Consumer Scores and Price Discrimination*

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Abstract

We study the implications of aggregating a consumer’s purchase history into a score that proxies for her unobserved willingness to pay. The consumer interacts with a sequence of firms in a stationary Gaussian setting. Each firm in turn relies on the consumer’s current score—a noisy, linear aggregate of past purchases quantity signals discounted exponentially—to learn about her preferences and to set prices. A strategic consumer reduces her demand to drive prices below the naive-consumer benchmark. For consumers with high average willingness to pay, the gains from low prices dominate the losses from better tailored prices, and hence tracking is beneficial. The equilibrium informativeness of a score is maximized by overweighing past signals relative to Bayes’ rule with disaggregated data. Hidden scores—those only observed by the firms—reduce demand sensitivity, increase expected prices, and reduce consumer surplus.

Keywords: price discrimination; purchase histories; consumer scores; persistence; transparency; ratchet effect.

JEL codes: C73, D82, D83.

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1 Introduction

Consumer scores are metrics that use all available data about individual customers—age, ethnicity, gender, household income, zip code, and purchase histories—to quantify and predict their profitability, health risk, job security, or credit worthiness. The best-known example is the FICO credit score used by every lender in the US. Another prominent, though less well-known, example is the Customer Lifetime Value (CLV) that many firms assign to their own customers in order to personalize prices, products, advertising messages, and various perks.\(^1\) In addition to being deployed internally, consumer scores are also traded. Several data brokers (e.g., Acxiom, Equifax, and Experian) develop scores that collect and process information from a variety of sources and sell these scores to companies, which, in turn, use them to refine their market-segmentation strategies. The transmission of the information contained in such scores thus creates a link between a consumer’s interaction with one firm and the terms of her future transactions with other firms and industries.\(^2\)

With the exception of credit markets, consumer scores are not regulated, nor are they available to consumers. As such, some consumers ignore these links across transactions, and even the most sophisticated consumers cannot perfectly forecast the impact of their actions. Awareness of the mechanisms at play is nonetheless increasing quickly over time, thanks to recent regulatory efforts aimed at improving the transparency of firms’ information (e.g., the European Union’s General Data Protection Regulation and, in a sense, the Chinese Social Credit System). With consumer awareness increasing at a fast pace, it is essential to understand how both technological and market forces affect incentives. In particular, if the final use of information impacts the distribution of surplus, the mechanisms by which consumer data are collected, aggregated and transmitted can affect the terms of the transactions in which the data are generated, and thus, the informational content of the data itself.\(^3\)

In this paper, we study the welfare consequences of aggregating purchase histories into scores that are used for third-degree price discrimination.\(^4\) We examine how these conse-

\(^1\)CLV scores’ composition is a rich area of marketing research (Dwyer, 1989; Berger and Nasr, 1998). The Wall Street Journal (2018) reports, “every consumer has at least one CLV score, more likely several.” NPR provides more information in its Planet Money podcast, https://www.npr.org/sections/money/2018/11/07/665392227/your-lifetime-value-score.

\(^2\)For example, the Equifax Discretionary Spending Limit Index is a number 1–1000 that “helps marketers differentiate between two households that look the same in terms of income and demographics but likely have considerably different spending power.” Similarly, information about a consumer’s sporting goods purchases or eating habits can become part of a predictive score for a health insurer (The Economist, 2012).

\(^3\)Many European consumers are already aware of market-segmentation practices. In a recent survey (European Commission, 2018), 62% of respondents knew of the personalized ranking of online offers (based on past behavior or contextual information), and 44% knew of instances of personalized pricing.

\(^4\)Price discrimination is implicitly used in a number of consumer markets in the form of personalized coupons, discounts, and fees (Dubé and Misra, 2017). Another related prominent practice is product steer-
quences depend on the consumers’ degree of sophistication (do they know they are being scored?) and on the availability of the sellers’ information (can they check their score?).

Our approach embeds a continuous-time model of the ratchet effect into an information design framework, which enables us to examine how data aggregation and transparency impact a strategic consumer’s incentives. In our model, a long-lived consumer faces a different monopolist at every instant of time. The consumer’s preferences are quadratic in the quantity demanded and linear in her privately observed willingness to pay, which is captured by a stationary Gaussian process. At any instant of time, an (unmodeled) data broker observes a signal of the consumer’s current purchase distorted by Brownian noise and updates a one-dimensional aggregate of past signals that we refer to as the score process. Only the current value of the score is revealed to each monopolist, who uses it to set prices.  

We contrast the cases of naive consumers who ignore the link between the current purchased quantity and future prices (Section 3); strategic consumers who understand how firms react to the score and directly observe their score at all times (Sections 4-6); and strategic consumers who cannot observe their scores (Section 7).

Overview of the Results (1.) Price discrimination based on purchase histories unambiguously harms naive consumers but can benefit strategic consumers. When consumers are naive, both consumer and total surplus are decreasing (and producer surplus is increasing) in the precision of the firms’ information. More strikingly, the aggregation of information into scores does not protect consumers at all: the precision of the information transmitted by the firm-optimal score is the same as if firms observed the entire (disaggregated) history of the consumer’s purchase signals (Proposition 1).

By contrast, a strategic consumer can manipulate future prices by reducing her quantity demanded and hence lower her score. To grasp why the consumer can benefit from the distribution of her score—even if firms ultimately use the information so-gained against her—consider the following two-period example. The consumer interacts with two firms sequentially. Firm 1 sets price $P_1$ using prior information only, while firm 2 privately observes a signal of the consumer’s first-period quantity before choosing price $P_2$. Because, the consumer recognizes the impact of her first period quantity choice on the second period price, she drives firm 2’s signal downwards by adopting a lower demand function than myopically optimal. Firm 1 anticipates the consumer’s manipulation incentives and lowers its price.

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5The Wall Street Journal (2018) reports, “At some retailers, the higher the [CLV score] number, the less likely you are to receive bigger discounts [...] some stores hold back discounts from higher-value customers until they are at risk of losing them. Why waste a 25% offer when the person is going to buy anyway?”

6This is a well-known property of static models with linear demand (Robinson, 1933; Schmalensee, 1981).
Figure 1: Strategic Demand Reduction

Figure 1 illustrates the equilibrium outcome: the consumer buys a lower quantity than she would like \((Q' < Q)\) but also pays a lower price \((P' < P)\). It is intuitive that a small amount of demand reduction is beneficial in the first period: the consumer gives up the marginal unit of consumption but receives an infra-marginal discount. Countering the impact of lower prices today are the losses from lower consumption today and from tailored prices tomorrow. However, if the consumer’s average willingness to pay is sufficiently high, so is the average quantity demanded (e.g., \(Q'\) in Figure 1 above). When firms reduce prices, discounts are applied to a large number of units. In this case, consumers benefit from lower prices more than they are hurt by personalized pricing.

Having developed some intuition for the effects of strategic demand reduction, we turn to the role of information aggregation and transparency in shaping equilibrium outcomes.\(^7\)

(2.) Firms can manage the ratchet effect and limit the resulting information loss by using relatively persistent scores. As we saw in Figure 1, the ratchet effect drives the average equilibrium price and quantity levels down relative to the naive case. Furthermore, higher consumer types (who expect to buy large quantities in the immediate future) have a stronger incentive to reduce their demand to drive prices downward. This reduces the sensitivity of the consumer’s actions to her type, the informativeness of the purchase signals, and thus the firms’ ability to price discriminate based on the score.

However, firms can use the aggregation of purchase histories to induce consumers to reveal more information about their preferences. With exponential scores, the question of how to aggregate information reduces to how heavily to discount past quantity signals. The

\(^7\)Relative to the two-period example, our stationary model has two advantages: it eliminates end-game effects that can artificially influence policy implications, and it is considerably more manageable than any finite-horizon version that allows for information aggregation and endogenous learning. Continuous time allows a tractable analysis of how the ratchet effect varies across information structures.
score’s discount rate is, in turn, inversely related to its persistence.

A unique score reveals the same amount of information in equilibrium as when firms observe the full history of noisy purchase signals. Because of the ratchet effect, however, that score does not maximize the firms’ learning in equilibrium: by overweighing past signals, a more persistent score correlates less with the consumer’s current type, thereby incentivizing the consumer to signal her preferences more aggressively. On the margin, the latter effect dominates: learning is optimized by a score that conceals some information about the consumer’s behavior in exchange for more precise purchase signals (Proposition 6). If the underlying signal technology is sufficiently precise, such a score is also more profitable.8

(3.) Making scores available to consumers makes demand more price-sensitive, reduces equilibrium prices, and increases consumer welfare. Strategic demand reduction helps a consumer induce lower prices (as in Figure 1) while limiting the amount of information transmitted by her purchases. In fact, if the underlying signal technology is sufficiently precise, a strategic consumer who has access to her score is then better off than a naive one. For this mechanism to operate successfully, however, score transparency is essential. To make this point, we examine the case in which the consumer is strategic but the score is hidden.

When scores are hidden, the firms’ beliefs are private, and prices acquire a signaling value: observing a high price realization signals to the consumer that the firms’ beliefs are high and that prices will be high in the near future. Because the consumer then expects to purchase relatively few units, she is less inclined to manipulate her score by reducing her current quantity, everything else being equal. Thus, with hidden scores, the consumer’s demand becomes less sensitive to price relative to the observable case (Proposition 9).

In equilibrium, firms exploit the reduced sensitivity of demand by making prices more responsive to the score. This exacerbates the ratchet effect, resulting in lower average quantities. Moreover, for each level of the score’s persistence, consumer surplus falls below the observable-score level, provided the underlying signal technology is precise enough. Having access to their score is therefore beneficial to consumers beyond increasing awareness. Indeed, consumer surplus can even be lower with hidden scores than with naive consumers. This result is relevant to policy: regulations promoting consumer awareness and score transparency have complementary roles, and one without the other may be detrimental to consumer welfare (Proposition 12).

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8 Aggregating purchase histories into scores can improve equilibrium learning, but introducing noise in the original signals cannot: a marginal increase in the score’s persistence yields a second-order loss in information but a first-order gain in the quality of the available signals that more than compensates for their inefficient aggregation. Conversely, introducing noise has a first-order negative effect that trumps the associated increase in consumer responsiveness.
Applied Relevance  When assessing our model’s policy implications, we must first determine to which extent real-life consumers can be expected to strategically manipulate the prices and products they are offered. In Section 8, we argue that lack of information by consumers—not lack of sophistication—is the most significant barrier to observing in practice the ratchet effects we uncover in our model.

To substantiate the critical role of information, we first present anecdotal evidence from consumer markets. In Section 8.1, we describe the cases of “shopping cart abandonment,” a popular tactic to obtain coupons from online retailers for near-complete purchases, and several strategies used to receive lower personalized prices from ride-sharing services such as Lyft or Uber. In both examples, consumers are able to anticipate the impact of their behavior. Consistent with our hypothesis, they take costly actions (e.g. delayed purchases, suboptimal routes) to misrepresent their true willingness to pay and obtain lower prices.

We then take a deep dive into the market for online display advertising, which we describe in detail in Section S.1 of the Supplementary Appendix. This business-to-business setting is an ideal environment to test the predictions of our model: (a) sellers are website publishers who can access summary statistics (analogous to scores) about each buyer’s past behavior and use them to set dynamic, personalized prices for advertising space; (b) buyers are sophisticated advertisers who are aware of the mechanism that links current purchases and future prices; and (c) buyers can evaluate the consequences of their actions in real time. In online advertising auctions, the ratchet effect arises when sellers attempt to learn the true distribution of advertisers’ valuations from their past bids. In Section 8.2, we describe the strategies adopted by sellers in order to alleviate the ratchet effect. These strategies are consistent with our model’s predictions. In particular, sellers use information aggregates to set reserve prices, and some sellers opt to abandon price discrimination altogether.

There are, of course, important differences between online advertisers and everyday consumers. As such, the online display advertising market serves as a benchmark for our model’s forces. In Section 8.3, we discuss what we can learn from these settings about the role of transparency policies. In particular, several features of business-to-consumer markets (e.g., consumers’ experience with managing their credit scores) facilitate the application of our paper’s policy recommendation to allow consumers to observe their scores. This policy is in line with recent regulatory intervention in the European Union.

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9It is much harder for consumers to learn about the underlying link between the terms of trade across different sellers and hence to behave strategically in seemingly independent transactions. A notable exception is the case of FICO credit scores, where the link between the borrower’s current behavior and the terms offered by different future lenders is regulated, transparent, and hence quite salient.
Related Literature   This paper builds on the literature on behavior-based price discrimination (Villas-Boas, 1999; Taylor, 2004), the results of which are surveyed in Fudenberg and Villas-Boas (2006), Fudenberg and Villas-Boas (2015), and Acquisti, Taylor, and Wagman (2016). Closest to our work is the two-period model of Taylor (2004) with observable actions and stochastic types. Taylor (2004) finds that ratchet forces result in lower equilibrium prices when consumers are strategic, as in our Figure 1 above. Qualitatively, our results differ because the noisy signals in our model imply that the consumer’s actions affect the information available to the firms. Therefore, the score’s persistence and transparency levels affect the firms’ ability to learn in a non-trivial way.\footnote{Cummings, Ligett, Pai, and Roth (2016) and Shen and Villas-Boas (2017) study models where second-period advertising is targeted on the basis of the consumer’s first-period purchase. These papers highlight a trade-off similar to ours, where the value of targeted advertising to the consumer impacts the equilibrium price of the first-period good, and hence the amount of information revealed by the consumer.}

Our score process is an instance of the linear Gaussian rating introduced by Hörner and Lambert (2017), who study information design in the Holmström (1999) career concerns model. Relative to their setting, we maintain the assumptions of short-lived firms and additive signals but introduce two additional features. First, our consumer is privately informed, which makes the informational content of the score endogenous. Second, we allow for an interaction in the consumer’s payoff function between her action and the firms’ beliefs, which implies that optimal actions depend on the level of the firms’ beliefs. This dependence makes the transparency question critical in our setting because the consumer’s incentives now depend on whether she knows her own score.

The force driving the dynamics of our model is the ratchet effect (e.g., Freixas, Guesnerie, and Tirole, 1985, Laffont and Tirole, 1988, and, more recently, Gerardi and Maestri, 2016), which has received experimental (Charness, Kuhn, and Villeval, 2011; Cardella and Depew, 2018) as well as empirical validation (Bellemare and Shearer, 2014). The ratchet effect also underscores the analysis of privacy in settings with multiple principals. Calzolari and Pavan (2006) consider the case of two principals, and Dworczak (2017) that of a single transaction followed by an aftermarket.\footnote{The ratchet effect appears, with a different interpretation or motivation, in relational contracts (Halac, 2012; Fong and Li, 2016) and in dynamic games with symmetric uncertainty (Cisternas, 2017b).} Relative to all these papers, the presence of noise in our model and the restriction to linear pricing limit the ratchet effect and allow consumers to potentially benefit from information transmission.

The marketing literature (Lewis, 2005) has already suggested the idea that the dynamics of strategic behavior must be incorporated into consumer valuation methods such as CLV. Finally, the ratchet effect in online advertising markets is the subject of a growing literature in operations research that we discuss in section S.1 in the Supplementary Appendix.
2 Model

We develop a continuous-time model with a long-lived consumer and a family of short-run firms. The model is motivated by a discrete-time setting with two key features. First, the consumer faces a different monopolist in every period. Second, within each period, the consumer and the current firm play a stage game with sequential moves: having observed the current score, the monopolist posts a unit price for its product, and the consumer then chooses which quantity to buy. It is instructive to begin with the observable (or transparent) case: the consumer can observe her score directly.

Players, types, and payoffs  Directly in continuous time, consider a long-lived consumer who interacts with a continuum of firms over an infinite horizon. The consumer discounts the future at rate $r > 0$ and, at any instant in time $t \geq 0$, consuming $Q_t = q$ units of the good at price $P_t = p$ results in a flow utility

$$u(\theta, p, q) := (\theta - p)q - \frac{q^2}{2},$$

where $\theta_t = \theta$ is the consumer’s type at $t$, understood as a measure of her willingness to pay at that point in time. We assume throughout that the type process is stationary and mean reverting, with mean $\mu > 0$, speed of reversion $\kappa > 0$, and volatility $\sigma_\theta > 0$, i.e.,

$$d\theta_t = -\kappa(\theta_t - \mu)dt + \sigma_\theta dZ^\theta_t, \ t > 0,$$

where $(Z^\theta_t)_{t \geq 0}$ is a Brownian motion. In particular, $(\theta_t)_{t \geq 0}$ is Gaussian, and by stationarity,

$$E[\theta_t] = \mu \quad \text{and} \quad \text{Cov}[\theta_t, \theta_s] = \frac{\sigma_\theta^2}{2\kappa} e^{-\kappa|t-s|}, \ \text{for all } t, s \geq 0.$$  

Each firm interacts with the consumer for only one instant, and only one firm operates at any time $t$; we refer to the monopolist operating at $t$ simply as firm $t$. Production costs are normalized to zero, and hence firm $t$’s ex post profits are given by $P_t Q_t, \ t \geq 0$.

