

# Learning to Disagree in a Game of Experimentation

## Supplementary Material

### Proof of Theorem 2

As mentioned in the paper, the proof of this theorem is rather tedious, and the interested reader might want to consult a mathematica file with some of the omitted algebraic operations, available on the authors' websites (`theorem2proof.nb`).

The logic of the argument is as follows. Suppose another equilibrium exists. Because on any interval over which a player's opponent does not switch with positive probability, a player's cost is convex, there is at most one time during such an interval at which he is willing to switch. Because of Lemma 6, we know that each player's equilibrium strategy must include in its support at least two switching times. If the support of a player's strategy is a dense subset of some interval, then so must be his opponent's (because of convexity, as explained), and continuity of the cost function then implies that this support is precisely  $[0, \bar{\tau}]$ , as defined in Theorem 1, and the equilibrium is the one described there. Hence, we might assume that there exists at least two times  $t_1, t_3$ , with  $0 < t_1 < t_3$ , such that, say, player 1's strategy assigns positive probability of switching at times  $t_1$  and  $t_3$ , and at no time in between. This however implies (convexity again) that there is some time  $t_2 \in (t_1, t_3)$  and some time  $t_0 < t_1$  such that player 2 is willing to switch at time  $t_0$  and  $t_2$ , but no time in between (and 1 does not switch at any time in  $(t_0, t_1)$  either).<sup>1</sup> We then derive a contradiction, showing that independently of how players behave at times not in  $[t_0, t_4]$ , the necessary (first- and second-order) conditions cannot hold simultaneously at those four dates.

As before, we work exclusively with log-likelihood ratios  $\ell$  rather than belief  $p$ . As mentioned (see (5) in section 3) on any interval  $[t_1, t_2]$  over which  $\bar{F}^j$  is a nonzero constant, we may write

$$\begin{aligned} \nu_t^j &= \frac{e^t \bar{F}_t^j}{1 + \int_0^t e^s \bar{F}_s^j ds} \\ &= \frac{\bar{F}_t^j}{\bar{F}_t^j + \left( F_0^j + \int_0^t e^s dF_s^j \right) e^{-t}} = (1 + C^j e^{-t})^{-1}, \end{aligned}$$

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<sup>1</sup>More precisely, either there is such a  $t_0 < t_1$ , or a  $t_4 > t_3$  in the support of 2's strategy, but relabelling the players if necessary, we may as well assume it is  $t_0 < t_1$ .

for the constant  $C^j := \left( F_0^j + \int_0^t e^s dF_s^j \right) / \bar{F}_t^j$ , which implies that, over such an interval,

$$\int_t^{\bar{t}} \nu_t^j dt = \ln(C^j + e^t).$$

Note that, by definition of the constant  $C^j$ , if  $\bar{F}^j$  is constant (and nonzero) over  $[t_1, t_2]$  and  $[t_3, t_4]$ , with  $t_2 < t_3$ , then the constant associated with the interval  $[t_3, t_4]$  is higher than the constant associated with the interval  $[t_1, t_2]$ .

This gives

$$\int_t^{\bar{t}} e^{\int_t^s (\nu_\tau^{-i} - \mu - I) d\tau} ds = -\frac{e^{-\phi(s-t)}}{C^j + e^t} \left( \frac{e^s}{\phi - 1} + \frac{C^j}{\phi} \right) \Big|_{s=t}^{\bar{t}}.$$

If  $\bar{F}^j = 0$  over this interval, we have

$$\int_t^{\bar{t}} e^{\int_t^s (\nu_\tau^{-i} - \mu - I) d\tau} ds = -\int_t^{\bar{t}} e^{-\phi(s-t)} ds = -\frac{e^{-\phi(s-t)}}{\phi} \Big|_{s=t}^{\bar{t}},$$

which we can view with some abuse as a some special case of the previous formula with  $C^j = +\infty$ . Note that this reduces to  $\phi^{-1}$  for  $\bar{t} = +\infty$ .

