \in R\ (\forall i)$, let the state of agent $x_i \in R$ be $x_i = \theta_i - \omega t$. As a result

$$\dot{x}_i = K_i \sum_{j \in N_i} \sin(x_j - x_i)$$

In the domain

$$\{x_1, x_2, \ldots, x_N \in R \,\big|\, |x_i - x_j| < \pi\}(i, j \in \{1, 2, \ldots, N\}) \tag{6}$$

let a relative Lyapunov function candidate be

$$\phi(x_i, x_j) = 1 - \cos(x_j - x_i) \tag{7}$$

Then the gradient of the relative Lyapunov function is

$$\nabla_{x_i} \phi(x_i, x_j) = -\sin(x_j - x_i)$$

The negative interactive force $-K_i \sin(x_j - x_i)$ between agents i and j is a quasi-gradient field of the relative Lyapunov function. According to the ideas behind Corollary 2, one can conclude that the system achieves consensus.

Note that there is no global asymptotic swarm stability for this system, and the swarm stability here is only regional. If consensus is regarded as some equilibrium, then the Kuramoto system has multiple equilibria. The initial states must locate inside the domain (6) to ensure consensus, as shown in Fig. 2 by a simple example with three agents. Otherwise, the relative Lyapunov function candidate (7) will be inappropriate, with the static state being non-consensus, as shown in Fig. 3. This instance illustrates the complexity of swarm stability analysis for nonlinear systems.

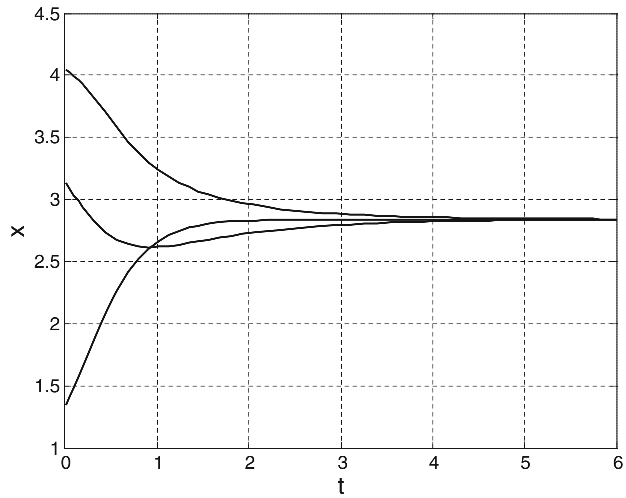

Fig. 2 Consensus trajectories of Kuramoto model with initial states $[3.13, 1.34, 4.04]^T$

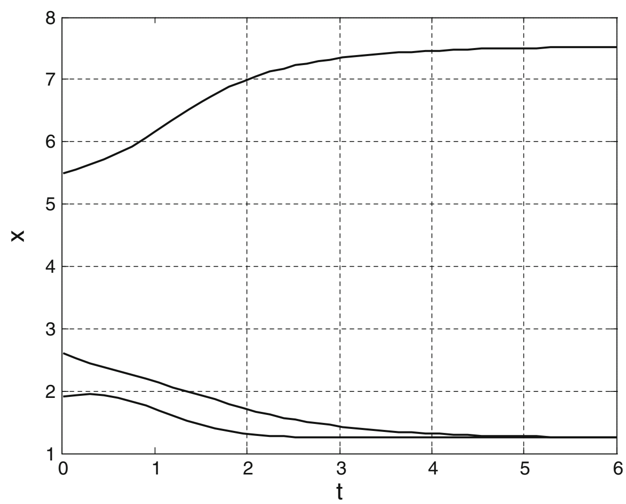

Fig. 3 Non-consensus trajectories of same Kuramoto model with initial states $[2.61, 1.92, 5.49]^T$

4 Swarm Stability of Nonlinear Compartmental Systems

In this section, we shall concern the swarm stability of a specific type of multi-agent system called compartmental system. The dynamics of each agent of dth order is:

$$\dot{x}_i = \sum_{j=1}^{N} w_{ij}(t) f_{ij}(x_i, x_j, t) \qquad (i \in \{1, 2, \ldots, N\}) \tag{8}$$

where $w_{ij}(t) = w_{ji}(t) \geq 0$, and $f_{ij}(x_i, x_j, t)$ represents the nonlinear interaction between agents i and j. Suppose the two rational assumptions below are satisfied.