Score process and information  At any $t \geq 0$, firm $t$ only observes the current value $Y_t$ of a score process $(Y_t)_{t \geq 0}$ that is provided by an (unmodeled) intermediary. By contrast, when scores are transparent, the consumer observes the entire history of scores $Y^t := (Y_s : 0 \leq s \leq t)$ in addition to past prices and quantities and type realizations.\(^{13}\)

\(^{12}\)Stationarity requires $\theta_0 \sim N(\mu, \sigma_\theta^2/2\kappa)$ independent of $(Z^\theta_t)_{t \geq 0}$.

\(^{13}\)In Section 5, we show that allowing each firm $t$ to observe the whole history $Y^t, t \geq 0$ is one particular instance of our model. In Section 7, in turn, firm $t$ observes only $Y_t$, and the score is hidden to the consumer.
Building a score process is a two-step procedure that involves data collection followed by data aggregation. We assume that the intermediary collects information about the consumer using a technology that records purchases with noise. Specifically, the intermediary observes

\[ d\xi_t = Q_t dt + \sigma_\xi dZ_t^{\xi}, \quad t > 0, \]

where \((Z_t^{\xi})_{t \geq 0}\) is a Brownian motion independent of \((Z_t^\theta)_{t \geq 0}\), \(Q_t\) is the realized purchase by the consumer at \(t \geq 0\), and \(\sigma_\xi > 0\) is a volatility parameter.

The intermediary operationalizes the data by aggregating every history of the form \(\xi^t := (\xi_s : 0 \leq s < t)\) into a real number \(Y_t\) that corresponds to the consumer’s time-\(t\) score, \(t \geq 0\). Building on Hörner and Lambert (2017), we restrict attention to the family of exponential scores, i.e., to Itô processes of the form

\[ Y_t = Y_0 e^{-\phi t} + \int_0^t e^{-\phi(t-s)} d\xi_s, \quad t \geq 0, \quad (4) \]

where \(\phi \in (0, \infty)\). Under this specification, the consumer’s current score is a linear function of the contemporaneous history of recorded purchases, and lower values of \(\phi\) lead to scores processes that exhibit more persistence, as past information is discounted less heavily in those cases. In differential form, the score process satisfies

\[ dY_t = -\phi Y_t dt + d\xi_t = (Q_t - \phi Y_t) dt + \sigma_\xi dZ_t^{\xi}, \quad t > 0. \quad (5) \]

Finally, the prior is that \((\theta_0, Y_0)\) is normally distributed; the exact distribution is determined in equilibrium so that the joint process \((\theta_t, Y_t)_{t \geq 0}\) is stationary Gaussian along the path of play. In what follows, the expectation operator \(\mathbb{E}[\cdot]\) is with respect to such prior, while \(\mathbb{E}_0[\cdot]\) conditions on the realized value of \((\theta_0, Y_0)\). The former is the relevant operator for studying welfare, while the latter is used in the equilibrium analysis. The conditional expectations of the consumer and firm \(t\) are denoted by \(\mathbb{E}_t[\cdot]\) and \(\mathbb{E}_t[\cdot|Y_t]\), respectively.

**Strategies and equilibrium concept** A strategy for the consumer specifies, for each \(t \geq 0\), a quantity \(Q_t \in \mathbb{R}\) to purchase as a function of the history of prices, types, and score values, \((\theta_s, P_s, Y_s : 0 \leq s \leq t)\). Instead, firm \(t\) chooses a price \(P_t \in \mathbb{R}\) that conditions on \(Y_t\) only, \(t \geq 0\). A strategy for the consumer is linear Markov if \(Q_t = Q(p, \theta_t, Y_t)\) for all \(t \geq 0\), where \(Q : \mathbb{R}^3 \to \mathbb{R}\) is linear and \(p\) is the current posted price (i.e., \(Q(\cdot, \theta_t, Y_t)\) is the demand at the history \((\theta_t, Y_t))\). Similarly for firm \(t\), \(P_t = P(Y_t)\) where \(P : \mathbb{R} \to \mathbb{R}\) is linear, \(t \geq 0\).

We focus on Nash equilibria in linear Markov strategies with the property that \((\theta_t, Y_t)_{t \geq 0}^{15}\)

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15See Lemma A.1 in Appendix A.
is stationary Gaussian. From this perspective, given a linear pricing rule $P(\cdot)$, an admissible strategy for the consumer is any process $(Q_t)_{t \geq 0}$ taking values in $\mathbb{R}$ and satisfying (i) progressive measurability with respect to the filtration generated by $(\theta_t, Y_t)_{t \geq 0}$, (ii) $\mathbb{E}_0 \left[ \int_0^T Q_s^2 ds \right] < \infty$ for all $T > 0$, and (iii) $\mathbb{E}_0 \left[ \int_0^\infty e^{-rt}(|\theta_t Q_t - Q_t^2/2| + |P_t(Y_t)Q_t|)dt \right] < \infty$.

Requirement (i) states that, at histories where firms have chosen prices as prescribed by any candidate equilibrium, the history $(\theta_s, Y_s : 0 \leq s \leq t)$ captures all the information that is relevant for future decision-making; (ii) and (iii) are purely technical.\footnote{Requirement (iii) is a mild strengthening of the condition $\mathbb{E}_0 \left[ \int_0^\infty e^{-rt}|u(\theta_t, P(Y_t), Q_t)|dt \right] < \infty$ that is usually imposed for verification theorems to hold (Sections 3.2 and 3.5 in Pham (2009)). In particular, it rules out strategies with the unappealing property of yielding high payoffs by making expenditures very negative (provided they exist). Finally, under (ii), the dynamic (5) admits a strong solution given any initial condition; therefore, the consumer’s best-response problem is well-defined (Section 3.2 in Pham (2009)).}

Definition 1. A pair $(Q, P)$ of linear Markov strategies is a Nash equilibrium if:

(i) when firms price using $P(\cdot)$, the policy $(\theta, Y) \mapsto Q(P(Y), \theta, Y)$ maximizes

$$\mathbb{E}_0 \left[ \int_0^\infty e^{-rt}u(\theta_t, P(Y_t), Q_t)dt \right]$$

subject to (2) and (5), among all admissible strategies $(Q_t)_{t \geq 0}$; and

(ii) at any time-$t$ history such that $Y_t = y$, $p = P(y)$ maximizes $p\mathbb{E}[Q(p, \theta_t, y)|Y_t = y]$.

A linear pair $(Q, P)$ is a stationary linear Markov equilibrium if, in addition, the type-score process $(\theta_t, Y_t)_{t \geq 0}$ induced by $(\theta, Y) \mapsto Q(P(Y), \theta, Y)$ is stationary Gaussian.

In a linear Markov (Nash) equilibrium, the optimality of the consumer’s strategy is verified only when firms set prices according to $P_t = P(Y_t)$ for all $t \geq 0$, i.e., on the path of play. Examining deviations from a prescribed price is, however, critical for determining the sensitivity of demand, i.e., the weight that a linear $Q$ attaches to the current posted price $p$. In Section 4.2, we select a value for such weight, thus refining our solution concept to provide an analog of Markov perfect equilibrium. Finally, the stationarity notion encompasses two ideas: the (controlled) score must admit a proper long-run distribution, and such a distribution must prevail from time zero. These two properties allow us to perform a meaningful welfare analysis that is also time invariant.

Remark 1 (Markov equilibrium with a long-lived firm). If the consumer instead faces one long-lived firm, our Markov equilibrium remains such if the firm is restricted to choosing prices that condition only on the current value of the score. This result also holds when the histories of purchase signals are observed, without any restriction on the firm’s strategy.\footnote{In the long-lived firm’s best-response problem to a linear Markov strategy by the consumer, the only}
To assess the equilibrium consequences of persistence and transparency, we describe the benchmark case of naive consumers. In particular, consider a consumer with preferences as in (1) who ignores the link between her current action and future prices. Given a posted price $p$, maximizing the consumer’s flow payoff yields a demand with a unit slope $Q(p) = \theta - p$. Each firm $t$ observes the consumer’s score $Y_t$ prior to setting the monopoly price. Letting $M_t := \mathbb{E}[\theta_t|Y_t]$, the equilibrium quantity and price are given by

$$Q_t = \theta_t - M_t/2 \quad \text{and} \quad P_t = M_t/2.$$ 

The ex ante expected profit and consumer surplus levels are given by

$$\Pi_Y^{\text{static}} = \frac{1}{4} (\mu^2 + \text{Var}[M_t]) \quad \text{and} \quad CS_Y^{\text{static}} = \frac{1}{2} \text{Var}[\theta] + \frac{\mu^2}{8} - \frac{3}{8} \text{Var}[\theta],$$

where the (ex ante) variability of the posterior mean, $\text{Var}[M_t]$, measures the precision of the firms’ information.

Because demand is linear, the average price and quantity levels (both equal to $\mu/2$) are independent of the information structure. The welfare consequences of using scores to price discriminate are thus fully determined by the firms’ ability to learn from such signals. On the one hand, better information increases firms’ profits by allowing them to better tailor the price to the consumer’s type. On the other hand, with a constant average quantity, total surplus must fall with greater price discrimination because the correlation between the type and the price reduces the degree of correlation between the type and the quantity purchased. Therefore, the consumer must be unambiguously worse off.

When the consumer is naive, each firm $t$ would ideally like to access the full set of disaggregated purchase signals $\xi^t := (\xi_s : 0 \leq s < t)$. This benchmark can be attained by an exponential score. Specifically, as we state in Section 5 (for more general purchase processes), there exists a specific value $\phi > 0$ such that the stationary posterior expectation that arises under the observation of disaggregated data in this case, $\mathbb{E}[\theta_t|\xi^t]$, is affine in

$$\int_0^t e^{-\phi(t-s)} d\xi_s, \quad t > 0.$$  

State variable is the firm’s belief about the consumer’s type. However, because the price and the type enter additively in the consumer’s strategy, a firm’s choice of a price today does not affect its future beliefs, thus rendering myopic behavior optimal. When each firm $t$ only observes $Y_t$, therefore, the measurability restriction on the long-lived firm’s strategy makes this setting strategically equivalent to the one studied. Finally, when the histories of recorded purchases are observed, the measurability restriction is vacuous.
Learning from disaggregated data ($\xi_s : 0 \leq s < t$) or from the contemporaneous value of a (stationary) score ($\hat{\phi}$) of persistence thus leads to identical beliefs.

The value $\hat{\phi} > 0$ denotes the (optimal) weight that the Kalman filter uses to discount past information when purchases follow $Q_t = \theta_t - E[\theta_t|\xi^t]/2$. Critically, by definition of the Kalman filter, discounting past recorded purchases with an exponential weight $\hat{\phi}$ maximizes firms’ learning. We summarize the results for the naive case in the following:

**Proposition 1** (Naive Benchmark).

1. Consumer and total surplus are decreasing in the precision of the firms’ information.
2. Firm profits are increasing in the precision of the firms’ information.
3. The precision of the firms’ information is maximized by observing the disaggregated history of signals.
4. Observing the contemporaneous value of a score process of persistence $\hat{\phi} > 0$ is equivalent to observing the disaggregated history of signals.

Two corollaries are distilled from this result. First, scores of persistence $\phi \neq \hat{\phi}$ hinder the firms’ learning, as information then ceases to be aggregated optimally (given the characteristics of the type and purchases processes). This is not qualitatively different from adding noise to the technology $(\xi_t)_{t \geq 0}$: since the behavior of a naive consumer is fixed, the signal-to-noise ratio in $(\xi_t)_{t \geq 0}$ worsens. Second, because the naive benchmark is equivalent to a repetition of static interactions (albeit with varying information on the firms’ side), the only channel through which a strategic consumer can benefit from information collection is by changing her demand in a dynamic environment.

A strategic consumer understands that the presence of a score process paves the way for the *ratchet effect*: larger purchases today lead to increased future prices due to the inherent persistence in the score process. In Section 4, for a given score of persistence $\phi > 0$, we aim to characterize a quantity process

$$Q_t = \alpha \theta_t + \beta M_t + \delta \mu$$

that generalizes $Q_t = \theta_t - M_t/2$ in the naive benchmark. The strategic effects of varying a score’s persistence are then reflected in the tuple $(\alpha, \beta, \delta)$ depending on $\phi > 0$, and its discrepancy with the naive values $(1, -1/2, 0)$ captures the strength of the ratchet effect.

In Proposition 4, we present the general version of these statements for purchase processes of the form $Q_t = \alpha \theta_t + \beta M_t + \delta \mu, t \geq 0$. In particular, we determine the persistence level $\nu(\alpha, \beta)$ that maximizes the precision of the firms’ information. In the naive case, we have $\hat{\phi} = \nu(1, -1/2)$. 

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18In Proposition 4, we present the general version of these statements for purchase processes of the form $Q_t = \alpha \theta_t + \beta M_t + \delta \mu, t \geq 0$. In particular, we determine the persistence level $\nu(\alpha, \beta)$ that maximizes the precision of the firms’ information. In the naive case, we have $\hat{\phi} = \nu(1, -1/2)$. 

12
The economic implications of consumer sophistication are twofold. First, the precision of
the firms’ information no longer fully determines the welfare consequences of price discrim-
ination: \((\alpha, \beta, \delta)\) encode demand adjustments in response to scores of different persistence,
which make the average quantity purchased and price paid no longer constant (the level
implications of the ratchet effect). Second, varying \(\phi > 0\) not only affects the score’s in-
formativeness directly through the way in which information is aggregated: it also does it
indirectly by endogenizing the signal-to-noise ratio in the purchase signal via \(\alpha\), the weight
on \(\theta_t\) in (6), depending on \(\phi\) (the informational implications of the ratchet effect). In par-
ticular, adding noise to the recorded purchases and sub-optimally aggregating the available
information in a statistical sense cease to have the same implications for learning.

4 Equilibrium Analysis with Transparent Scores

In this section we characterize a stationary linear Markov equilibrium with realized purchases
as in (6) for a consumer who observes the score process \((Y_t)_{t \geq 0}\) directly. We proceed in three
steps: (i) we characterize the firms’ (stationary Gaussian) beliefs when learning from the
score; (ii) we determine the sensitivity of the consumer’s demand that pins down the firms’
monopoly price; and (iii) we solve the consumer’s dynamic optimization problem.

4.1 Stationary Beliefs

Stationarity imposes two restrictions in our model: each firm \(t\) must use the same rule
to update its beliefs, and the process \((\theta_t, Y_t)\) has to admit a long-run distribution that is
initialized at time 0 via an appropriate choice of \((\theta_0, Y_0)\).

If \((\theta_t, Y_t)\) is Gaussian, however, the projection theorem for Gaussian random variables
yields the following linear updating rule,

\[
M_t := \mathbb{E}[\theta_t | Y_t] = \mu + \frac{\text{Cov}[\theta_t, Y_t]}{\text{Var}[Y_t]} (Y_t - \mathbb{E}[Y]),
\]

where \(\text{Cov}[\theta_t, Y_t]/\text{Var}[Y_t]\) and \(\mathbb{E}[Y]\) are independent of time. Therefore, the firms’ posterior
mean belief takes the form \(M_t = \mu + \lambda [Y_t - \bar{Y}]\) for some \(\lambda\) and \(\bar{Y}\) in \(\mathbb{R}\).

Lemma A.1 in Appendix A shows that the previous restrictions reduce to the following
conditions: a long-run Gaussian distribution exists when \(\phi - \beta \lambda > 0\), and \(\lambda\) must satisfy

\[
\lambda = \frac{\alpha \sigma^2 \beta (\phi - \beta \lambda)}{\alpha^2 \sigma^2 + \sigma^2 \kappa (\phi - \beta \lambda + \kappa)}, \quad (7)
\]
The first condition states that the score must have a positive rate of decay when \( M_t = \mu + \lambda [Y_t - \bar{Y}] \). The second constraint simply reflects that the score \( Y_t \) used to construct \( M_t \) contains past beliefs that depend themselves on \( \lambda \).

The regression coefficient \( \lambda \) measures the responsiveness of beliefs to changes in the score, and it plays a central role in our analysis. In fact, using \( M_t = \mu + \lambda [Y_t - \bar{Y}] \), we can recast the problem of controlling a score as one of controlling a belief, namely,

\[
\begin{align*}
    dM_t &= \left[ -\phi \left( M_t - \mu + \lambda \bar{Y}\right) + \lambda Q_t \right] dt + \lambda \sigma_t \xi_t dt + \lambda \sigma_t \xi Z_t, \quad t \geq 0.
\end{align*}
\]  

Thus, the consumer’s choice of quantity affects the current firm’s belief linearly with a slope of \( \lambda \), and this effect decays at rate \( \phi \). The following result underscores a key tension between the short- and long-term response of beliefs as the persistence of the scores varies:

**Lemma 1** (Persistence and Sensitivity). \( \lambda \) that solves (7) is strictly increasing in \( \phi \).

Intuitively, the more a score overweighs past information, the less it correlates with the current type, and hence beliefs respond less to new information as captured by \( \lambda \), and vice-versa. In particular, attempting to endow beliefs (a fortiori, prices) with persistence by making scores more persistent themselves is not without cost: the short-term response of beliefs is diminished.