We may ignore payoff-irrelevant  $\nu$  constants and rewrite the cost of stopping at time  $t$  as

$$\mathcal{C}_t^i := \frac{e^{-\mu t}}{\mu} \left( \mu \gamma e^{\ell t} \int_t^\infty e^{\int_t^s (\nu_\tau^{-i} - \phi) d\tau} ds - 1 \right).$$

We need to show that, given any value of  $\ell = \ell_{t_1}$ , there exists no positive real numbers  $C^1$ ,  $C^2$  and  $D^2 \geq C^2$  (corresponding to the constant for the formula for  $\nu^1$  on  $(t_1, t_3)$ , and the two constants for the formulas for  $\nu^2$  on  $(t_1, t_2)$  and  $(t_2, t_3)$  respectively) such that

$$\mathcal{C}_{t_1}^1 = \mathcal{C}_{t_3}^1, \quad \frac{d\mathcal{C}_{t_1}^1}{dt} \Big|_{t=t_1} = \frac{d\mathcal{C}_{t_3}^1}{dt} \Big|_{t=t_3} = \frac{d\mathcal{C}_{t_2}^2}{dt} \Big|_{t=t_2} = 0,$$

yet

$$\frac{d^2\mathcal{C}_{t_1}^1}{dt^2} \Big|_{t=t_1} \geq 0, \quad \frac{d^2\mathcal{C}_{t_3}^1}{dt^2} \Big|_{t=t_3} \geq 0, \quad \frac{d^2\mathcal{C}_{t_2}^2}{dt^2} \Big|_{t=t_2} \geq 0,$$

where derivatives at  $t_1$  and  $t_3$  are right- and left-hand derivatives, respectively. We first develop explicit formulas for these quantities. Note that, for  $t \in \{t_1, t_3\}$  and  $i = 1$ ,

$$\begin{aligned} \mathcal{C}_t^{i'} &= \gamma e^{\ell t - \mu t} \left( (\mu + 1) \int_t^\infty e^{\int_t^s (\nu_\tau^{-i} - \phi) d\tau} ds - 1 \right) - \mu \mathcal{C}_t^i = \mathcal{C}_t^i + e^{-\mu t} \left( \frac{\mu + 1}{\mu} - \gamma e^{\ell t} \right) \\ &= \left( 1 - \gamma e^{\ell t} + \gamma e^{\ell t} \int_t^\infty e^{\int_t^s (\nu_\tau^{-i} - \phi) d\tau} ds \right) e^{-\mu t}, \end{aligned}$$

and

$$\mathcal{C}_t^{i''} = \mathcal{C}_t^{i'} - (\mu + 1)e^{-\mu t} - \gamma(\nu_t^{-i} + 1 - I - \mu)e^{\ell t - \mu t}.$$

Hence, because  $\mathcal{C}_t^{i'}|_{t=t_1} = \mathcal{C}_t^{i'}|_{t=t_3} = 0$ , yet  $\mathcal{C}_t^i|_{t=t_1} = \mathcal{C}_t^i|_{t=t_3}$ , we must have

$$e^{-\mu t_1} \left( \frac{\mu + 1}{\mu} - \gamma e^{\ell t_1} \right) = e^{-\mu t_3} \left( \frac{\mu + 1}{\mu} - \gamma e^{\ell t_3} \right),$$

or, writing  $\ell_k$  for  $\ell_{t_k}$ ,

$$e^{-\mu(t_3-t_1)} = \frac{\mu + 1 - \gamma\mu e^{\ell_1}}{\mu + 1 - \gamma\mu e^{\ell_3}}.$$

Let

$$K_3 := \int_{t_3}^{\infty} e^{\int_{t_3}^s (\nu_{\tau}^{-i} - \phi) d\tau} ds.$$

Note that  $K_3 \geq \int_{t_3}^{\infty} e^{-\int_{t_3}^s \phi d\tau} ds = \phi^{-1}$ .