Assumption 1 The interaction between any neighboring pair is skew-symmetric:

$$f_{ij}(x_i, x_j, t) = -f_{ji}(x_j, x_i, t) \qquad (i, j \in \{1, 2, \ldots, N\})$$

Assumption 2 The interactions are independent of translation:

$$f_{ij}(x_i, x_j, t) = f_{ij}(x_i - x_j, 0, t) \qquad (i, j \in \{1, 2, \ldots, N\})$$

Remark 4 Compartmental system [23] has been widely studied in biological, chemical and economic fields. Each compartment contains certain substance, with the substance flowing from neighbor to neighbor.

The attraction mechanism can be regulated by relative Lyapunov functions.

Definition 5 (*Attraction*) Suppose there is relative Lyapunov function $V(\alpha, \beta) \geq 0$ with $\alpha, \beta \in R^d$ s.t. $V(\alpha, \beta) = V(\beta, \alpha)$, $V(\alpha, \beta) = 0 \Leftrightarrow \alpha = \beta$, and $\frac{\partial V(\alpha,\beta)}{\partial \alpha}$ is a linear function as to α. If for certain $i, j \in \{1, 2, \ldots, N\}$,

$$\frac{\partial V(\alpha, \beta)}{\partial \alpha}\bigg|_{(\xi_1,\xi_2)} f_{ij}(\xi_1, \xi_2, t) < 0 \qquad (\xi_1 \neq \xi_2)$$

then the neighboring pair (i, j) are attractive. If

$$\frac{\partial V(\alpha, \beta)}{\partial \alpha}\bigg|_{(\xi_1,\xi_2)} f_{ij}(\xi_1, \xi_2, t) \leq 0 \qquad (\xi_1 \neq \xi_2)$$

then the neighboring pair (i, j) are weakly attractive.

Remark 5 Relative Lyapunov function is more general than the ordinary concept of distance. Even if two neighboring agents are attractive, their Euclidean distance in the state space could still increase. In this sense, the approach proposed in the current paper is advantageous over certain other works such as Moreau's [3] because his technical assumptions implicate that the distance between any two neighbors must always decrease.

The asymptotical swarm stability of system (8) has been studied in [7]. Some of its main results are cited here.

Lemma 1 [7] *For dynamical multi-agent system (8), with Assumption 1, the sum of the agent states is constant.*

Proposition 3 [7] *With Assumptions 1 and 2, a multi-agent system (8) with all neighboring pairs attractive is globally uniformly asymptotically swarm stable iff G is jointly connected.*

The condition for swarm stability is much looser than that for asymptotic swarm stability.

Proposition 4 *With Assumptions 1 and 2, a multi-agent system (8) is globally uniformly swarm stable if all neighboring pairs are weakly attractive.*

Proof According to Lemma 1, the average of agent states $x_0 = (\sum_{i=1}^{N} x_i)/N$ is constant. Let $e_i = x_i - x_0$ and $\Gamma = \sum_{i=1}^{N} V(e_i, 0)$. Its derivative is

$$\dot{\Gamma} = \sum_{i=1}^{N} \dot{V}(e_i, 0) = \sum_{i=1}^{N} \left(\frac{\partial V(\alpha, \beta)}{\partial \alpha}\bigg|_{(e_i,0)} \dot{e}_i \right)$$
$$= \sum_{i=1}^{N} \left(\frac{\partial V(\alpha, \beta)}{\partial \alpha}\bigg|_{(e_i,0)} \sum_{j=1}^{N} f_{ij}(e_i, e_j, t) \right)$$

Rearranging the summing terms according to the edges in G with considering the assumptions yields

$$\dot{\Gamma} = \sum_{(i,j)\in G} w_{ij}(t) f_{ij}(e_i, e_j, t) \left(\frac{\partial V(\alpha, \beta)}{\partial \alpha}\bigg|_{(e_i,0)} - \frac{\partial V(\alpha, \beta)}{\partial \alpha}\bigg|_{(e_j,0)} \right)$$
$$= \sum_{(i,j)\in G} w_{ij}(t) f_{ij}(e_i - e_j, 0, t) \frac{\partial V(\alpha, \beta)}{\partial \alpha}\bigg|_{(e_i-e_j,0)}$$

Because all neighboring pairs are weakly attractive, $\dot{\Gamma} \leq 0$. Thus, Γ never increases. For $\forall i \in \{1, 2, \ldots, N\}$ and $\forall t_0$, $V(e_i(t), 0) \leq \Gamma(t_0)$ $(t > t_0)$. This implicates that the system is globally uniformly swarm stable.

Comparing Proposition 4 with Proposition 3, one can see a notable difference lies in that swarm stability requires no connectivity for the graph topology.