### 4.2 Sensitivity of Demand and Monopoly Pricing

In this section, we determine the sensitivity of the consumer’s demand and characterize the firms’ monopoly price. Because the score is observed by the consumer and the firms adopt a linear strategy, the consumer can perfectly anticipate the candidate equilibrium price. The sensitivity of demand is then determined by the (optimal) change in the consumer’s quantity demanded in response to a price deviation \( p \neq P(Y_t) \). This poses a challenge in continuous time: imposing optimality of the consumer’s strategy at such off-path histories does not pin down her response to a deviation, as every firm operates over a zero-measure set.

To overcome this challenge, we refine our stationary linear Markov equilibrium concept by requiring that prices be supported by the limit sensitivity of demand along a natural sequence of discrete-time games indexed by their period length. Along such a sequence, as the period length shrinks to zero, the limit demand sensitivity is equal to \(-1\).

Heuristically, we consider a discrete-time version of our model in which the period length given by \( \Delta > 0 \) is small. Given any posted price \( p \), we can write the consumer’s continuation
value $V_t$ recursively with $M_t$ as a state,

$$
V_t = \max_q \left[ (\theta_t - p)q - \frac{q^2}{2} \right] \Delta + \mathbb{E}_t[V_{t+\Delta}]
= \max_q \left[ (\theta_t - p)q - \frac{q^2}{2} \right] \Delta + V_t + \frac{\partial V_t}{\partial M_t} \left[ -\phi \left( M_t - \mu + \lambda \bar{Y} \right) + \lambda \bar{q} \right] \Delta t + \text{other terms.}
$$

When $\Delta$ is sufficiently small, the missing terms that are affected by $q$ on the right-hand side have only second-order effects on the consumer’s payoff; therefore, the impact of quantities on the continuation value becomes asymptotically linear. Furthermore, because firms do not observe past prices, the continuation game is unaffected by the actual choice of $p$. The consumer’s best reply is then given by

$$
Q_t = \theta_t - p + \lambda \frac{\partial V_t}{\partial M_t},
$$

where $\frac{\partial V_t}{\partial M_t}$ is independent of the posted price, which leads to a slope of demand of value $-1$. In other words, the incentives to manipulate the firms’ beliefs affect the intercept but not the slope of the demand function. Henceforth, except for the case of hidden scores, a stationary linear Markov equilibrium is understood to have unit demand sensitivity.\(^\text{19}\)

Having pinned down the sensitivity of demand independently of the remaining equilibrium coefficients, we obtain a clean characterization of the monopoly price process along the path of play of any stationary linear Markov equilibrium.

**Lemma 2 (Monopoly Price).** Consider a stationary linear Markov equilibrium in which the quantity demanded follows (6). Then, prices are given by

$$
P_t = (\alpha + \beta)M_t + \delta \mu, \ t \geq 0.
$$

The intuition is simple: because demand has unit slope, the monopoly price along the path of play of such an equilibrium satisfies $P_t = \mathbb{E}[Q_t|Y_t], \ t \geq 0$.

Equipped with this result, we can formulate the consumer’s best-response problem to a price process $P_t$ with parameters $(\alpha, \beta, \delta)$ as a linear-quadratic optimization problem,

$$
\max_{(Q_t)_{t\geq0}} \mathbb{E} \left[ \int_0^\infty e^{-rt} \left( (\theta_t - P_t)Q_t - \frac{Q_t^2}{2} \right) dt \right]
$$

\(^\text{19}\)In section S.4 in the Supplementary Appendix, we examine a sequence of discrete-time games that employ the usual discretized version of diffusions in which noise is scaled by $\sqrt{\Delta}$. We show that along this sequence (i) linear best replies on the path of play are also optimal after observing off-path prices and (ii) the weight that linear best replies attach to the current price converges to $-1$ as the period length goes to zero.
subject to

\[
\begin{align*}
  d\theta_t &= -\kappa(\theta_t - \mu)dt + \sigma_\theta dZ_\theta^t \\
  dM_t &= (-\phi[M_t - \mu + \lambda \bar{Y}] + \lambda Q_t)dt + \lambda \sigma_z dZ_z^t \\
  P_t &= (\alpha + \beta)M_t + \delta \mu,
\end{align*}
\]

where \( \lambda \) satisfies (7).

### 4.3 Stationary Linear Markov Equilibria

To characterize stationary linear Markov equilibria, we use standard dynamic-programming tools. In a nutshell, we look for a quadratic value function and impose the condition that the firms correctly anticipate the consumer’s behavior. This yields a sub-system of equations for the equilibrium coefficients \((\alpha, \beta, \delta)\) that is coupled with equation (7) to pin down the equilibrium sensitivity of beliefs \( \lambda \); we then look for a solution that satisfies the stationarity condition \( \phi - \beta \lambda > 0 \). As we show in the proof of Theorem 1, there is a unique such solution to this system, which in turn allows us to establish the existence and uniqueness of an equilibrium in this class. Furthermore, the equilibrium can be computed in closed form, up to the solution of a single algebraic equation for the coefficient \( \alpha \).

**Theorem 1** (Existence and uniqueness). For any \( \phi > 0 \), there exists a unique stationary linear Markov equilibrium. In this equilibrium, \( 0 < \alpha < 1 \) is characterized as the unique solution to the equation

\[
a = 1 + \frac{\Lambda(\phi, \alpha, B(\phi, \alpha))aB(\phi, \alpha)}{r + \kappa + \phi}, \quad a \in [0, 1]. \tag{11}
\]

Moreover, \( \beta = B(\phi, \alpha) < 0 \), \( \delta = D(\phi, \alpha) \in \mathbb{R} \), and \( \lambda = \Lambda(\phi, \alpha, B(\phi, \alpha)) > 0 \), where \( B : (0, \infty) \times [0, 1] \to \mathbb{R} \), \( D : (0, \infty) \times [0, 1] \to \mathbb{R} \), and \( \Lambda : (0, \infty) \times [0, 1] \times (-\infty, 0) \to \mathbb{R} \) are defined in (A.7), (A.12), and (A.8).

Figure 2 illustrates the equilibrium coefficients \((\alpha, -\beta, \delta)\), their naive benchmark levels, and the average equilibrium price (and quantity)

\[
\mathbb{E}[P_t] = \mathbb{E}[Q_t] = [\alpha + \beta + \delta] \mu. \tag{12}
\]
4.4 Strategic Demand Reduction

The consumer’s strategic behavior is encoded in $\frac{\partial V(\theta, M)}{\partial M}$ in the first-order condition (9). With a quadratic value function, moreover, this term is linear in the consumer’s type and the current firm’s belief. Intuitively, this derivative must capture the strength of the \textit{ratchet effect} that is expected to arise in this environment. The next result confirms this intuition by obtaining an transparent representation for this term.

\textbf{Proposition 2 (Value of Future Savings).} The equilibrium prices and quantities satisfy

$$Q_t = \theta_t - P_t - \Psi_t, \text{ where}$$

$$\Psi_t := \lambda E_t \left[ \int_t^\infty e^{-(r+\phi)(s-t)}(\alpha + \beta)Q_s ds \right], \ t \geq 0. \quad (13)$$

To see why $\Psi_t$ is a measure of the value of future savings, recall that, by the Envelope Theorem, the future benefit of a marginal reduction in today’s quantity along the “optimal trajectory” is equal to the net present value of the associated reduction in future prices, holding the future quantities constant. In particular, lowering $Q_t$ by one unit reduces the posterior belief $M_{t+dt}$ by $\lambda$ and the contemporaneous price $P_{t+dt}$ by $\lambda(\alpha + \beta)$; the impact on subsequent prices then vanishes at the rate $\phi$ at which beliefs discount the initial change.\textsuperscript{20}

We conclude that $\lambda \partial V/\partial M$ takes the form of the value of future savings in (14) and measures the strength of the ratchet effect: the larger the savings, the stronger the incentive to engage in a downward deviation from the static Nash equilibrium. Moreover, using $Q_s = \alpha \theta_s + \beta M_s + \delta \mu$ and taking expectations under the prior distribution of $(\theta_t)_{t \geq 0}$ in

\textsuperscript{20}While we prove the result only in equilibrium, (14) is an optimality condition and thus holds at a greater level of generality: if a best-response to a price process that is affine in the belief process exists, standard variational arguments show that an expression analogous to (14) must hold.
(14), the average strength of this effect is given by

$$E[P_t] = \lambda \frac{(\alpha + \beta)(\alpha + \beta + \delta)\mu}{r + \phi}. \quad (15)$$

As we shall establish shortly, both $E[P_t] = (\alpha + \beta + \delta)\mu$ and $\alpha + \beta$ are strictly positive. Therefore, the right-hand side of (15) is strictly positive, and hence the average quantity demanded contracts. Coupled with the fact that $E[P_t] = E[Q_t]$, it is easy to conclude from (14) that the average price lies below the static level $\mu/2$, as in Figure 2 (right panel).

These incentives for demand reduction are not, however, uniform across consumer types. In particular, the value of future savings in Proposition 2 satisfies

$$\frac{\partial \Psi_t}{\partial \theta_t} = \frac{\lambda \alpha \beta}{r + \kappa + \phi}. \quad (16)$$

Comparing equation (16) with the right-hand side of equation (11) from Theorem 1, we conclude that the equilibrium coefficient $\alpha(\phi)$ is simply the (static) unit weight attached to the type $\theta$, diminished exactly by the sensitivity of the value of future savings to $\theta$. From this perspective, the denominator of the right-hand side of (16) indicates that the impact of $\theta_t$ on future types $\theta_s$, $s > t$, depreciates at rate $\kappa$, so the impact of a marginal change in today’s type on future savings decays at the augmented rate of $r + \kappa + \phi$. The numerator, in turn, indicates that an increase in the current type $\theta_t$ positively affects not only future types $\theta_s$ but also future belief realizations $M_s$, $s > t$.

We now use this intuition to derive properties of the equilibrium as a function of the score’s persistence, making explicit the dependence of $(\alpha, \beta, \delta)$ on $\phi$ when required.

**Proposition 3** (Equilibrium Properties).

1. **Uninformative scores:** $\lim_{\phi \to 0, \infty} (\alpha(\phi), \beta(\phi), \delta(\phi)) = (1, -1/2, 0)$; $\lim_{\phi \to 0, \infty} E[(P_t - \mu/2)^2] = 0$.

2. **Bounds on the strength of ratchet effect:** for all $\phi > 0$,

$$1/2 < \frac{r + \kappa + \phi}{r + \kappa + 2\phi} < \alpha(\phi) < 1; \quad -\alpha(\phi)/2 < \beta(\phi) < 0; \quad \text{and} \quad E[P_t] \in (\mu/3, \mu/2). \quad (17)$$

3. **Strategic demand reduction across types:** $\alpha(\cdot)$ is strictly quasiconvex.

4. **Effect of noise:** $\alpha(\phi)$ and $E[P_t]$ are increasing in $\sigma_{\varepsilon}/\sigma_\theta$ for all $\phi > 0$.

Part (i) shows that the equilibrium coefficients and price all converge to the static benchmark when the score becomes uninformative: the value of future savings (14) vanishes as
\( \phi \to 0 \) and \( +\infty \). In fact, if the score has no memory \( (\phi \to \infty) \), new information is forgotten instantaneously: in this case, the consumer’s actions have no impact on future prices, and hence behaving myopically is optimal. Likewise, if the score is fully persistent \( (\phi \to 0) \), it places an arbitrarily large weight on arbitrarily old information that is uncorrelated with the current type: in this case, the firms’ beliefs are simply not sensitive to new information, leading to the same conclusion.\(^{21}\)

Part (ii) formalizes the intuition that the benefits of a downward deviation are greater for higher types \((\alpha < 1)\): because types are persistent, a high \( \theta \) is more likely to buy larger quantities in the future, and hence is more willing to invest in strategic demand reduction. The equilibrium coefficient \( \alpha \) and the expected price level are also bounded from below.\(^{22}\)

Part (iii) shows that the sensitivity of the value of manipulation to the consumer’s type is strongest for an intermediate level of persistence \( \phi \). Consider equation (16) and recall the sensitivity-persistence tradeoff (Lemma 1): if the score is persistent (i.e., \( \phi \) is low), strategic demand reduction has a long-lasting but small effect on prices because \( \lambda \) is also low. Conversely, because \( \lambda \) is bounded due to the noise in the score, the net present value of future savings vanishes for all types \( \theta \) as \( \phi \) grows without bound.

Finally, part (iv) considers the effects of noise in the purchase signals on the value of demand reduction: as the exogenous noise increases, beliefs become less responsive to changes in the score; therefore, the consumer’s incentives to manipulate decrease, and equilibrium prices rise in expectation. Furthermore, the incentives to manipulate decrease more quickly for higher types, which increases the equilibrium \( \alpha \).

5 Equilibrium Learning

The firms’ ability to price discriminate depends on how much they are able to learn from the score. In this section, we show that the precision of the firms’ information is maximized by relatively persistent scores: scores that discount past signals insufficiently heavily relative to Bayesian updating when the purchase histories are observable. Unlike in the naive benchmark, the ratchet effect introduces a wedge between maximizing firms’ learning and the use of disaggregated data.

\(^{21}\)Formally, the proof of the proposition shows that \( \lim_{\phi \to \infty} \text{Var}[Y_t] = 0 \) and \( \lim_{\phi \to 0} \lambda(\phi) = 0 \).

\(^{22}\)The firms’ conjecture of \( \alpha \) and the consumer’s choice of \( \alpha \) are strategic substitutes. Thus, if firms believed that the quantity signals were uninformative \((\alpha = 0)\), the consumer would then have no reason to manipulate them (i.e., she would choose \( \alpha = 1 \)).
The extent of firms’ learning is summarized by the following quantity:\(^{23}\)

\[
\frac{\text{Cov}[\theta_t, Y_t]}{\text{Var}[\theta_t] \text{Var}[M_t]} = \frac{\alpha \lambda(\phi, \alpha(\phi), \beta(\phi))}{\phi + \kappa - \beta \lambda(\phi, \alpha(\phi), \beta(\phi))} := G(\phi, \alpha(\phi), \beta(\phi)) \in [0, 1]. \quad (18)
\]

The function \(G\) highlights the two channels through which a score’s persistence affects learning: directly via \(\phi\) determining the weight attached to past signals (and its resulting impact on \(\lambda\)), and indirectly via the coefficients of the consumer’s strategy. The presence of equilibrium effects thus opens the possibility of a score that gives up on its optimality as a statistical filter, i.e., as an optimal aggregate given the underlying signals, in exchange for an improvement in the quality of such signals by affecting the consumer’s behavior.

The question of how to maximize learning is ultimately one of how to optimally aggregate information accounting for these two channels. It is natural to use the case of disaggregated signals \(\xi^t := (\xi_s : 0 \leq s < t)\) as a reference point. To this end, start by holding the consumer’s behavior \((\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_-\) fixed, and define

\[
\nu(\alpha, \beta) := \kappa + \frac{\gamma(\alpha) \alpha(\alpha + \beta)}{\sigma_\xi^2}, \quad (19)
\]

where \(\gamma(\alpha) > 0\) is the steady state variance of beliefs when the histories of \((\xi_t)_{t \geq 0}\) are observable.\(^{24}\) We now establish an equivalence result between learning from the history of disaggregated signals and from the current level of a score with persistence \(\nu(\alpha, \beta) > 0\).

**Proposition 4** (Learning from Disaggregated Histories). Consider a process \((Q_t)_{t \geq 0}\) as in (6) with \(\alpha + \beta > 0\).

1. \(\nu(\alpha, \beta) > 0\) is the unique maximizer of \(G(\cdot, \alpha, \beta)\).

2. If firms observe the histories of \((\xi_t)_{t \geq 0}\) and their beliefs are stationary, then the posterior mean process is affine in a stationary Gaussian score (4) with \(\phi = \nu(\alpha, \beta)\).

3. If firms only observe the current value of a stationary Gaussian score with \(\phi = \nu(\alpha, \beta)\), their beliefs coincide with those that arise from observing the histories \(\xi^t, t \geq 0\).