The conditions  $\mathcal{C}_t^{i'}|_{t=t_1} = \mathcal{C}_t^{i'}|_{t=t_3} = 0$  are equivalent to

$$1 - \frac{1}{\gamma}e^{-\ell_3} = K_3 \geq \frac{1}{\phi},$$

and

$$1 - \frac{1}{\gamma}e^{-\ell_1} = \int_{t_1}^{t_2} e^{\int_{t_1}^s (\nu_{\tau}^{-i} - \phi) d\tau} ds + e^{\int_{t_1}^{t_2} (\nu_{\tau}^{-i} - \phi) d\tau} \int_{t_2}^{t_3} e^{\int_{t_2}^s (\nu_{\tau}^{-i} - \phi) d\tau} ds + e^{\int_{t_1}^{t_3} (\nu_{\tau}^{-i} - \phi) d\tau} K_3,$$

so that

$$1 - \frac{1}{\gamma}e^{-\ell_1} = \int_{t_1}^{t_2} e^{\int_{t_1}^s (\nu_{\tau}^{-i} - \phi) d\tau} ds + e^{\int_{t_1}^{t_2} (\nu_{\tau}^{-i} - \phi) d\tau} \int_{t_2}^{t_3} e^{\int_{t_2}^s (\nu_{\tau}^{-i} - \phi) d\tau} ds + e^{\int_{t_1}^{t_3} (\nu_{\tau}^{-i} - \phi) d\tau} \left( 1 - \frac{1}{\gamma}e^{-\ell_3} \right).$$

Equivalently, we must have

$$1 \geq \frac{1}{\gamma}e^{-\ell_3} + \frac{1}{\phi},$$

and

$$\begin{aligned} 1 - \frac{1}{\gamma}e^{-\ell_1} &= \int_{t_1}^{t_2} e^{\int_{t_1}^s (\nu_{\tau}^{-i} - \phi) d\tau} ds + e^{\int_{t_1}^{t_2} (\nu_{\tau}^{-i} - \phi) d\tau} \int_{t_2}^{t_3} e^{\int_{t_2}^s (\nu_{\tau}^{-i} - \phi) d\tau} ds + e^{\int_{t_1}^{t_3} (\nu_{\tau}^{-i} - \phi) d\tau} \left( 1 - \frac{1}{\gamma}e^{-\ell_3} \right) \\ &= \frac{1}{C^2 + e^{t_1}} \left( \left( \frac{C^2}{\phi} + \frac{e^{t_1}}{\phi - 1} \right) - e^{-\phi(t_2-t_1)} \left( \frac{C^2}{\phi} + \frac{e^{t_2}}{\phi - 1} \right) \right) \\ &\quad + e^{-\phi(t_2-t_1)} \frac{C^2 + e^{t_2}}{D^2 + e^{t_2}} \left( \left( \frac{D^2}{\phi} + \frac{e^{t_2}}{\phi - 1} \right) - e^{-\phi(t_3-t_2)} \left( \frac{D^2}{\phi} + \frac{e^{t_3}}{\phi - 1} \right) \right) \\ &\quad + e^{-\phi(t_3-t_1)} \frac{C^2 + e^{t_2}}{D^2 + e^{t_2}} (D^2 + e^{t_3}) \left( 1 - \frac{1}{\gamma}e^{-\ell_3} \right). \end{aligned}$$

Rearranging,

$$\begin{aligned} \left(1 - \frac{1}{\gamma}e^{-\ell_1} - \frac{1}{\phi}\right) (1 + C^2e^{-t_1}) &= \frac{1}{\phi(\phi - 1)} - \frac{(D^2 - C^2)e^{(1-\phi)(t_2-t_1)}}{(D^2 + e^{t_2})\phi(\phi - 1)} - \frac{C^2 + e^{t_2}}{D^2 + e^{t_2}} \frac{e^{(1-\phi)(t_3-t_1)}}{\phi(\phi - 1)} \\ &+ e^{(1-\phi)(t_3-t_1)} \frac{C^2 + e^{t_2}}{D^2 + e^{t_2}} (1 + D^2e^{-t_3}) \left(1 - \frac{1}{\gamma}e^{-\ell_3} - \frac{1}{\phi}\right). \end{aligned}$$