Example 3 Consider a nonlinear time-varying compartmental system of second order with four agents:

$$\begin{cases} \dot{x}_{i1} = \sum_{j=1}^{4} w_{ij}(t)(x_{i1} - x_{j1} \\ \qquad + 3(x_{i2} - x_{j2})(x_{i1} - x_{j1})^2) \\ \dot{x}_{i2} = \sum_{j=1}^{4} w_{ij}(t)(x_{j1} - x_{i1})^3 \end{cases} \qquad (i = 1, \ldots, 4)$$

with $x_i = \left[x_{i1} \ x_{i2} \right]^T \in R^2$ the state of agent i. This system satisfies (A1)~(A2). If a relative Lyapunov function candidate is selected as

$$V(x_i, x_j) = x_i^T P x_i + x_j^T P x_j - x_i^T P x_j - x_j^T P x_i$$

with $P = diag([\,1\ 3\,])$, then evidently, any neighboring agents are weakly attractive. According to Proposition 4, it is swarm stable if with an undirected jointly connected graph.

As an instance with randomly switching edge weights of the graph, the swarm stable trajectories of four agents in the

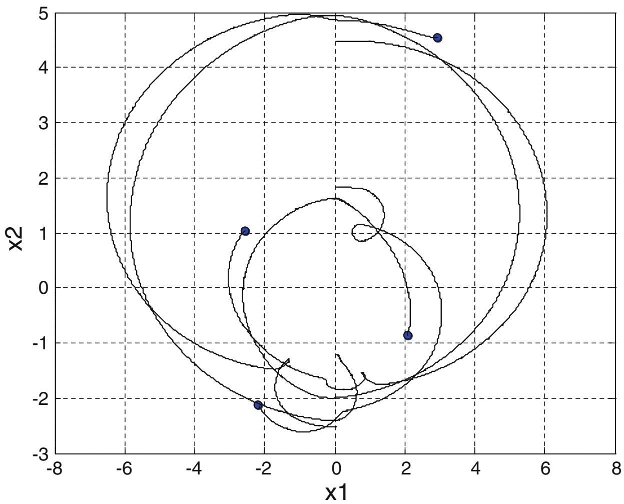

Fig. 4 Swarm stable trajectories of four agents in a compartmental system ($t \in [0, 20]$)

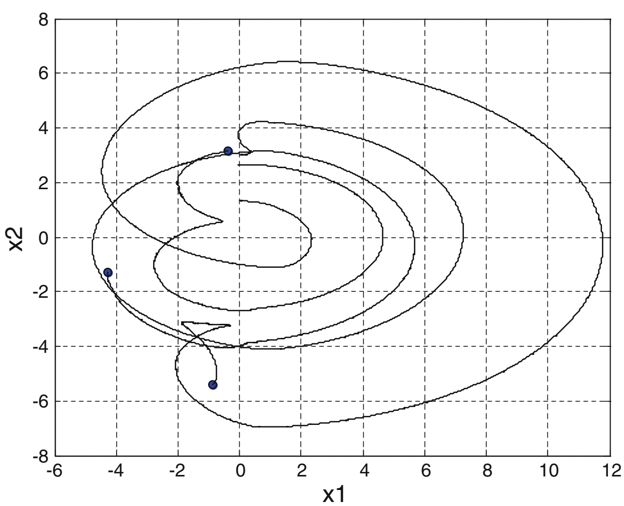

Fig. 5 Three independent relative trajectories ($t \in [0, 20]$)

phase plane are shown in Fig. 4. To clarify the variations of relative distances between agents, the trajectories of three independent relative states are shown in Fig. 5, which are $x_1 - x_2$, $x_2 - x_3$, and $x_3 - x_4$. It is clear that they are bounded. However, there is no consensus. Note that the thick dots in the following figures represent the starting positions of trajectories. One can sense that the motion of such a nonlinear time-varying system is rather sophisticated. □

5 Conclusions

Criteria for swarm stability of nonlinear dynamical multi-agent systems of high order are studied. The idea in this paper is based on the notion that the convergent property of a multi-agent system can be guaranteed by some dissipative property of an image system in certain abstract space, which is unique and different from those in the literature concerning consensus problems of linear systems. New concepts such as relative Lyapunov function and quasi-gradient field are concomitant with the discussions. The original motivation is from attempting to apply Lyapunov's second method to dynamical multi-agent systems. The approach proposed can potentially deal with the swarm stability of nonlinear systems that may be time-varying or heterogeneous. However, the results derived so far are still theoretic and restricted, probably because of the internal limitation of Lyapunov's second method and the complexity of compound nonlinear systems. It is usually difficult to determine an appropriate relative Lyapunov function for any particular scenario. This paper is just a preliminary endeavor along this direction. There are still many problems awaiting us to offer the solutions. In the future, we shall further attempt to seek some more practical techniques, e.g. considering systems with specific configurations and transforming the nonlinear models into canonical forms.

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