The persistence level \(\nu(\alpha, \beta)\) aggregates the data generated by a linear strategy \(Q\) with coefficients \((\alpha, \beta)\) without loss of information, for a given fixed behavior; i.e., it defines what an optimal score in a statistical sense would look like.\(^{25}\) Given a score with generic persistence

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\(^{23}\)By the projection theorem, \(\text{Var}[\theta_t|Y_t] = \text{Var}[\theta_t](1 - \text{Cov}[\theta_t, Y_t]/\text{Var}[\theta_t] \text{Var}[M_t])\).

\(^{24}\)\(\gamma(\alpha)\) is the unique positive root of \(x \mapsto \alpha^2 x^2/\sigma_\xi^2 + 2\kappa x - \sigma_\phi^2 = 0\).

\(^{25}\)This score discounts past signals with a weight higher than \(\kappa\), the weight with which the type process discounts past shocks to taste, reflecting the identification problem faced by the firms while observing \((\xi_t)_{t \geq 0}\).
\( \phi > 0 \), however, a strategic consumer need not choose the coefficients \((\alpha(\phi), \beta(\phi))\) for which the score is an optimal filter in a statistical sense, i.e., for which \( \phi = \nu(\alpha(\phi), \beta(\phi)) \). The importance of these fixed points is clear: if an equilibrium in which the firms have access to disaggregated signals exists, then, from the previous proposition, the weight with which the associated beliefs discount past purchase signals must solve \( \phi = \nu(\alpha(\phi), \beta(\phi)) \).

**Definition 2 (Non-concealing score).** A score with persistence \( \phi > 0 \) is non-concealing if

\[
\phi = \nu(\alpha(\phi), \beta(\phi)).
\]  \((20)\)

We now establish the existence and uniqueness of a solution to \((20)\) and derive its implications for signaling.

**Proposition 5 (Uniqueness of a Non-Concealing Score and Signaling).**

(i) There exists a unique \( \phi^* \in \mathbb{R}_+ \) solving \( \phi = \nu(\alpha(\phi), \beta(\phi)) \).

(ii) The coefficient \( \alpha(\cdot) \) is strictly decreasing at the fixed point \( \phi = \phi^* \).

The previous result suggests that learning can be enhanced by choosing scores that are more persistent than the unique non-concealing score, despite such scores concealing some information about the consumer’s behavior (as \( \phi \neq \nu(\alpha(\phi), \beta(\phi)) \) in those cases).\(^{26}\) Without fear of confusion, we let \( G(\phi) := G(\phi, \alpha(\phi), \beta(\phi)) \), denote the equilibrium gain function.

**Proposition 6 (Equilibrium Learning).**

(i) The equilibrium gain function \( G(\phi) \in [0,1] \) is maximized in \((0, \phi^*)\).

(ii) The function \( G(\phi; \sigma_\xi) \) is decreasing in \( \sigma_\xi \) for all \( \phi > 0 \).

By the definition of an optimal filter, changing the persistence of the score has only a second-order effect on learning, holding \((\alpha, \beta)\) constant.\(^{27}\) Increasing \( \alpha \), however, has a first-order effect on learning, as the score is now more sensitive to the consumer’s type. Thus, the indirect effect on the consumer’s incentives to reveal information drives the firms’ learning around \( \phi^* \), while the direct effect of \( \phi \) dominates away from the optimal filter \( \phi^* \). Figure 3 plots \( G \) as a function of \( \phi - \nu(\phi) \): its maximum is located to the left of the vertical axis.

\(^{26}\)Part (i) establishes the uniqueness of a stationary linear Markov equilibrium when the histories of \((\xi_t)_{t\geq0}\) are observable as a byproduct. A unique non-concealing score then results from the strategic substitutability between the firms’ conjectured actions and the consumer’s actual choices, as discussed in footnote 23.

\(^{27}\)Marginally increasing \( \beta \) at \((\phi^*, \alpha(\phi^*), \beta(\phi^*))\), in turn, has no first-order effect on the amount of information transmitted either: \( \beta \) is the coefficient on \( M_t \) in the consumer’s strategy, and at \( \phi^* \), the score perfectly accounts for the contribution of the beliefs to the recorded purchases.

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That a reduction in $\phi$ from $\phi^*$ increases the equilibrium coefficient $\alpha$ for all discount rates is somewhat surprising: even a very patient consumer finds it optimal to attach a higher weight to her type, despite the consequences that more persistent scores can have for long-term prices. Opposing this force is the fact that a score that attaches an excessive weight to past signals also correlates less with the consumer's current type (sensitivity-persistence tradeoff). This results in a reduced sensitivity of beliefs (and hence, of prices) to changes in the score, as illustrated in Figure 4 below. In turn, less sensitive prices make the consumer less concerned about purchasing large quantities.²⁸

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²⁸The trade-off between persistence and sensitivity also arises in signal-jamming models with symmetric uncertainty. See, for example, Cisternas (2017a) in the context of career concerns.
as derived in (16). From Section 4.4, the sensitivity of the value of future savings to the consumer’s current type reflects both the direct impact of a shock to \( \theta_t \) on future types (which decays at a rate \( \kappa \)) and its indirect impact on future prices (which depreciates at rate \( r + \phi \)) via the change in the quantity demanded.

With a linear relationship between \((Y_t)_{t \geq 0}\) and \((\theta_t)_{t \geq 0}\), the equilibrium gain function \( G(\phi) \) is also akin to an impulse response, where a shock to a past type \( \theta_s, s < t \), has an impact on the past score \( Y_s \) that depreciates at rate \( \phi \). However, the type shock itself depreciates at rate \( \kappa \). Loosely speaking then, the gain function is akin to the \textit{undiscounted} impulse response of the marginal value of future savings to a shock to \( \theta_t \).

The gain function and the marginal value of future savings then differ only in that discounting gives the immediate future more relevance in the latter. As a result, the sensitivity-persistence tradeoff is tilted in favor of the sensitivity effect. This, in turn, leads a consumer with any degree of time preference to become more responsive to her type when the impact of quantities on future prices is \textit{backloaded} relative to the non-concealing score \( \phi^* \). Thus, aggregation in the direction of greater persistence facilitates price discrimination.

Finally, part (ii) in Proposition 6 shows how noise and persistence have qualitatively different effects on equilibrium learning when consumers are strategic. We know from Propositions 3 and 5 that both increasing \( \sigma_\xi \) and reducing \( \phi \) below \( \phi^* \) induce increases \( \alpha \). However, while distorting persistence away from the optimal filter triggers sufficiently strong equilibrium effects, adding noise to the purchase signals reduces \( G \) unambiguously: the intuition is that, unlike moving \( \phi \) around \( \phi^* \) (a choice variable), increasing \( \sigma_\xi \) (a parameter) does have a negative first-order effect on learning, which trumps the increase in \( \alpha \). In other words, adding exogenous noise to purchase signals is an inferior means to manage the ratchet effect, relative to a sub-optimal endogenous aggregation of such signals.\(^{29}\)

6 Welfare Analysis

In this section, we examine the welfare consequences of the score’s persistence. We show that firms can profit from relatively persistent scores, and that such scores do not necessarily hurt consumers relative to a setting without price discrimination. Omitting the dependence of \( P_t \) and \( M_t \) on \( \phi \), and using that \( \mathbb{E}[Q_t|Y_t] = P_t \), firm \( t \)’s ex ante profits are given by

\[
\Pi(\phi) := \mathbb{E}[P_t Q_t] = \mathbb{E}[P_t^2] = \mathbb{E}[P_t]^2 + \text{Var}[P_t], \ t \geq 0. \tag{21}
\]

\(^{29}\)In Section S.2.2 of the Supplementary Appendix we establish a stronger version of this result for the case of publicly observable signal histories: the equilibrium gain function \( G(\phi^*(\sigma_\xi), \sigma_\xi) \) is decreasing in \( \sigma_\xi \), even though \( \phi^*(\cdot) \) is itself decreasing, which partly contrasts with the increase in the noise. Thus, additional noise is not conducive to more learning in a world without scores.
A similar calculation yields the ex ante flow consumer surplus,

$$CS(\phi) = \mathbb{E}[P_1]\left(\mu - \frac{3}{2}\mathbb{E}[P_1]\right) + L(\phi)\text{Var}[P_1] + \alpha(\phi)\left(1 - \frac{\alpha(\phi)}{2}\right)\text{Var}[\theta_t],$$

(22)

where

$$L(\phi) := \frac{\alpha(\phi)^2}{2} + \beta(\phi) - \frac{3}{2} \frac{\alpha(\phi)^2}{2} - \frac{3}{2} < 0, \text{ for all } \phi > 0.$$  

We know from Proposition 3 that \(\mathbb{E}[P_1] > \mu/3\). Therefore, consumer surplus is increasing, and producer surplus is increasing, both in the expected price level and in the firms’ ability to tailor prices based on the information contained in the score, as measured by the ex ante variability of the price.\(^{30}\) Moreover, these two moments respond very differently to the score’s persistence \(\phi\) and to the average willingness to pay \(\mu\). Indeed, we have

$$\mathbb{E}[P_t] = (\alpha(\phi) + \beta(\phi) + \delta(\phi))\mu \quad \text{and} \quad \text{Var}[P_t] = (\alpha(\phi) + \beta(\phi))^2\text{Var}[M_t] = (\alpha(\phi) + \beta(\phi))^2\text{Var}[\theta_t]G(\phi).$$

On the one hand, the expected price is largest for uninformative scores (\(\phi = 0\) and \(\phi \to \infty\)): the ratchet effect benefits the consumer through a lower expected price for all informative scores. On the other hand, the variance of the price inherits all of the properties of the equilibrium gain function \(G\) derived in Proposition 6. In particular, it is maximized by some \(\phi \in (0, \phi^*)\), i.e., by an informative and persistent score.\(^{31}\)

The effect of the expected price on consumer surplus is proportional to \(\mu^2\): the benefit of low prices is higher when the average willingness to pay \(\mu\) is high, and discounts are applied to a larger number of units. Instead, the costs of price discrimination for the consumer are independent of her average willingness to pay \(\mu\). Thus, consumers with a high \(\mu\) benefit more from the availability of information than those with low \(\mu\), and firms derive a net benefit from informative scores only if \(\mu\) is low enough. See Figures 5 and 6.

The presence of several nonlinear terms in the expressions for consumer surplus and firms’ profits makes the full characterization of the associated optimal degrees of persistence as a function of primitives a daunting task. For this reason, we specialize some of our welfare

\(^{30}\)Moreover, \(\alpha(\phi) \leq 1\) implies that the third term is increasing in \(\alpha\): by shading down her demand, the consumer moves away from her static optimum, reducing her surplus.

\(^{31}\)The derivation of the expressions for \(CS(\phi)\) and \(\Pi(\phi)\) can be found in section S.2.3 in the Supplementary Appendix, as well as the properties of \(\text{Var}[P_t]\). The latter can be interpreted as the value of information to the firms: since \(\text{Var}[P_t] = \mathbb{E}[P_tQ_t] - (\mathbb{E}[P_t] \cdot \mathbb{E}[Q_t])\), the variance \(\text{Var}[P_t]\) measures the supplemental profits relative to pricing under the consumer’s equilibrium strategy, with the knowledge of the prior distribution only. As such, it is a good proxy for a monopolist data broker’s profit. See Bergemann and Bonatti (2019) for a model of intermediation in the market for consumer information.
results to the noiseless limit $\sigma_\xi \searrow 0$, in which further insights can be obtained.\textsuperscript{32}

Let $\phi^c$ and $\phi^f$ denote the consumer and firm optimal persistence levels, respectively.

**Proposition 7** (Optimal Persistence). For all $\sigma_\xi > 0$:

(i) there exists $\mu_f^c > 0$ such that $\phi^f$ is interior if $\mu < \mu_f^c$; and there exists $\mu_c^f > 0$ such that $\phi^c$ is interior if $\mu > \mu_c^f$;

(ii) there exists $\mu_c^f > 0$ such that $\phi^c \in \{0, \infty\}$ if $\mu < \mu_c^f$; and there exists $\bar{\mu}_f > 0$ such that $\phi^f \in \{0, \infty\}$ if $\mu > \bar{\mu}_f$.

In the pointwise limit of $\Pi(\cdot)$ and $CS(\cdot)$ as $\sigma_\xi \searrow 0$:

(iii) there exists a firm optimal $\phi^f < \lim_{\sigma_\xi \searrow 0} \phi^*(\sigma_\xi) = \infty$ for all $\mu$;

(iv) there exist $\bar{\rho} > \rho > 0$ such that, if $r/\kappa \in [\rho, \bar{\rho}]$, then $\phi^f$ and $\phi^c$ are continuous and monotone. Moreover, $\phi^f > 0$ if and only if $\mu < \sqrt{r}/\kappa$ and $\phi^c > 0$ if and only if $\mu > 3\sqrt{r}/\kappa$.

\textsuperscript{32}When $\sigma_\xi = 0$, all coefficients can be solved for in closed form, and the model is of class $C^1$ around that point (section S.3 in the Supplementary Appendix).
Parts (i) and (ii) establish sufficient conditions for the informative and uninformative optima, respectively. Part (iii) formalizes our intuition that firms benefit from aggregating information into (excessively) persistent scores. Figure 7 illustrates the result in part (iv): there exists a range of $\mu$ over which all market participants prefer uninformative scores, which is intuitive, because information reduces total surplus.

![Figure 7: Optimal Persistence, $(\sigma_\theta, \sigma_\xi, \kappa, r) = (1, 0, 1, 0.8)$](image)

Finally, we contrast the expected consumer surplus in the strategic case with the naive-consumer benchmark (Section 3) in the noiseless limit case. Intuitively, a naive consumer, who demands the static quantity as in Section 3, suffers the costs of tailored prices without reaping the benefits of lower average prices. Thus, regardless of whether firms or consumers would prefer informative or uninformative scores, consumers should be better off acting strategically than following the myopic demand $Q_t = \theta_t - p$.

**Proposition 8** (Naïve vs. Strategic Consumers). *In the limit as $\sigma_\xi \searrow 0$, consumer surplus is larger when consumers are strategic than when they are naive for all $\phi > 0$.*

In Section 7, we show that this result is more subtle than it seems: strategic consumers who cannot observe their score would sometimes be better off acting myopically.  

### 7 Hidden Scores

In this section, we study the case in which the score $Y_t$ is observed by firm $t$ but hidden to the consumer for all $t \geq 0$. The goal of this exercise is twofold: first, to better understand the

33When $\sigma_\xi = 0$, we have $\phi^* \to \infty$ but $G(\phi^*) \to 1$ due to $\lambda \to \infty$. Therefore, the non-concealing score maximizes learning in this case, but not profits.

34Proposition 8 can be strengthened to hold in neighborhoods of $\sigma_\xi = 0$ for compact sets of persistence levels: given $[\underline{\phi}, \bar{\phi}] \subset (0, \infty)$, there exists $\bar{\sigma}_\xi > 0$ such that, for all $0 < \sigma_\xi < \bar{\sigma}_\xi$, $CS^{strategic}(\phi; \sigma_\xi) > CS^{naive}(\phi; \sigma_\xi)$ for all $\phi \in [\underline{\phi}, \bar{\phi}]$. This is because the equilibrium variables are of class $C^1$ as functions of $(\phi, \sigma_\xi) \in (0, \infty) \times [0, \infty)$, and so they converge uniformly over compact sets as $\sigma_\xi \searrow 0$ (refer to section S.3.3 in the Supplementary Appendix for details). The same applies to Proposition 11 and 12 in the next section.
mechanism by which directly observing her score can help a strategic consumer; second, to predict the welfare implications of growing consumer sophistication (i.e., concerns regarding discriminatory practices) under alternative information structures.

It is easy to adapt our model from Section 2 to the case where the consumer does not directly observe her score: a strategy for the consumer is linear Markov if it is a linear function of \((\theta_t, p)\) only, where \(p\) is the contemporaneous price. A linear strategy for the firms is as in the baseline model, i.e.,

\[
Q(\theta, p) = \delta^h \mu + \alpha^h \theta + \zeta^h p \quad \text{and} \\
P(Y) = \pi^h_0 + \pi^h_1 Y,
\]

where the superscript \(h\) stands for hidden. The corresponding concepts of admissible strategies, equilibrium, and stationarity are all straightforward modifications of those introduced in Section 2.\(^{35}\) We again focus on stationary linear Markov equilibria.

Hidden scores have important strategic implications. First, both the firms and the consumer can now signal their private information. In particular, if \(\pi^h_1 \neq 0\), the firm’s strategy is invertible, and hence the consumer perfectly learns her score along the path of play; the consumer then has the same information as in the observable case on the equilibrium path. However, by signaling the level of the consumer’s current score, today’s price provides information about future firms’ beliefs, and hence about future prices—this informational channel deeply affects the consumer’s incentives.