Finally, the second-order conditions read (from above, given the foc's), for  $t = t_1, t_3$ :

$$\phi - 1 - \nu_t^{-i} \geq \frac{\mu + 1}{\gamma} e^{-\ell_t},$$

or

$$\phi - 1 \geq \frac{\mu + 1}{\gamma} e^{-\ell_1} + \frac{1}{1 + C^2e^{-t_1}},$$

and

$$\phi - 1 \geq \frac{\mu + 1}{\gamma} e^{-\ell_3} + \frac{1}{1 + D^2e^{-t_3}}.$$

Finally, we might want to use that

$$e^{\ell_3 - \ell_1} = e^{(1-I)(t_3-t_1)} \frac{C^2 + e^{t_2}}{C^2 + e^{t_1}} \frac{D^2 + e^{t_3}}{D^2 + e^{t_2}}.$$

Note that  $t_2$  enters via the ratio  $\rho_2 := (C^2 + e^{t_2})/(D^2 + e^{t_2})$  (up to one instance). Hence we may rewrite the three equations as

$$\begin{aligned} \left(1 - \frac{1}{\gamma}e^{-\ell_1} - \frac{1}{\phi}\right) (1 + C^2e^{-t_1}) &= \\ \frac{1}{\phi(\phi - 1)} - (1 - \rho_2) \frac{e^{(1-\phi)(t_2-t_1)}}{\phi(\phi - 1)} - \rho_2 \frac{e^{(1-\phi)(t_3-t_1)}}{\phi(\phi - 1)} &+ e^{(1-\phi)(t_3-t_1)} \rho_2 (1 + D^2e^{-t_3}) \left(1 - \frac{1}{\gamma}e^{-\ell_3} - \frac{1}{\phi}\right), \\ e^{\ell_3 - \ell_1} (1 + C^2e^{-t_1}) &= e^{(2-I)(t_3-t_1)} \rho_2 (1 + D^2e^{-t_3}). \end{aligned}$$

and

$$e^{-\mu(t_3-t_1)} = \frac{\mu + 1 - \gamma\mu e^{\ell_1}}{\mu + 1 - \gamma\mu e^{\ell_3}}.$$

Combining the first and second,

$$\begin{aligned} \phi(\phi - 1) \left(1 - \frac{1}{\gamma}e^{-\ell_1} - \frac{1}{\phi} - e^{-(1+\mu)(t_3-t_1)} e^{\ell_3 - \ell_1} \left(1 - \frac{1}{\gamma}e^{-\ell_3} - \frac{1}{\phi}\right)\right) (1 + C^2e^{-t_1}) &= \\ 1 - (1 - \rho_2) e^{(1-\phi)(t_2-t_1)} - \rho_2 e^{(1-\phi)(t_3-t_1)}, \end{aligned}$$

Finally, from the other player's point of view, we have as a first-order condition

$$1 - \frac{e^{-\ell_2}}{\gamma} =: K_2 \geq \phi^{-1},$$

where

$$K_2 := \int_{t_2}^{\infty} e^{\int_{t_2}^s (\nu_{\tau}^{-i} - \phi) d\tau} ds.$$

We may improve this bound as follows. Set  $t_1 = 0$  wlog henceforth. Consider the definition of  $K_2$ ; break the corresponding integral into the intervals  $[t_2, t_3]$  and  $[t_3, \infty)$ ; and use the fact that  $\nu_t^1 \geq 0$  for all  $t \geq t_3$ . We then get:

$$(1 + C^1 e^{-t_2}) \left( 1 - \frac{1}{\phi} - \frac{e^{-\ell_2}}{\gamma} \right) \geq \frac{1 - e^{-(\phi-1)(t_3-t_2)}}{\phi(\phi-1)}.$$

Note also that  $1 - 1/\phi = e^{-\ell^*}/\gamma$ . The second-order condition at  $t_2$  gives

$$\phi - 1 \geq \frac{\mu + 1}{\gamma} e^{-\ell_2} + \frac{1}{1 + C^1 e^{-t_2}}.$$

Combining, we have

$$1 - \frac{e^{-\ell_2}}{\gamma} \geq \phi^{-1}, \text{ and } 1 \geq \phi - 1 - \frac{\mu + 1}{\gamma} e^{-\ell_2} \geq 0.$$

Note that the first inequality gives

$$\frac{e^{-\ell_2}}{\gamma} \leq \frac{\phi - 1}{\phi},$$

which implies, given that  $\phi \geq \mu + 1$ ,

$$\frac{e^{-\ell_2}}{\gamma} \leq \frac{\phi - 1}{\mu + 1},$$

which is simply the inequality  $\phi - 1 - \frac{\mu+1}{\gamma} e^{-\ell_2} \geq 0$ . So we only have

$$\frac{\phi - 2}{\mu + 1} \leq \frac{e^{-\ell_2}}{\gamma} \leq \frac{\phi - 1}{\phi}.$$

In addition, we have that

$$e^{\ell_2 - \ell_1} = \frac{1 + C^1 e^{-t_2}}{1 + C^1 e^{-t_1}}.$$

All this can be summarized as follows. Can we have simultaneously:

$$(1 + C^1 e^{-t_2}) \frac{e^{-\ell^*} - e^{-\ell_1} e^{-\delta_2}}{\gamma} \geq \frac{1 - e^{-(\phi-1)(t_3-t_2)}}{\phi(\phi-1)},$$

and (“foc at  $t_1$  combined with foc at  $t_3$ ”)

$$(1 + C^2 e^{-t_1}) \frac{e^{-\ell^*} - e^{-\ell_1}}{\gamma} = \frac{1}{\phi(\phi-1)} \left( 1 - \frac{D^2 - C^2}{D^2 + e^{t_2}} e^{-(\phi-1)(t_2-t_1)} - \frac{C^2 + e^{t_2}}{D^2 + e^{t_2}} e^{-(\phi-1)(t_3-t_1)} \right) + e^{-(\phi-1)(t_3-t_1)} \frac{C^2 + e^{t_2}}{D^2 + e^{t_2}} (1 + D^2 e^{-t_3}) \frac{e^{-\ell^*} - e^{-\ell_1} e^{-\delta_3}}{\gamma}, \quad (1)$$

for some  $t_3 \geq t_2 \geq t_1 := 0$ ,  $D^2 \geq C^2 \geq 0$ ,  $C^1 \geq C^2$ , where  $\ell_1, \delta_2 := \ell_2 - \ell_1 \leq 0$ ,  $\delta_3 := \ell_3 - \ell_1 \leq 0$  solve and (“learning gives  $\ell_3$ ”)

$$\frac{C^2 + e^{t_2}}{D^2 + e^{t_2}} (1 + D^2 e^{-t_3}) e^{-\delta_3} = e^{(I-2)(t_3-t_1)} (1 + C^2 e^{-t_1}), \quad (2)$$

(“learning gives  $\ell_2$ ”)

$$(1 + C^1 e^{-t_2}) e^{-\delta_2} = 1 + C^1 e^{-t_1}, \quad (3)$$

and finally (“payoff equality between  $t_1$  and  $t_3$ ” slightly rearranged)

$$\frac{e^{-\ell_1}}{\gamma} = \frac{\mu}{\mu + 1} \frac{1 - e^{\delta_3} e^{-\mu(t_3-t_1)}}{1 - e^{-\mu(t_3-t_1)}}. \quad (4)$$

Note that  $C''_{t_2} \geq 0$  is implied by  $C'_{t_2} = 0$  and  $C^2_{t_3} \geq C^2_{t_2}$ . This is because, given that player 1 does not quit on  $(t_1, t_3)$ , if  $C^2$  is convex at some  $t$  in the interval, it is convex at all lower  $t$  in this interval.