Second, the price sensitivity of demand \(\zeta^h\) is now determined along the equilibrium path (unlike in the case of observable scores, where off-path prices are required). The intuition comes again from discrete time: if a score process is hidden and has full-support noise, then (i) the consumer is not able to predict the next period’s price using today’s observation and (ii) any price realization is possible. Thus, the (discrete time) price process induced by a linear strategy exhibits intra-temporal variation that identifies the slope of demand.\(^{36}\)

\(^{35}\)There are two changes only: \(Y_t\) is suppressed in the notion of a linear Markov strategy for the consumer, and admissible strategies are conditioned on \((\theta_t, P_t)_{t \geq 0}\) rather than on \((\theta_t, Y_t)_{t \geq 0}\). The latter change is innocuous in the consumer’s best-response problem to a linear pricing strategy—we work with \((\theta_t, P_t)_{t \geq 0}\) as states for consistency only.

\(^{36}\)In continuous time, the price process induced by a linear Markov pricing strategy will have continuous paths, so deviations can be detected. Because with full-support noise this issue arises only in continuous time, we refine our equilibrium in the continuous-time game by assuming that the firms conjecture that the consumer responds to a (intra-temporal) deviation with a sensitivity that coincides with the (inter-temporal) sensitivity of the quantity demanded along the path of play of a candidate Nash equilibrium. Thus, as in discrete time, the same candidate dynamic policy \(Q(\theta, p)\) is used by the firms in their pricing problem.
7.1 Equilibrium Analysis with Hidden Scores

Turning to equilibrium analysis, it is immediate that any equilibrium must entail $\zeta^h < 0$. Hence, firm $t$ sets the monopoly price $P(Y_t) = -[\delta^h \mu + \alpha^h M_t(Y_t)]/[2\zeta^h]$. We therefore seek to characterize an equilibrium in which the on-path purchase process is of the form

$$Q_t = \delta^h \mu + \alpha^h \theta_t + \zeta^h \left[ \frac{-\delta^h \mu + \alpha^h M_t}{2\zeta^h} \right] = \frac{\delta^h}{2} \mu + \alpha^h \theta_t + \beta^h M_t,$$

(25)

where $\beta^h := -\alpha^h/2$ and $M_t = \mu + \lambda^h [Y_t - \bar{Y}^h]$ for some $\lambda^h$ and $\bar{Y}^h$, $t \geq 0$. In particular, the realized prices and quantities satisfy $P_t = -E[Q_t|Y_t]/\zeta^h$ along the path of play.\(^{37}\)

As in Theorem 1, the quest for stationary linear Markov equilibria reduces to a single equation for the coefficient $\alpha^h$ on the consumer’s type. This equation is identical to (11) in the observable case, replacing $B(\phi, \alpha) \in (-\alpha/2, 0)$ with $-\alpha/2.\(^{38}\)

**Proposition 9 (Existence and Uniqueness).** There exists a unique stationary linear Markov equilibrium. In this equilibrium, $\alpha^h \in (0, 1)$ and the price sensitivity of demand is given by

$$\zeta^h = -\frac{2(r + 2\phi)}{2(r + 2\phi) + \alpha^h \lambda^h} \in \left(-1, -\frac{r + 2\phi}{r + 3\phi}\right).$$

(26)

Proposition 9 uncovers a key implication of scores’ opacity: demand is less price sensitive than in the transparent case (i.e., $\zeta^h > -1$). The intuition comes from the informational content of prices when scores are hidden: by informing the consumer that her score is high, a high price today is a signal of high prices tomorrow, and hence of low quantities purchased in the future. This, in turn, diminishes the scope for scaling back current purchases to reduce the price. Conversely, the advantage of reducing prices is greater when prices are low, and the consumer is likely to buy more units in the near future. More formally, because the consumer’s value function is convex in the current price, the marginal value of manipulating future prices downward is decreasing in $p$ as shown in Figure 8 below.

The price-signaling effect reduces the incentives for downward quantity deviations, relative to the observable case, at any given price. However, as we show in Proposition 10 below, the direct effect of lower demand sensitivity drives down the average quantity traded via higher posted prices. To unify notation, we rewrite the realized demand and prices along

\(^{37}\)Since the quantity demanded (25) has the same structure as (6), the characterization of stationary beliefs in Section 4.1 applies to the hidden case with the additional restriction that $\beta = -\alpha^h/2$.

\(^{38}\)Furthermore, several properties of the baseline model extend to this setting. For instance: $\alpha^h(\phi)$ is quasiconvex; there exists a unique non-concealing rating $\phi^{*, h}$ determine by $\phi^{*, h} = \nu(\alpha^h(\phi^{*, h}), -\alpha^h(\phi^{*, h})/2)$; and the gain function $G$ is maximized to the left of $\phi^{*, h}$ when $\sigma_\xi > 0$. Refer to Appendix B for more details.
the equilibrium path in the observable case in \((\theta, P)\) space, instead of \((\theta, M)\), as follows:

\[
Q_t = \delta^o \mu + \alpha^o \theta_t + \zeta^o P_t \quad \text{and} \quad P_t = \pi_0^o + \pi_1^o Y_t. \tag{27}
\]

Let \((Q^o, P^o)\) and \((Q^h, P^h)\) denote the average (quantity, price) pairs in the observable and hidden cases, respectively.

**Proposition 10** (Role of Transparency: strategies). *In equilibrium, for all \(\phi > 0\):

(i) **Sensitivity of price to score**: \(\pi^h_1(\phi) > \pi^o_1(\phi) > 0\).

(ii) **Sensitivity of demand to type**: \(1 > \alpha^o(\phi) > \alpha^h(\phi) > 0\)

(iii) **Average prices and quantities**: \(\mu/2 > P^h(\phi) > P^o(\phi) = Q^o(\phi) > Q^h(\phi) > \mu/4\).

Facing a demand that is *less sensitive* to price than in the observable case (i.e., \(0 > \zeta^h > -1\)), each firm charges a price that is both higher and *more sensitive* to changes in the score, relative to the observable case (i.e., \(\pi^h_1(\phi) > \pi^o_1(\phi) > 0\)).

When prices are more sensitive to the score, the ratchet effect is stronger, i.e., \(Q^o > Q^h\). To understand this result, we adjust the representation of the value of future savings (Proposition 2) to the hidden case. By the envelope theorem, we have

\[
Q_t = \theta_t - P_t - \pi^h_1 \mathbb{E}_t \left[ \int_t^\infty e^{-(r+\phi)(s-t)} Q_s ds \right], \quad t \geq 0.
\]

Since \(\pi^h_1 > \pi^o_1 = \lambda(\alpha + \beta)\), the equilibrium value of a downward deviation increases. These incentives are also stronger for higher types, explaining the ranking of signaling coefficients.
Finally, while average prices remain below the no-information case, they are higher than when scores are observed by the consumer: the lower sensitivity of demand more than compensates for the strengthening of the ratchet effect.

7.2 Welfare Comparison: Observable vs. Hidden Scores

The properties of the equilibrium average prices and quantities strongly suggest that consumers are worse off without transparency. By contrast, it is not clear that firms will be better off: the stronger ratchet effects (low average quantities and reduced signaling) can outweigh the benefits of high prices.

We are again able to confirm these intuitions in the noiseless limit case \( \sigma_x \lesssim 0 \), by taking advantage of closed-form expressions in the solutions to both models when \( \sigma_x = 0 \).

Proposition 11 (Role of Transparency: welfare). In the limit as \( \sigma_x \lesssim 0 \):

(i) consumer surplus is larger for all \( \phi > 0 \) when scores are observable;

(ii) there exists \( \bar{\phi} > 0 \) such that for all \( \phi > \bar{\phi} \) the firms’ ex ante profits are higher in the observable case than in the hidden case.

The key behind part (i) is that observing the score allows consumers to disentangle current from future prices: the observation of an abnormally high price no longer implies that future prices will be high too. Thus, while transparent scores do not add to the consumers’ information in equilibrium, they do enable consumers to eliminate the signaling effect of prices; this, in turn, results in a demand that is more price sensitive, which translates into lower prices and increased consumer surplus. Intuitively, a consumer is able to “turn down” a bad offer (literally speaking, to buy considerably less quantity) if the price is too high, in the anticipation of future discounts. In contrast, the consumer is not able to anticipate such lower future prices when scores are hidden.

Part (ii) follows from two effects that take place in the noiseless limit case. First, the signaling coefficient in the hidden case is strictly below the observable counterpart, uniformly for large \( \phi \): thus, for low persistence, more informative signals in the observable case offset the benefits of prices that are more sensitive to the score in the hidden case. Second, the lower quantity in the hidden case reduces profits relative to the observable case for all \( \phi > 0 \).

\(^{39}\)We thank an anonymous referee for this suggestion. Observe that the hidden-scores model exhibits a discontinuity at \( \sigma_x = 0 \). Specifically, by keeping track of her purchases, the consumer always knows her score when there is no noise: the hidden and observable models then coincide. However, because the sensitivity of demand is determined differently in the hidden case, the limits of the (hidden) coefficients as \( \sigma_x \lesssim 0 \) differ from their observable counterparts. Refer to Appendix C for a more detailed discussion, and to section S.3 in the Supplementary Appendix for all the details.
Thus, transparency always benefits a strategic consumer, and can even yield a Pareto improvement. We conclude our analysis by showing that, somewhat surprisingly, eliminating consumer naiveté without providing transparent scores is not necessarily beneficial to the consumer. In particular, strategic consumers with a low average willingness to pay $\mu$ can be hurt by hidden scores relative to the naive case.

**Proposition 12 (Naive vs. Strategic).** Let $\rho := r/\kappa < 4$ and

$$\mu < \min\left\{ \frac{4 - \rho}{2\rho}, \frac{\sqrt{2\rho(2\rho + 1)}}{\rho + 1} \right\}.$$  

Then, in the limit $\sigma_\xi \searrow 0$, consumer surplus with hidden scores is larger for all $\phi$ when consumers are naive than when they are strategic.

In other words, unless the discount rate is high or the ratchet effect is very costly (high $\mu$), committing to myopic behavior would benefit the consumer, because it preserves the price sensitivity of demand that is lost when scores become hidden; put differently, committing to ignore the information embedded in the price can be valuable to the consumer. These results warn against awareness policies that are not combined with transparency regulation.

Finally, when evaluating transparency policies, one must account for the endogeneity of the score’s persistence. To strengthen the idea, Figure 9 depicts consumer surplus in the hidden and observable case (when $\mu = 0$), evaluated at the firm-optimal $\phi$ in each case, as a function of $\rho$. It shows (albeit numerically) that even allowing a data broker to adjust the weight of past signals in the composition of the score (of course, restricted to the exponential class) does not overturn the benefits of transparency for consumers.

![Figure 9: Consumer Surplus at the Firm-Optimal $\phi^f(\rho)$, $\sigma_\xi = 0$](image)
8 Applications

Our results have two main policy implications for the aggregation and the transmission of consumer data. First, the results on optimal persistence in Section 6 do not support blanket regulations that eliminate data collection and transmission: even adverse uses of information (such as price discrimination) can have positive equilibrium effects for consumers. In particular, when purchase histories are tracked, strategic consumers implicitly demand compensation for the information they reveal. Sellers might then limit the scope for dynamic price discrimination, for example through less informative scores, in order to mitigate ratchet forces.\footnote{This is in sharp contrast to any information obtained by the firms from exogenous sources. As in our naive case, this information is bound to benefit firms and to harm consumers if it is later used against them.} Second (and perhaps most surprising), the results in Section 7 suggest the need for joint policy interventions: while transparency policies are always beneficial, they may be in fact necessary to avoid the negative consequences of consumer sophistication. In other words, partial regulation that promotes awareness may backfire if it only makes consumers aware of the informational content of prices.

The credibility of these policy recommendations relies on the critical assumption that strategic consumers—who are aware of the underlying mechanisms—will manipulate their scores. Here we argue that lack of information, not consumer naiveté, is the main obstacle to score manipulation in practice. In order to validate our hypothesis, we document consumer responses to dynamic price discrimination in a number of real-world settings. We begin with two examples from consumer markets; we then turn to the market for online display advertising—a business-to-business (B2B) market that fits both our applications and our results. Finally, we discuss the external validity of the evidence from online advertising, and in particular, how our transparency recommendation can be easily incorporated into regulation for business-to-consumer (B2C) markets.

8.1 Examples from Consumer Markets

Shopping cart abandonment is a well-known problem in marketing. It occurs whenever online shoppers fail to complete a purchase after selecting a product. There is widespread evidence of this practice: for example, Salescycle reports that approximately 77% of carts go unfinished (for a myriad reasons).\footnote{See \url{https://blog.salecycle.com/infographics/infographic-the-remarketing-report-q3-2018/}.} Abandonment often triggers an email by the vendor with a promotional offer. This practice suggests the profitability of searching for a specific product, then waiting for a lower price.\footnote{“Let it be: If you can bear to wait, try leaving your item in the shopping cart for a day. The retailer might send you an email offering a discount on that item. Or you might find a discount in your social feeds,”} Salesforce reports that approximately 40% of consumers
open emails from stores about their abandoned shopping carts—a response rate far higher than for regular marketing material, which is consistent with consumers’ willingness to buy at a lower price. More strikingly, online merchants are clearly aware of the potential value of “abandoned cart coupons.” They are also wary of the ratchet effect. Consider the marketing agency ActiveCampaign offers the following warning:\footnote{See \url{https://www.activecampaign.com/blog/abandoned-cart-coupon/}.}

“If you regularly use coupons and discounts to move merchandise, your business starts to rely on those coupons and discounts. People wait for sales before they buy [...] you train them to expect deals.”

**Rideshare services** Most consumers in the US are aware that prices on rideshare platforms such as Uber are personalized on the basis of individual and market characteristics, including past behavior on the app (\textit{The Guardian}, 2018). In an attempt to lower their prices, many consumers experiment with various techniques to get around the algorithm. These strategies include requesting and then rejecting quotes for unnecessary rides, so to simulate greater price sensitivity; changing the destination address mid-route, which leverages discrepancies the pricing algorithm; and other dubious techniques.\footnote{See \url{https://therideshareguy.com/uber-is-ripping-off-frequent-riders-and-heres-how-to-avoid-it/}.}

While these examples do not rely on scores to guide the prices set by different sellers, they are consistent with the role of information: both in the case of shopping carts and rideshare services, consumers are aware of the potential for dynamic price discrimination, and they do attempt to manipulate prices.

### 8.2 Online Display Advertising

The ideal setting to illustrate the strategic demand reduction mechanism at work in our model is the market for online display (e.g., banner) advertising. In this market, any publisher or website owner can be a seller of advertising space; and the demand for space comes from advertisers who wish to reach a targeted population of final consumers. The market is economically relevant: worldwide spending on online display ads totaled \$53 billion in 2018 and made up 20\% of all digital advertising revenues.\footnote{Source: Statista DMO 2019, available at \url{https://www.statista.com/outlook/216/100/digital-advertising/worldwide}.} Supplementary Appendix S.1 provides...
a comprehensive description of this setting. In this section, we summarize its properties and observed outcomes as they relate to our results.

The market for display advertising shares the following key economic properties with our model: (i) sellers price discriminate, both across buyers and over time, by choosing personalized reserve prices in real-time auctions for advertising space; (ii) reserve prices are tailored based on summary statistics about each buyer’s past bids;\(^\text{46}\) (iii) information about past auction outcomes is aggregated and distributed by Supply-Side Platforms (SSPs)—technology platforms help sellers manage their space; (iv) buyers participate in a large number of auctions, many of which close at the reserve price, so that their problem is well-approximated by the continuous choice of flow advertising volume; and (v) reserve prices are not public, but large advertisers estimate them in real time by running a series of A/B tests that enable them to estimate the optimal level of bid shading.

A growing body of theoretical and empirical work recognizes the importance of the ratchet effect in this market. In Supplementary Appendix S.1, we discuss an extensive literature in operations research and computer science describing the advertisers’ optimal demand-reduction strategies and the robustness of the sellers’ strategies to the ratchet effect. This is not only a matter of academic interest: the ratchet effect influences practical algorithm and market design by technology platforms in online advertising markets. Lahaie, Munoz Medina, Sivan, and Vassilvitskii (2018) develop tests based on bid perturbations to identify the relationship between past bids and future reserve prices. Similarly, the demand-side platform Criteo advertises its ability to reduce bids when (static) second-price auction mechanics are manipulated by the seller. These demand-reduction techniques are explicitly described in the paper by Abeille, Calauzènes, Karoui, Nedelec, and Perchet (2018).

Publishers and SSPs are heterogeneous in their approach to mitigating the ratchet effect. Consistent with our model, some sellers view the value of information as sufficiently high to encourage dynamic personalized pricing, while others prefer to limit the value of strategic behavior. For example, the Rubicon Project (a supply-side platform that places almost $1 billion in advertising spending per year) has recently stopped allowing reserve prices to adjust based on past behavior specifically citing concerns over buyer manipulation,

> Using buy-side dimensions when setting price floors results in buyer distrust [...] in the long term, this approach will result in changes in bidding behavior and reduced liquidity.\(^\text{47}\)

\(^{46}\)Hummel (2018) notes, “If a publisher signs up for an online ad network such as the Google Display Network, that publisher will be given reports about revenue and the number of ads shown that the publisher could then use in refining reserve prices. However, the publisher would not be told the individual bids that the advertisers had made in each auction.”