It follows that the relevant inequality is  $C^2_{t_3} \geq C^2_{t_2}$ . Rewriting it implies that we have, for some  $L_3$ ,

$$L_3 \geq \frac{1}{\phi}, L_3 \geq \frac{1 + C^1 e^{-t_2}}{1 + C^1 e^{-t_3}} e^{-(2-\phi)(t_3-t_2)} \left( 1 - \frac{e^{-\ell_2}}{\gamma} \right),$$

and

$$(1 + C^1 e^{-t_2}) \frac{e^{-\ell^*} - e^{-\ell_2}}{\gamma} = \frac{1 - e^{-(\phi-1)(t_3-t_2)}}{\phi(\phi-1)} + e^{-(\phi-1)(t_3-t_2)} (1 + C^1 e^{-t_3}) \left( L_3 - \frac{1}{\phi} \right).$$

(Here,  $L_3$  is the unknown  $\int_{t_3}^{\infty} e^{\int_{t_3}^s (\nu_{\tau}^1 - \phi) d\tau} ds$ .) To summarize, having set  $t_1 = 0$ ; we must show that there exists no  $T_2 = e^{-t_2}$ ,  $T_3 = e^{-t_3}$ ,  $C^1, C^2$  such that, given that  $\ell_1$  solves (4),  $\delta_2$  solves (3),  $\delta_3$  solves (2),  $D^2$  solves (1), and we have

1.  $1 > T_2 > T_3 > 0$ ;
2.  $\ell_3 \geq \ell^*$ ;
3.  $D^2 \geq C^2 \geq 0$ ,  $C^1 \geq C^2$  (these two inequalities follow from the definition of  $\nu^{-i}$ );

4. (soc at  $t_3$ ) it holds that

$$\phi - 1 \geq \frac{\mu + 1}{\gamma} e^{-\ell_3} + \frac{1}{1 + D^2 e^{-t_3}}; \quad (5)$$

5. (“soc” at  $t_3$  vs.  $t_2$ ) Both inequalities

$$(1 + C^1 e^{-t_2}) \frac{e^{-\ell^*} - e^{-\ell_2}}{\gamma} \geq \frac{1 - e^{-(\phi-1)(t_3-t_2)}}{\phi(\phi-1)}, \quad (6)$$

and

$$(1 + C^1 e^{-t_2}) \frac{e^{-\ell^*} - e^{-\ell_2}}{\gamma} \geq \frac{1 - e^{-(\phi-1)(t_3-t_2)}}{\phi(\phi-1)} + \quad (7)$$

$$e^{-(\phi-1)(t_3-t_2)} (1 + C^1 e^{-t_3}) \left( \frac{1 + C^1 e^{-t_2}}{1 + C^1 e^{-t_3}} e^{-(2-\phi)(t_3-t_2)} \left( 1 - \frac{e^{-\ell_2}}{\gamma} \right) - \frac{1}{\phi} \right).$$

As mentioned, the details showing that these inequalities cannot hold simultaneously are contained in the mathematica file `theorem2proof.nb`.

## Correlation through the FGM Copula

We introduce correlation in the normal form through recommendations over switching times. Specifically, for the case of two players, let  $F(t_1, t_2)$  denote the joint distribution of switching-time recommendations. More details and specific calculations are in the annotated Mathematica file `correlated.nb` available on the authors’ websites.

Throughout, we assume symmetry of this distribution; we introduce an (arbitrarily small amount of) background learning, i.e.,

$$\dot{\ell}_t = -\bar{u} + u_t^1 + u_t^2,$$

with  $\bar{u} > 2$ ; we also normalize

$$\ell^0 = \ell^{**} := \ln \left( \frac{\mu + 1}{\gamma(\phi - 2)} \right);$$

and we define  $\phi := 2 + \mu$ .

Using the expression derived in the proof of Lemma 2, the expected cost of switching to the risky arm at time  $t$ , when given recommendation  $t'$ , is given by

$$C(t, t') = \frac{e^{-\mu t}}{\mu} \left( \mu \gamma e^{\ell t} \int_t^\infty e^{-\phi(s-t)} e^{\int_t^s \nu_\tau d\tau} ds - 1 \right),$$

where

$$\begin{aligned}\ell_t &= \ell^0 + (1-u)t + \int_0^t \nu_\tau d\tau, \\ e^{\int_t^s \nu_\tau d\tau} &= \frac{1 + \int_0^s e^\tau \bar{F}(\tau | t') d\tau}{1 + \int_0^t e^\tau \bar{F}(\tau | t') d\tau},\end{aligned}$$

and  $\bar{F}(\tau | t')$  denotes the complementary distribution function conditional on receiving recommendation  $t'$ .

The cost can be written in terms of the distribution as follows:

$$\begin{aligned}C(t, t') &= \underbrace{\gamma e^{\ell^0} \frac{e^{(2-\phi)t}}{\phi-1} - \frac{e^{-\mu t}}{\mu}}_{K(t)} - \frac{\gamma e^{\ell^0}}{\phi} \left( \int_0^t e^{(1-\phi)t} e^s F(s | t) ds + \int_t^\infty e^t e^{(1-\phi)s} F(s | t) ds \right) \\ &= K(t) - \frac{\gamma e^{\ell^0}}{\phi} \left( \int_0^t e^{(1-\phi)t} e^s F(s | t) ds + \int_t^\infty e^t e^{(1-\phi)s} F(s | t) ds \right).\end{aligned}$$

Now we can write the IC constraint as

$$\left. \frac{\partial C(t, t')}{\partial t'} \right|_{t'=t} = 0.$$

Therefore, we obtain

$$\begin{aligned}0 &= K'(t) - \frac{\gamma e^{\ell^0}}{\phi} \left( (1-\phi) \int_0^t e^{(1-\phi)t} e^s F(s | t) ds + \int_t^\infty e^t e^{(1-\phi)s} F(s | t) ds \right) \\ &= K'(t) - K(t) + C(t) + \gamma e^{\ell^0} \int_0^t e^{(1-\phi)t} e^s F(s | t) ds,\end{aligned}$$

where  $C(t) := C(t, t)$ . We then express everything in terms of probability densities. The objective becomes

$$C(t, t) = K(t) - \gamma e^{\ell^0} \frac{e^{(2-\phi)t} F(t | t)}{\phi-1} + \frac{\gamma e^{\ell^0}}{\phi} \left( \int_0^t e^{(1-\phi)t} e^s f(s | t) ds - \frac{1}{\phi-1} \int_t^\infty e^t e^{(1-\phi)s} f(s | t) ds \right).$$

The IC constraint becomes

$$-K'(t) = \gamma e^{\ell^0} \frac{\phi-2}{\phi-1} e^{(2-\phi)t} F(t | t) + \frac{\gamma e^{\ell^0}}{\phi} \left( (1-\phi) \int_0^t e^{(1-\phi)t} e^s f(s | t) ds - \frac{1}{\phi-1} \int_t^\infty e^t e^{(1-\phi)s} f(s | t) ds \right)$$

Writing the conditional distribution in terms of the joint and marginal distributions yields the following expression for the IC constraint:

$$\begin{aligned}0 &= \frac{(\mu+1)e^{t(2-\phi)} \int_0^t f(t, s) ds}{\phi-1} + f(t) \left( \frac{(\mu+1)(2-\phi)e^{t(2-\phi)}}{(\phi-2)(\phi-1)} + e^{-\mu t} \right) \\ &+ \frac{(\mu+1) \left( (1-\phi) \int_0^t e^{s+t(1-\phi)} f(t, s) ds - \frac{\int_t^{\bar{\tau}} e^{s(1-\phi)+t} f(s, t) ds}{\phi-1} \right)}{(\phi-2)\phi},\end{aligned}$$



where  $\bar{\tau}$  is the upper bound on the support of the equilibrium strategy.