\(^{47}\)See the Rubicon Project’s white paper “Maintaining the Equilibrium: How Dynamic Price Floors
8.3 Policy Discussion

There are, of course, several differences between large advertisers—who optimize their bids using machine-learning techniques to estimate the dynamics of reserve prices—and individual consumers in less structured markets. In that sense, the market for online advertising is the ideal setting where to identify to the implications of the ratchet effect. However, several features of consumer markets facilitate the translation of our results into policy. First, the gap in the strategic ability of firms and consumers is not as large as one may think—there is a wide distribution of firm sophistication levels. Second, the consumers’ experience with credit scores makes the transparency discussion particularly salient even in markets with a small number of strategic actors.

FICO consumer credit scores are used to determine the borrowing conditions (interest rates, maximum amounts) for US consumers. It is well-known that consumers can manipulate credit scores in the short run to improve their borrowing opportunities. Indeed, countless websites explain how to improve your score: do not apply for credit cards, avoid bad credit, consolidate your debt. These sites offer score simulations for consumers to check the consequences of a hypothetical default on a bill payment. Most relevant to our results, the FICO Score Open Access program is designed such that the score the customer sees exactly matches a score version the lender uses within their risk management decisions. The credit bureaus themselves advertise the transparency of their products as enabling consumers to “stop overpaying for credit.”

Now consider a hidden score, i.e., the Equifax Discretionary Spending Index (DSI), ranging from 1 to 1000, whose objective is to “Better understand how much customers can spend [in order to] create targeted promotions that appeal to them.” Because such a score is used for market segmentation, allowing consumers to access their score in real time would enable them to estimate their outside option. This idea resonates with Ezrachi and Stucke (2016):

Moreover, as more online retailers engage in dynamic, differential pricing, it will be harder for consumers to discover a general market price and to assess their outside options. [...] Even for more alert customers, firms can seek to reduce their incentive and ability to search for outside options.

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48 See https://www.fico.com/en/products/fico-score-open-access/. In the world of credit scoring, transparency is not universally considered a positive feature. According to the German credit-score system SCHUFA, “Publicly known scoring methods involve the danger of manipulation aiming to improve score values irrespective of whether the actual creditworthiness is improved” (Rona-Tas and Hiß, 2008). Similar attempts characterize the Chinese experience, where “Sesame Credit is basically a big data gamified version of the Communist Party’s surveillance methods,” and “We’re also bound to see the birth of reputation black markets selling under-the-counter ways to boost trustworthiness” (Wired Magazine, 2017).

49 See https://www.equifax.com/business/discretionary-spending-index/.
This is the mechanism at work in our model with hidden scores: the forces that make a strategic consumer more price sensitive cannot operate unless the consumer can check her score and assess the likelihood of purchasing in the future.\footnote{Data brokers beyond regulated credit agencies make few attempts at improving the transparency of their information. Two exceptions are the Bluekai Registry \url{http://www.bluekai.com/registry/} and Acxiom’s \textit{About the Data} initiative, which reveal to consumers which interest groups they belong to. Even so, consumers have no means of accessing the value of their scores being sold to merchants.}

Transparency policies have proved effective in other settings (e.g., the analysis of overdraft fees in Grubb, 2015). Because scores ultimately used for price discrimination can be traded, an actionable policy recommendation is that consumers be allowed to freely access them. This recommendation is aligned with the Right to Access—enabling consumers to learn in real time what firms know about them—introduced by the General Data Protection Regulation (GDPR) in the EU.

\section{Conclusions}

We have explored the allocation, informational, and welfare consequences of scoring consumers based on their purchase histories and using the information so-gained to price discriminate. Our analysis placed special emphasis on score persistence and transparency, two themes of critical importance for recent regulatory efforts aimed at protecting consumers.

Our model clearly makes a number of simplifying assumptions, the strongest of which is perhaps the restriction to a continuous score with exponential weights. One advantage of the simple class of scores we consider is its flexibility to accommodate different stage games, and thus shed light on other uses of consumer-level information. For example, one could examine a game where scores are used to tailor products of varying quality to the consumer’s tastes. Alternatively, our model can be amended to capture other forms of score-based discrimination, such as racial profiling. For example, Brayne (2017) describes the role of risk and merit scores in driving law enforcement’s “stratified surveillance” practices—yet another form of segmentation.

The most promising future direction involves formalizing a market for summary statistics towards an understanding of the endogenous dissemination of consumer information. In such a market, the mechanisms by which information is sold, and whom it is sold to, will influence the consumer’s incentives to reveal it. We pursue this avenue of research in ongoing work.
Appendix A: Proofs

Proofs for Section 3

Proof of Proposition 1. Parts 1 and 2 follow from the expressions for consumer and producer surplus. Part 3 follows directly from the weight on $\theta_t$ not responding to the firms’ information structure. Finally, 4 is a special case of Proposition 4.

Proofs for Section 4

The statements in Section 4.1 follow from the next

Lemma A.1 (Stationarity and Beliefs). A process $(\theta_t, Y_t)_{t \geq 0}$ with $(Q_t)_{t \geq 0}$ as in (6) and $M_t = \mathbb{E}[\theta_t | Y_t]$ for all $t \geq 0$ is stationary Gaussian if and only if:

(i) $M_t = \mu + \lambda[Y_t - \bar{Y}]$, with $\bar{Y} = \mu (\alpha + \beta + \delta) / \phi$ and $\lambda = \frac{\alpha \sigma_y^2 (\phi - \beta \lambda)}{\alpha^2 \sigma_y^2 + \sigma_x^2 \kappa (\phi - \beta \lambda + \kappa)}$;

(ii) the score process (4) is mean reverting: $\phi - \beta \lambda > 0$;

(iii) $(\theta_0, Y_0) \sim \mathcal{N}([\mu, \bar{Y}]^T, \Gamma)$ is independent of $(Z_{t0}^\theta, Z_{t0}^\xi)_{t \geq 0}$, where the long-run covariance matrix $\Gamma$ is given in (A.2).

Proof of Lemma A.1. Suppose that $(\theta_t, Y_t)_{t \geq 0}$ is stationary Gaussian. By stationarity, $\mathbb{E}[Y_t]$, and Cov[$\theta_t, Y_t$]/Var[$Y_t$] are independent of time; let $\bar{Y}$ and $\lambda$ denote their respective values (to be determined). Moreover, by normality,

$$M_t := \mathbb{E}[\theta_t | Y_t] = \mu + \lambda [Y_t - \bar{Y}], \ t \geq 0.$$ 

Let $\hat{\delta} := \delta \mu + \beta (\mu - \lambda \bar{Y})$ and $\hat{\beta} = \beta \lambda$. We can then write the quantity demanded (6) as

$$Q_t = \delta \mu + \alpha \theta_t + \beta M_t = \hat{\delta} + \alpha \theta_t + \hat{\beta} Y_t, \ t \geq 0.$$ 

Using that $d\xi_t = Q_t dt + \sigma_x dZ_t^\xi$, we can conclude that $(\theta_t, Y_t)_{t \geq 0}$ evolves according to

$$d\theta_t = -\kappa (\theta - \mu) dt + \sigma_\theta dZ_t^\theta,$$
$$dY_t = [-\phi - \hat{\beta}] Y_t + \hat{\delta} + \alpha \theta_t] dt + \sigma_x dZ_t^\xi, \ t > 0.$$ 

The previous system is linear, and thus admits an analytic solution. Specifically, letting

$$X := \begin{bmatrix} \theta \\ Y \end{bmatrix}, \ A_0 := \begin{bmatrix} \kappa \mu \\ \hat{\delta} \end{bmatrix}, \ A_1 := \begin{bmatrix} \kappa & 0 \\ -\alpha & \phi - \hat{\beta} \end{bmatrix}, \ B := \begin{bmatrix} \sigma_\theta & 0 \\ 0 & \sigma_x \end{bmatrix} \text{ and } Z := \begin{bmatrix} Z_t^\theta \\ Z_t^\xi \end{bmatrix},$$
we can write \( dX_t = [A_0 - A_1 X_t]dt + BdZ_t \), \( t > 0 \), which has as unique (strong) solution

\[
X_t = e^{-A_1 t} X_0 + \int_0^t e^{-A_1 (t-s)} A_0 dt + \int_0^t e^{-A_1 (t-s)} BdZ_s, \quad t \geq 0,
\]

(A.1)

where \( e^{A_1 t} \) denotes the matrix exponential (Section 1.7 in Platen and Bruti-Liberati (2010)).

From the additive structure of (A.1), \( X_t \) is Gaussian for all \( t \geq 0 \) if and only if \( X_0 \) is Gaussian. But this implies that \( X_0 \) must be independent of \( Z := (Z_t)_{t \geq 0} \) for \( Z \) to be a Brownian motion under the (null-sets augmented) filtration generated by \( Z \) and \( X_0 \).

Letting \( \mathcal{N}(\mu, \Gamma) \) denote the stationary distribution of \( X_t, \quad t \geq 0 \), it follows that \( \mu \in \mathbb{R}^2 \) and the \( 2 \times 2 \) covariance matrix \( \Gamma \) must satisfy the equations

\[
\mathbb{E}[X_t] = \mu \iff e^{-A_1 t} \mu + [A_1^{-1} - e^{-A_1 t} A_1^{-1}] A_0 = \mu \quad \text{and}
\]

\[
\text{Var}[X_t] = \Gamma \iff e^{-A_1 t} \Gamma e^{-A_1 t} + e^{-A_1 t} \text{Var} \left[ \int_0^t e^{A_1 s} BdZ_s \right] e^{-A_1 t} = \Gamma,
\]

where \( \text{Var}[\cdot] \) and \( T \) denote the covariance matrix and transpose operators, respectively.

Observe that the first condition leads to \( \mu = A_1^{-1} A_0 \) provided \( A_1 \) is invertible. This, in turn, happens when \( \phi - \beta \lambda \neq 0 \)—we assume this in what follows. Regarding the second condition, differentiating it and using that \( \text{Var} \left[ \int_0^t e^{A_1 s} BdZ_s \right] = \int_0^t e^{A_1 s} B^2 e^{A_1 s} ds \) yields

\[
-A_1 \Gamma - \Gamma A_1^T + B^2 = 0.
\]

Using that \( \hat{\mu} = (\mathbb{E}[\theta_t], \mathbb{E}[Y_t])^T = (\mu, \hat{\gamma})^T \), and that \( \Gamma_{11} = \text{Var}[\theta_t], \Gamma_{12} = \Gamma_{21} = \text{Cov}[\theta_t, Y_t] \) and \( \Gamma_{22} = \text{Var}[Y_t] \), it is then easy to verify that the previous system has as solution

\[
\hat{\mu} = \begin{bmatrix} \mu \\ \hat{\gamma} \end{bmatrix} \quad \text{and} \quad \Gamma = \begin{bmatrix} \sigma_\theta^2 & \frac{\alpha \sigma_\theta^2}{2\kappa} \\ \frac{\alpha \sigma_\theta^2}{2\kappa} & \alpha^2 \sigma_\gamma^2 + \frac{\alpha \sigma_\gamma^2}{2\kappa} + \frac{(\phi - \beta \lambda + \kappa)^2}{2\kappa(\phi - \beta \lambda + \kappa)^2} \end{bmatrix},
\]

(A.2)

To guarantee that the previous expressions indeed correspond to the first two moments of stationary Gaussian process, however, we must verify that \( \Gamma \) is both positive semi-definite and finite. Since \( \sigma_\theta^2 / 2\kappa > 0 \), positive semi-definiteness reduces to

\[
\det(\Gamma) \geq 0 \iff \frac{\sigma_\theta^2 (\phi - \hat{\beta})^2 + \alpha^2\sigma_\gamma^2 \kappa}{(\phi - \hat{\beta} + \kappa)^2(\phi - \hat{\beta})} > 0 \iff \phi - \frac{\sigma_\theta^2 (\phi - \hat{\beta})}{\alpha^2\sigma_\gamma^2 \kappa} \geq 0.
\]

Because \( \phi - \beta \lambda \neq 0 \), however, it follows that \( \phi - \beta \lambda \) must be strictly positive. As a byproduct,

\[^{51}\text{Denote such filtration by} (\mathcal{G}_t)_{t \geq 0}. \text{In the absence of independence, there must} t \geq 0 \text{such that} Z_t \text{is not independent of} \mathcal{G}_0; \text{but this violates the independent-increments requirement of a Brownian motion.}\]
To finish the proof, we find $\lambda$ and $\bar{Y}$ that are consistent with Bayes’ rule given a score process that is driven by (6). Using (A.2),

$$
\lambda = \frac{\text{Cov}[\theta_t, Y_t]}{\text{Var}[Y_t]} = \frac{\alpha \sigma_\gamma^2 (\phi - \beta \lambda)}{\alpha^2 \sigma_\gamma^2 + \kappa \sigma_\zeta^2 (\phi + \kappa - \beta \lambda)}
$$

and

$$
\bar{Y} = \frac{\mu [\alpha + \beta + \delta]}{\phi}
$$

where the last equality follows from $\bar{Y} = [\tilde{\delta} + \alpha \mu] / (\phi - \beta \lambda)$ and $\tilde{\delta} = \delta \mu + \beta (\mu - \lambda \bar{Y})$; this proves (i). The converse part of the Proposition is true by the previous constructive argument. This concludes the proof.

Proof of Lemma 1. It follows from partially differentiating (7) with respect to $\phi$. □

Proof of Lemma 2. Consider a linear Markov strategy $Q(p, \theta, Y)$ for the consumer with weight equal to $-1$ on the contemporaneous price. Because the time-$t$ monopolist assumes that past purchases followed (6), we have that $M_t = \mu + \lambda [Y_t - \bar{Y}]$, $t \geq 0$, where $\bar{Y}$ and $\lambda$ are given in (i) in Lemma A.1. Thus, we can write $Q(p, \theta_t, M_t) = q_0 + \alpha \theta_t + q_2 M_t - p$ for some coefficients $q_0, \alpha$ and $q_2$. Importantly, the weight that the strategy attaches to the contemporaneous price does not change under this linear transformation.

The monopolist operating at time $t$ therefore solves

$$
\max_p p \mathbb{E}[q_0 + \alpha \theta_t + q_2 M_t - p | Y_t] \Leftrightarrow P(M_t) = \frac{q_0}{2} + \frac{\alpha + q_2}{2} M_t,
$$

which leads to a realized purchase

$$
Q_t = q_0 + \alpha \theta_t + q_2 M_t - P(M_t) = \frac{q_0}{2} + \alpha \theta_t + \frac{q_2 - \alpha}{2} M_t, t \geq 0.
$$

We conclude that when demand is linear with unit sensitivity, if realized purchases are given by $Q_t = \delta \mu + \alpha \theta_t + \beta M_t$, contemporaneous prices satisfy $P_t = \delta \mu + (\alpha + \beta) M_t, t \geq 0$. Importantly, once the coefficients $(\alpha, \beta, \delta)$ are determined, simple algebra shows that prices are supported by the linear Markov strategy

$$
Q(p, \theta_t, Y_t) = 2 \delta \mu + [\mu - \lambda \bar{Y}] [\alpha + 2 \beta] + \alpha \theta_t + \lambda [\alpha + 2 \beta] Y_t - p,
$$

This concludes the proof. □
Proof of Theorem 1. Under the set of admissible strategies defined in Section 2, Verification Theorem 3.5.3 in Pham (2009) applies. Specifically, we look for a quadratic solution
\[ V = v_0 + v_1\theta + v_2M + v_3M^2 + v_4\theta^2 + v_5\theta M \]
to the HJB equation
\[
\begin{align*}
rv(\theta, M) &= \sup_{q \in \mathbb{R}} \left\{ (\theta - [(\alpha + \beta)M + \delta\mu])q - q^2/2 - \kappa(\theta - \mu)V_{\theta} \right. \\
&\quad \left. + \left[ \lambda q - \phi(M - \mu + \lambda\bar{Y}) \right] \frac{\partial V}{\partial M}(\theta, M) + \frac{\lambda^2 \sigma_z^2}{2} \frac{\partial^2 V}{\partial M^2} + \frac{\sigma_\theta^2}{2} \frac{\partial^2 V}{\partial \theta^2} \right\}
\end{align*}
\]
subject to standard transversality conditions. To find a stationary linear Markov equilibrium, however, (i) we impose the fixed-point condition that the optimal policy is of the form
\[ \delta\mu + \alpha\theta + \beta M, \]
and (ii) with the use of the equation for \( \lambda \) (equation (7)), find coefficients that satisfy the stationarity condition \( \phi - \beta\lambda > 0 \) (part (ii) in Lemma A.1).

To this end, observe that the first-order condition of the HJB equation reads
\[
q = \theta - [\delta\mu + (\alpha + \beta)M] + \lambda[v_2 + 2v_3M + v_5\theta]
\]
\[
= -\delta\mu + \lambda v_2 + [1 + \lambda v_5]\theta + [2\lambda v_3 - (\alpha + \beta)]M
\]
which leads to the following system matching-coefficient conditions:
\[
\delta\mu = -\delta\mu + \lambda v_2, \quad \alpha = 1 + \lambda v_5, \quad \text{and} \quad \beta = 2\lambda v_3 - (\alpha + \beta).
\]
(A.3)

By the Envelope Theorem, moreover,
\[
(r + \phi)[v_2 + 2v_3M + v_5\theta] = -(\alpha + \beta)[\delta\mu + \alpha\theta + \beta M] - \kappa(\theta - \mu)v_5
\]
\[
+ [\lambda(\delta\mu + \alpha\theta + \beta M) - \phi(M - \mu + \lambda\bar{Y})]2v_3,
\]
(A.4)

which yields the following system of equations
\[
\begin{align*}
(r + \phi)v_2 &= -(\alpha + \beta)\delta\mu + \kappa\mu v_5 + [\lambda\delta\mu + \phi(\mu - \lambda\bar{Y})]2v_3 \\
(r + 2\phi)2v_3 &= -(\alpha + \beta)\beta + 2v_3\lambda\beta \\
(r + \kappa + \phi)v_5 &= -(\alpha + \beta)\alpha + 2v_3\lambda\alpha.
\end{align*}
\]
(A.5)
Using that $v_2, v_3$ and $v_5$ can be written as a function of $\alpha, \beta$ and $\delta \mu$, (A.5) becomes

\[
\begin{align*}
(r + \phi \frac{2\delta \mu}{\lambda}) &= -(\alpha + \beta)\delta \mu + \kappa \mu \frac{\alpha - 1}{\lambda} + [\lambda \delta \mu + \phi (\mu - \lambda \bar{Y})] \frac{\alpha + 2\beta}{\lambda} \\
(r + 2\phi \frac{\alpha + 2\beta}{\lambda}) &= -(\alpha + \beta)\beta + \beta (\alpha + 2\beta) \\
(r + \kappa + \phi \frac{\alpha - 1}{\lambda}) &= -(\alpha + \beta)\alpha + \alpha (\alpha + 2\beta),
\end{align*}
\]

where we have assumed that $\lambda \neq 0$. In fact, since $\phi - \beta \lambda > 0$ in any stationary linear Markov equilibrium, the equation for $\lambda$ (i.e., (7)) implies that $\lambda \neq 0$ as long as $\alpha \neq 0$; but the latter is a corollary of the following lemma.

**Lemma A.2.** Any stationary linear Markov equilibrium must satisfy $\alpha \in (0, 1)$.

**Proof.** Consider a stationary linear Markov equilibrium with coefficients $(\alpha, \beta, \delta)$. Straightforward integration shows that the consumer’s equilibrium payoff is quadratic, and thus the system of equations (A.6) holds.

Suppose that $\alpha = 0$. From (7), $\lambda = 0$, and so $M_t = \mu$ for all $t \geq 0$; but this implies that prices do not respond to changes in the score, and hence, it is optimal for the consumer to behave myopically by choosing $Q_t = \theta - p$, a contradiction. If instead $\alpha < 0$, the last equation in (A.6) yields

\[
\phi - \beta \lambda = (r + \kappa) \left( \frac{1}{\alpha} - 1 \right) + \frac{\phi}{\alpha} < 0,
\]

which is a contradiction with the equilibrium being stationary ((ii) in Lemma A.1).

The case $\alpha = 1$ can be easily ruled out too: since $\lambda > 0$ in this case, the last equation in the system (A.6) yields that $\beta = 0$, but the second equation then implies that $\alpha = 0$, a contradiction. As a corollary, $\beta \neq 0$ in a stationary linear Markov equilibrium.

Suppose now that $\alpha > 1$. The last two equations of (A.6) can be used to solve for $\beta$ and thus to find an expression for $\lambda$ as a function of $\phi, \alpha$, and the parameters $r$ and $\kappa$. In addition, from the last equation in (A.6),

\[
L := \phi - \beta \lambda = \frac{\phi - \alpha (\kappa + r) + \kappa + r}{\alpha},
\]

and hence, we can solve for $\phi = \phi(\alpha, L)$. We conclude that in the equation for $\lambda$, (7), $\phi$ can be replaced by expressions that depend on $L$ and $\alpha$. Specifically, the resulting equation is

\[
\frac{\alpha L \sigma_\phi^2}{\kappa (\kappa + L) \sigma_\xi^2 + \alpha^2 \sigma_\phi^2} + \frac{(\alpha - 1)(\kappa + L + r)(3\alpha (\kappa + L + r) - 3\kappa + L - r)}{\alpha (2\alpha (\kappa + L + r) - 2\kappa - r)} = 0.
\]
By stationarity, $L > 0$. Since $\alpha > 1$, however, this implies that the left-hand side of this expression is strictly positive, which is a contradiction. Thus, $\alpha \in (0, 1)$. \hfill \Box

We continue with the proof of the proposition. From the proof of the previous lemma, $\beta \neq 0$. In the system \((A.6)\), we can multiply the second equation by $\alpha \neq 0$ and the third by $\beta \neq 0$ to obtain $(r + 2\phi)\alpha(\alpha + 2\beta) = (r + \kappa + \phi)\beta(\alpha - 1)$. From here, $\beta = B(\phi, \alpha)$ where

$$B(\phi, x) := \frac{-x^2(r + 2\phi)}{2(r + 2\phi)x - (r + \kappa + \phi)(x - 1)} \in \left(-\frac{x}{2}, 0\right) \text{ when } x \in (0, 1). \quad (A.7)$$

Moreover, since $\alpha \in (0, 1)$ and $\phi - \beta \lambda > 0$, it follows from (7) that $\lambda > 0$. However, when $\alpha > 0$ and $\beta < 0$, the unique strictly positive root of (7) is given by

$$\Lambda(\phi, \alpha, \beta) := \frac{\sigma_0^2\alpha(\alpha + \beta) + \kappa\sigma_z^2(\kappa + \phi) - \sqrt{[\sigma_0^2\alpha(\alpha + \beta) + \kappa\sigma_z^2(\kappa + \phi)]^2 - 4\kappa(\sigma_0\sigma_z)^2\alpha\beta\phi}}{2\beta\kappa\sigma_z^2}. \quad (A.8)$$

In particular, since $\alpha^2 + \alpha B(\phi, \alpha) = \alpha[\alpha + B(\phi, \alpha)] \geq \alpha^2/2 > 0$ when $\alpha \in [0, 1]$, $\sigma_0^2\alpha(\alpha + B(\phi, \alpha)) + \kappa\sigma_z^2(\kappa + \phi) > 0$ over the same range.

We conclude that $\lambda = \Lambda(\phi, \alpha, B(\phi, \alpha))$ in equilibrium, and so, using the last equation of \((A.6)\), we arrive to equation (11): namely, $\alpha \in (0, 1)$ must satisfy $A(\phi, \alpha) = 0$, where

$$A(\phi, x) := (r + \kappa + \phi)(x - 1) - \Lambda(\phi, x, B(\phi, x))xB(\phi, x), \ x \in [0, 1]. \quad (A.9)$$

We now establish the existence and uniqueness of a solution to this equation, along with regularity properties with respect to $\phi > 0$ and $\sigma_z^2 > 0$ that will be used later on.

**Lemma A.3.** For every $\phi > 0$ and $\sigma_z^2 > 0$, there exists a unique $\alpha \in (0, 1)$ satisfying the previous equation. Moreover, the induced function $\alpha : (0, \infty)^2 \to (0, 1)$ is of class $C^1$.

**Proof:** Fix $\phi > 0$ and $\sigma_z^2 > 0$; the dependence of $A$ on $\sigma_z^2 > 0$ is via $\Lambda$, and we omit it until needed. Observe that as $x \to 1$, $B(\phi, x) \to -1/2$ and $\lim_{x \to 1} \Lambda(\phi, x, B(\phi, x)) > 0$. Hence, $\lim_{x \to 1} A(\phi, x) > 0$. Similarly, as $x \to 0$, $B(\phi, x) \to 0$ and $B(\phi, x)\Lambda(\phi, x, B(\phi, x)) \to 0$. Hence, $\lim_{x \to 0} A(\phi, x) < 0$. The existence of $\alpha \in (0, 1)$ satisfying $A(\phi, \alpha) = 0$ follows from the continuity of $x \in [0, 1] \mapsto A(\phi, x)$ and the Intermediate Value Theorem.

To show uniqueness, we prove that $x \mapsto -\Lambda(\phi, x, B(\phi, x))xB(\phi, x)$ is strictly increasing in $[0, 1]$. To this end, notice first that since

$$H(\phi, x) := -\Lambda(\phi, x, B(\phi, x))B(\phi, x) > 0, \ x \in (0, 1), \quad (A.10)$$
However, from the expression for cancellation of $B$, straightforward algebra shows that
\[
\ell_x(\phi, \hat{x}) [\ell(\phi, \hat{x}) - \ell^2(\phi, \hat{x}) - 4\kappa\sigma_\phi^2 \sigma_\xi^2 B(\phi, \hat{x})\hat{x}\phi]^{1/2} = 2\kappa(\sigma_\phi^2 \sigma_\xi^2)[B_x(\phi, \hat{x}) \hat{x} + B(\phi, \hat{x})]\phi.
\]

Moreover, straightforward algebra shows that
\[
B_x(\phi, x) x = B(\phi, x) - \frac{x^2(r + 2\phi)(r + \kappa + \phi)}{[2x(r + 2\phi) - (r + \kappa + \phi)(x - 1)]^2} < 0 \quad \text{for} \quad x \in [0, 1],
\]
so $B_x(\phi, x) x + B(\phi, x) < 0$ for all $x \in [0, 1]$. It then follows that $\ell_x(\phi, \hat{x}) = \sigma_\phi^2[2x + B_x(\phi, \hat{x})\hat{x} + B(\phi, \hat{x})] > 0$, otherwise the left-hand side of (A.11) is positive, while the right-hand side is negative.

Isolating the square root and squaring both sides in the first-order condition leads to the cancellation of $\ell^2 \ell_x^2$ in (A.11). Dividing the resulting expression by $4\kappa(\sigma_\phi^2 \sigma_\xi^2)^2\phi$ then yields
\[
0 \quad = \quad \ell_x(\phi, \hat{x}) \left\{ \ell(\phi, \hat{x})[-B_x(\phi, \hat{x})\hat{x} - B(\phi, \hat{x})] + \ell_x(\phi, \hat{x})B(\phi, \hat{x}) \hat{x} \right\}_{\kappa := \frac{\ell_x(\phi, \hat{x})}{\ell(\phi, \hat{x})} - \frac{\ell_x^2(\phi, \hat{x})}{\ell(\phi, \hat{x})} - 1} \geq 0
\]

But since $\ell_x(\phi, \hat{x}) > 0$, we must have that $K < 0$. In particular, using that $\ell(\phi, x) = \sigma_\phi^2[x + B(\phi, x)] + \kappa\sigma_\xi^2(\phi + \kappa)$ and $\kappa\sigma_\xi^2(\phi + \kappa)[-B_x(\phi, \hat{x})\hat{x} - B(\phi, \hat{x})] > 0$, it must be that
\[
\sigma_\phi^2\{[\hat{x}^2 + \hat{x}B(\phi, \hat{x})][-B_x(\phi, \hat{x})\hat{x} - B(\phi, \hat{x})] + [2\hat{x} + \hat{x}B_x(\phi, \hat{x}) + B(\phi, \hat{x})\hat{x}B(\phi, \hat{x})]\} < 0
\]
\[
\iff \hat{x}^2[-\hat{x}B_x(\phi, \hat{x}) + B(\phi, \hat{x})] < 0.
\]

However, from the expression for $B_x(\phi, x) x$, we have that $-xB_x(\phi, x) + B(\phi, x) = x^2(r + 2\phi)(r + \kappa + \phi)/[2x(r + 2\phi) - (r + \kappa + \phi)(x - 1)]^2 \geq 0$, reaching a contradiction. The continuity of $H_x$ implies that $x \mapsto H(\phi, x)$ is strictly increasing.

To conclude, since $(0, 1) \times (0, \infty)^2 \mapsto A(x, \phi, \sigma_\xi^2)$ is of class $C^1$ and $\partial A/\partial x > 0$, our existence result allows us to apply the Implicit Function Theorem: namely, around any point $(\phi, \sigma_\xi^2) \in (0, +\infty)^2$ there exists a unique function, $(\phi, \sigma_\xi^2) \in (0, \infty)^2 \mapsto \alpha(\phi, \sigma_\xi^2) \in (0, 1)$ sat-
isfying the equation, and such function is of class \( C^1 \). However, since we already established that existence and uniqueness holds over the whole domain \((0, \infty)\), the local property of continuous differentiability trivially extends globally. This concludes the proof of the lemma. □

It remains to characterize \( \delta \). Recall that the first equation in (A.6) reads

\[
(r + \phi)\frac{2\delta \mu}{\lambda} = -(\alpha + \beta)\delta \mu + \kappa \mu \frac{\alpha - 1}{\lambda} + [\lambda \delta \mu + \phi(\mu - \lambda \bar{Y})] \frac{\alpha + 2\beta}{\lambda},
\]

where \( \bar{Y} = \mu[\delta + \alpha + \beta]/\phi \). Plugging this expression in the previous equation yields

\[
\left[ \frac{2(r + \phi)}{\lambda} + \alpha + \beta \right] \delta \mu = \mu \left[ \frac{\kappa(\alpha - 1)}{\lambda} + \frac{\alpha + 2\beta}{\lambda} \left[ \phi - (\alpha + \beta)\lambda \right] \right].
\]

Observe that since \( \alpha + \beta > 0 \), the bracket on the left-hand side is strictly positive. If \( \mu = 0 \) this equation is trivially satisfied, i.e., the price and quantity demanded along the path of play have no deterministic intercept (and \( v_2 = 0 \), leaving the rest of the system unaffected). If \( \mu \neq 0 \), we have that \( \delta = D(\phi, \alpha) \) where

\[
D(\phi, x) := \frac{\kappa(\alpha - 1) + [\alpha + 2B(\phi, \alpha)]\left[ \phi - (\alpha + B(\phi, \alpha))\Lambda(\phi, \alpha, B(\phi, \alpha)) \right]}{2(r + \phi) + (\alpha + B(\phi, \alpha))\Lambda(\phi, \alpha, B(\phi, \alpha))}, \tag{A.12}
\]

for \( (\phi, x) \in (0, \infty) \times (0, 1) \), which is well-defined for all values \( \phi > 0 \).

To conclude the proof of the theorem, there are two final steps:

1. Determination of the remaining coefficients. From the three matching coefficient conditions (A.3), \( v_2, v_3 \) and \( v_5 \) are determined using \( \delta, \alpha \) and \( \beta \) as follows:

\[
v_2 = \frac{2\delta \mu}{\lambda}, \quad v_3 = \frac{\alpha + 2\beta}{2\lambda} > 0, \quad \text{and} \quad v_5 = \frac{\alpha - 1}{\lambda} < 0.
\]

As for \( v_1 \) and \( v_4 \) (corresponding to \( \theta \) and \( \theta^2 \) in the value function) these can be obtained by differentiating the HJB equation with respect to \( \theta \). Specifically,

\[
(r + \kappa)[v_1 + 2v_4 \theta + v_5 M] = (\delta \mu + \alpha \theta + \beta M)[1 + v_5 \lambda] - v_5 \phi \left[ M - \mu + \lambda \bar{Y} \right] - 2v_4 \kappa (\theta - \mu)
\]

leads to the additional equations

\[
2(r + \kappa)v_4 = \alpha \cdot \left[ 1 + v_5 \lambda \right] - 2v_4 \kappa \Rightarrow v_4 = \frac{\alpha^2}{2(r + 2\kappa)}, \quad \text{and},
\]

\[
(r + \kappa)v_1 = \delta \mu \alpha + v_5 \phi (\mu - \lambda \bar{Y}) \Rightarrow v_1 = \frac{\delta \mu \alpha}{r + \kappa} + \frac{\phi (\mu - \lambda \bar{Y}) \alpha \beta}{(r + \kappa + \phi)(r + \kappa)}.
\]
The coefficient \( v_0 \) can be found by equating the constant terms in the HJB equation—since the value function is quadratic, there is no constraint on this coefficient.

2. Transversality conditions and admissibility of the candidate equilibrium strategy (6).

This is verified in the section S.2 in the Supplementary Appendix.