We now consider the marginal distribution over switching times in our symmetric equilibrium, which is given by

$$F(t) = 1 + \frac{e^{t(-\mu+\phi-2)} ((\phi-2)(\mu-\phi+1) + (\phi-1)(-\mu+\phi-2)e^{\mu t})}{\mu}.$$

We construct a new distribution by slightly perturbing the independent randomization according to a bivariate FGM copula. Thus, for a given marginal, the joint is given by

$$F(t_1, t_2) = F(t_1)F(t_2)(1 + \rho(1 - F(t_1))(1 - F(t_2))),$$

with parameter  $\rho \in [-1, 1]$ .

Under a FGM copula with parameter  $\rho$ , the incentive-compatibility constraint for obeying the recommendation to switch at time  $t$  is a functional equation that is linear in the marginal distribution  $F$ . In particular, the IC constraint can be written as the combination of two *linear* operators  $K_0$  and  $K_1$  as follows:

$$K_0(F) + \rho K_1(F) = 0,$$

where

$$\begin{aligned} K_0(F) &= \frac{(\mu+1) \left( \int_t^{\bar{\tau}} F(s) e^{s(-\phi)+s+t} ds + (\phi-1) \left( - \int_0^t F(s) e^{s-t\phi+t} ds \right) - \frac{1-(\phi-1)^2}{\phi-1} F(t) e^{t(2-\phi)} \right)}{(\phi-2)(\phi-1)\phi} \\ &+ e^{-\mu t} - \frac{(\mu+1)(1-F(t))e^{-t(\phi-2)}}{\phi-1}, \end{aligned}$$

and

$$K_1(F) = \frac{(\mu+1)(1-2F(t)) \left( (\phi-1) \int_0^t (F(s)-1)F(s) (-e^{s-t\phi+t}) ds + \int_t^{\bar{\tau}} (F(s)-1)F(s) e^{s(1-\phi)+t} ds \right)}{(\phi-2)\phi}.$$

Because our original distribution  $F_0(t)$  satisfies  $K_0(F_0) = 0$ , we look for a distribution  $F_1(\rho) = F_0 + \rho\tilde{F}$  that satisfies

$$K_0(F_0 + \rho\tilde{F}) + \rho K_1(F_0 + \rho\tilde{F}) = 0.$$

Simplifying using linearity, and letting  $\rho \rightarrow 0$  we obtain the following condition:

$$K_1(F_0) + K_0(\tilde{F}) = 0.$$

This condition captures the restriction that incentives (under a small amount of correlation) impose on the marginal. It identifies a distribution  $\tilde{F}$  we can use to (locally) modify our equilibrium distribution and preserve incentives. In the file `correlated.nb`, we differentiate total costs under the distribution  $F_1(\rho)$  around  $\rho = 0$ .

Clearly, no matter the degree of correlation  $\rho$ , no player can start experimenting before  $p^{**}$  or after  $p^*$ . The design variable is the degree of correlation, but requires adjusting the support of the marginal distribution to match  $p^*$  of the most pessimistic type. In particular, the mass point at time  $\bar{\tau}$  is now a function of  $\rho$ .

To then evaluate how the upper bound of the support and the mass point vary with  $\rho$ , we impose that the most pessimistic type (the one who receives recommendation  $t = \bar{\tau}$ ) must hold belief  $p_{\bar{\tau}} = p^*$ . The mass point is then given by the difference  $1 - F_1(\bar{\tau})$ . Adding up terms, the derivative of the total cost with respect to  $\rho$  is given by

$$\frac{dC}{d\rho} = \int_0^{\bar{\tau}} \left( C(t) f_1(t) + \left( \frac{dC(t)}{d\rho} + \frac{dC(t)}{d\bar{\tau}} \frac{d\bar{\tau}}{d\rho} \right) \right) dt + (1 - F_1(\bar{\tau})) \frac{dC(\bar{\tau})}{d\bar{\tau}} \frac{d\bar{\tau}}{d\rho}.$$

For any value of the remaining parameters  $(\mu, \phi)$ , the file `correlated.nb` shows that the derivative of the cost is negative, *i.e.*, positive correlation is beneficial.