This concludes the proof. \( \square \)

**Proof of Proposition 2.** Consider the partial differential equation (PDE)

\[-(\alpha + \beta)[\delta \mu + \alpha \theta + \beta M] + \mathcal{L}F(\theta, M) - (r + \phi)F(\theta, M) = 0\]

\[\lim_{r \to \infty} e^{-rt}\mathbb{E}_0[F(\theta^0_t, M^m_t)] = 0,\]

where \( \mathcal{L}F := -\kappa(\theta - \mu)F_{\theta} + [-\phi(M - \mu + \lambda Y) + \lambda(\delta \mu + \alpha \theta + \beta M)]F_M + \frac{\sigma^2}{2}F_{\theta \theta} + \frac{(\lambda \sigma Y)^2}{2}F_{MM} \)

and \((\theta^0_t, M^m_t)_{t \geq 0}\) is the type-belief process starting from \((\theta_0, M_0) = (\theta, M) \in \mathbb{R}^2\).

From the proof of Proposition 1, the previous equation admits as solution the function

\[V_M(\theta, M) = v_2 + 2v_3M + v_5\theta\]

where \(v_2, v_3\) and \(v_5\) are the coefficients of the consumer’s value function on \(M, M^2\), and \(M\theta\), respectively. In fact, display (A.4) shows that the previous function satisfies the PDE, while the transversality condition follows directly from \((\theta^0_t)_{t \geq 0}\) and \((M^m_t)_{t \geq 0}\) being mean reverting and \(V_M\) being linear.

Importantly, \(V_M(\cdot, \cdot)\) (i) is of class \(C^2\) and (ii) exhibits quadratic growth. Thus, the Feynman-Kac Representation Theorem (Remark 3.5.6 in Pham 2009) applies: namely,

\[V_M(\theta, M) = -\mathbb{E}_0\left[\int_0^\infty e^{-(r+\phi)t}(\alpha + \beta)Q_t \, dt\right], \forall t \geq 0,\]

where we used that \(Q_t = \delta \mu + \alpha \theta_t + \beta M_t\) in equilibrium. The result then follows from \(V_M(\theta_t, M_t) = -\mathbb{E}_0\left[\int_0^\infty e^{-(r+\phi)\tilde{t}}(\alpha + \beta)Q_t \, d\tilde{t}\right] = -\mathbb{E}_t\left[\int_t^\infty e^{-(r+\phi)(s-t)}(\alpha + \beta)Q_s \, ds\right]\) if \((\theta_t, M_t) = (\theta_0, M_0) = (\theta, M) \in \mathbb{R}^2\). This concludes the proof. \( \square \)

**Proof of Proposition 3.** (i) Limits. Let \(\ell(\phi, \alpha) := \alpha \sigma^2_\theta[\alpha + B(\phi, \alpha)] + \kappa \sigma^2_\xi(B(\phi, \alpha))\) and

\[J(\phi) := \sqrt{[\ell(\phi, \alpha(\phi))]^2 - 4\kappa(\sigma^2_\sigma B(\phi, \alpha)[\alpha(\phi)]^2 B(\phi, \alpha(\phi))} - \ell(\phi, \alpha(\phi))\]

With this in hand, observe that (11) (or, equivalently, \(A(\phi, \alpha(\phi)) = 0\), where \(A(\phi, x)\) is defined in (A.9)), becomes \((r + \kappa + \phi)(\alpha(\phi) - 1) + \alpha(\phi)J(\phi)/[2\kappa \sigma^2_\xi] = 0\).
Since \(\alpha(\phi) \in (0, 1)\) for all \(\phi > 0\), and \(0 < |B(\phi, \alpha)| < 1/2\) for all \(\alpha \in (0, 1)\) and \(\phi > 0\), we have that \(0 < -4\kappa(\sigma_x\sigma_\theta)^2\beta(\phi)\alpha(\phi)\phi \rightarrow 0\) as \(\phi \rightarrow 0\). In addition, because \(\alpha(\phi) \neq \beta(\phi) > 0\), it follows that \(\ell(\phi, \alpha) > \kappa^2\sigma_x^2\). Using that \(\beta(\phi) = B(\phi, \alpha(\phi))\) then yields,

\[
0 < J(\phi) = \frac{-4\kappa(\sigma_x\sigma_\theta)^2\beta(\phi)\alpha(\phi)\phi}{\sqrt{[\ell(\phi, \alpha(\phi))]^2 - 4\kappa(\sigma_x\sigma_\theta)^2\beta(\phi)\alpha(\phi)\phi + \ell(\phi, \alpha(\phi))}} < \frac{-4\kappa(\sigma_x\sigma_\theta)^2\beta(\phi)\alpha(\phi)\phi}{2\kappa^2\sigma_x^2}.
\]

We conclude that \(\lim_{\phi \to 0} \alpha(\phi)\) exists and takes value 1.

As for the limit to \(+\infty\), notice that since \(\ell(\phi, \alpha(\phi)) \geq \kappa^2\phi\) and \(\alpha(\phi)B(\phi, \alpha(\phi)) < 0\),

\[
0 < J(\phi) = \frac{-4\kappa(\sigma_x\sigma_\theta)^2B(\phi, \alpha(\phi))\alpha(\phi)\phi}{\sqrt{[\ell(\phi, \alpha(\phi))]^2 - 4\kappa(\sigma_x\sigma_\theta)^2B(\phi, \alpha(\phi))\alpha(\phi)\phi + \ell(\phi, \alpha(\phi))}} \leq \frac{-4\kappa(\sigma_x\sigma_\theta)^2B(\phi, \alpha(\phi))\alpha(\phi)}{2\kappa^2\sigma_x^2\kappa}.
\]

But since \(\alpha(\cdot)\) and \(B(\cdot, \alpha(\cdot))\) are bounded, \(J(\cdot)\) is bounded too. Thus, from \(A(\phi, \alpha(\phi)) = 0\),

\[
1 - \alpha(\phi) = \frac{\alpha(\phi)J(\phi)}{2\kappa^2\sigma_x^2} \frac{1}{(r + \kappa + \phi)} \rightarrow 0 \text{ as } \phi \rightarrow \infty.
\]

Regarding the limit values for \(\beta(\phi) = B(\phi, \alpha(\phi))\), these follow from the limit behavior of \(\alpha(\phi)\) and the definition of \(B\) given by (A.7). As for \(\delta(\phi)\), recall from (A.12) that

\[
\delta(\phi) = \frac{\kappa(\alpha(\phi) - 1) + [\alpha(\phi) + 2\beta(\phi)][\phi - (\alpha(\phi) + \beta(\phi))\lambda(\phi)]}{2(r + \phi) + (\alpha(\phi) + \beta(\phi))\lambda(\phi)}.
\]

From Lemma A.4 next, however, \(\lambda(\phi) \rightarrow 0\) as \(\phi \rightarrow 0\). Using that \(\alpha(\phi) \rightarrow 1\) and \(\alpha(\phi) + 2\beta(\phi) \rightarrow 0\) as \(\phi \rightarrow 0\), and that \(\alpha(\phi) + \beta(\phi) \rightarrow 0\), it direct that \(\delta(\phi) \rightarrow 0\) as \(\phi \rightarrow 0\). Also from the same lemma, \(\lambda(\phi) \rightarrow \sigma_\delta^2/\kappa\sigma_x^2\) as \(\phi \rightarrow \infty\). Thus, \([\phi - (\alpha(\phi) + \beta(\phi))\lambda(\phi)]/[2(r + \phi) + (\alpha(\phi) + \beta(\phi))\lambda(\phi)] \rightarrow 1/2\) as \(\phi \rightarrow \infty\). The limit \(\delta(\phi) \rightarrow 0\) as \(\phi \rightarrow \infty\) then follows from \(\alpha(\phi) \rightarrow 1\) and \(\alpha(\phi) + 2\beta(\phi) \rightarrow 0\) as \(\phi \rightarrow \infty\).

It remains to show the limit result on prices. To this end, we start with a preliminary

**Lemma A.4.** \(\lim_{\phi \to 0} \lambda(\phi) = 0\), \(\lim_{\phi \to 0} \lambda(\phi) = \sigma_\delta^2/\kappa\sigma_x^2\) and \(\lim_{\phi \to 0} \lambda(\phi)/\phi = 2\sigma_\delta^2/\sigma_\theta^2 + 2\sigma_x^2\kappa^2\).

**Proof.** \(\lim_{\phi \to 0} \lambda(\phi) = 0\) is direct consequence of the first bound in (A.15) which we establish shortly in the proof of part (ii) of the Proposition. Also, letting \(\ell(\phi, \alpha) := \sigma_\delta^2/\sigma_\theta^2 + 2\sigma_x^2\kappa^2\) +
\( \kappa \sigma_\xi^2 [\phi + \kappa] \), it is straightforward to verify that

\[
\lambda(\phi) = \frac{4 \kappa (\sigma_\xi \sigma_\eta)^2 \alpha(\phi)}{2 \kappa \sigma_\xi^2 \left( \sqrt{\frac{\ell(\phi, \alpha(\phi))}{\phi}} - \frac{4 \kappa (\sigma_\xi \sigma_\eta)^2 B(\phi, \alpha(\phi)) \alpha(\phi) + \ell(\phi, \alpha(\phi))}{\phi} \right)} \to \frac{4 \kappa (\sigma_\xi \sigma_\eta)^2}{2 \kappa \sigma_\xi^2 [\kappa \sigma_\xi^2 + \kappa \sigma_\xi^2]} = \frac{\sigma_\eta^2}{\kappa \sigma_\xi^2}
\]

as \( \phi \to \infty \), and thus the second limit holds. The third limit follows directly from the first equality in the previous display. This ends the proof of the lemma.

Using the lemma, we first show that \( \lim_{\phi \to \infty} \text{Var} [\lambda(\phi) Y_t] = \lim_{\phi \to 0} \text{Var} [\lambda(\phi) Y_t] = 0 \). Recall that \((\delta(\phi), \alpha(\phi), \beta(\phi)) \to (0, 1, -1/2)\) as \( \phi \to 0 \) and \( +\infty \). Also, from \((A.2)\),

\[
\text{Var}[Y_t] = \frac{1}{2(\phi - \beta(\phi) \lambda(\phi))} \left[ \sigma_\xi^2 + \frac{\alpha(\phi) \sigma_\eta^2}{\kappa \phi - \beta(\phi) \lambda(\phi) + \kappa} \right]. 
\]

By the previous lemma, therefore, \( \lim_{\phi \to \infty} \text{Var}[Y_t] = 0 \), and so \( \lim_{\phi \to \infty} \text{Var} [\lambda(\phi) Y_t] = 0 \). As for the other limit, we can write \((A.13)\) as

\[
\text{Var} [\lambda(\phi) Y_t] = \frac{1}{2(\phi - \beta(\phi) \lambda(\phi))} \lambda(\phi) \to \frac{\sigma_\xi^2 + \frac{\alpha(\phi) \sigma_\eta^2}{\kappa \phi - \beta(\phi) \lambda(\phi) + \kappa}}{\lambda(\phi) - \text{constant}} \to 0 \text{ as } \phi \to 0.
\]

The \( L^2 \)-limits then follow directly from the following results: \((\delta(\phi), \alpha(\phi), \beta(\phi)) \to (0, 1, -1/2)\) as \( \phi \to 0, \infty \); \( P_t = \delta \mu + (\alpha + \beta) M_t \) and \( M_t = \mu + \lambda [Y_t - \bar{Y}] \); \( \mathbb{E}[P_t] = \mu [\alpha(\phi) + \beta(\phi) + \delta(\phi)] \to \mu/2 \) as \( \phi \to 0 \) and \( +\infty \); and the triangular inequality.

(ii) Bounds. Observe that the bounds for \( \beta(\phi) \) were already determined from \((A.7)\) and \( \alpha(\phi) \in (0, 1) \). As for the lower bound for \( \alpha \), we will show the stronger result

\[
\max \left\{ \frac{r + \kappa + \phi}{r + \kappa + 2\phi}, \frac{r + \kappa + \phi}{r + \kappa + \phi + \sigma_\eta^2/2\kappa \sigma_\xi^2} \right\} \leq \alpha(\phi).
\]

The bound is tight in the sense that it converges to 1 when \( \phi \to 0 \) and \( +\infty \).

To obtain the bound, observe that from \((A.8)\), \( \lambda(\phi) \) satisfies

\[
\lambda(\phi) = \frac{2 \sigma_\eta^2 \alpha(\phi) \phi}{\sqrt{\ell^2(\phi, \alpha(\phi)) - 4 \kappa (\sigma_\xi \sigma_\eta)^2 \alpha(\phi) B(\phi, \alpha(\phi)) \phi + \ell(\phi, \alpha(\phi))}} < \frac{\sigma_\eta^2 \alpha(\phi) \phi}{\ell(\phi, \alpha(\phi))}, \quad (A.14)
\]
where \( \ell(\phi, \alpha(\phi)) := \sigma_\phi^2\alpha(\phi)[\alpha(\phi) + B(\phi, \alpha(\phi))] + \kappa\sigma_\xi^2[\phi + \kappa] \). From here, we get two bounds:

\[
\lambda < \frac{2\phi}{\alpha(\phi)} \quad \text{and} \quad \lambda < \frac{\sigma_\phi^2}{\kappa\sigma_\xi^2}.
\] (A.15)

In fact, since \( B(\phi, \alpha(\phi)) \leq -\alpha/2 \), then \( \ell(\phi, \alpha(\phi)) > \sigma_\phi^2\frac{\alpha(\phi)^2}{2} \); but using this in (A.14) leads to the first inequality in (A.15). Similarly, the second upper bound follows from (A.14) using that \( \alpha(\phi) < 1 \) and that \( \ell(\phi, \alpha(\phi)) > \kappa\sigma_\xi^2\phi \) due to \( \alpha + B(\phi, \alpha) > 0 \). In particular, \( \lambda(\phi) \) is bounded over \( \mathbb{R}_+ \), and it converges to zero as \( \phi \to 0 \), as promised.

Consider now the locus \( A(\phi, \alpha(\phi)) = 0 \). Using the first bound in (A.15) yields

\[
0 = (r + \kappa + \phi)(\alpha(\phi) - 1) + \frac{\lambda(\phi)}{\alpha(\phi)} \alpha(\phi) \left[ -B(\phi, \alpha(\phi)) \right] < (r + \kappa + \phi)(\alpha(\phi) - 1) + \phi\alpha(\phi)
\]

\[
\Rightarrow \alpha(\phi) > \frac{r + \kappa + \phi}{r + \kappa + 2\phi} > \frac{1}{2}, \quad \text{for all } \phi > 0.
\]

Similarly, using the second bound, \( 0 < (r + \kappa + \phi)(\alpha(\phi) - 1) + [\sigma_\theta\alpha(\phi)]^2/[2\kappa\sigma_\xi^2] \); the desired second bound for alpha follows from imposing that \( \alpha^2 < \alpha \) in the previous inequality.

We conclude this part by establishing the bounds for the expected price, and omit the dependence of \( (\alpha, \beta, \delta, \lambda) \) on \( \phi \) in the process. Observe that \( \mathbb{E}[P_t] = \delta \mu + (\alpha + \beta)\mathbb{E}[M_t] = [\delta + \alpha + \beta] \mu \). Now, adding the second and third equation in the system (A.6) yields \( (\alpha + 2\beta)(\alpha + \beta)\lambda = (r + 2\phi)(\alpha + 2\beta) + (r + \kappa + \phi)(\alpha - 1) + (\alpha + \beta)^2\lambda \). Thus, in equilibrium,

\[
\delta = \frac{\kappa(\alpha - 1) + [\alpha + 2\beta][\phi - (\alpha + \beta)\lambda]}{2(r + \phi) + (\alpha + \beta)\lambda} \quad \Rightarrow \quad \frac{\kappa(\alpha - 1) + (\alpha + 2\beta)\phi - (r + 2\phi)(\alpha + 2\beta) - (r + \kappa + \phi)(\alpha - 1) - (\alpha + \beta)^2\lambda}{2(r + \phi) + (\alpha + \beta)\lambda} \quad \Rightarrow \quad \frac{-(r + \phi)[2(\alpha + \beta) - 1] - (\alpha + \beta)^2\lambda}{2(r + \phi) + (\alpha + \beta)\lambda},
\]

from where it is easy to conclude that

\[
\mathbb{E}[P_t] = \mu[\alpha + \beta + \delta] = \mu\frac{r + \phi}{2(r + \phi) + (\alpha + \beta)\lambda}. \quad (A.16)
\]

In particular, \( \mathbb{E}[P_t] < \mu/2 \) when \( \mu \neq 0 \) follows directly from \( \lambda(\alpha + \beta) > 0 \).

On the other hand, from (A.14) and \( \ell(\phi, \alpha) > \sigma_\phi^2\alpha[\alpha + B(\phi, \alpha)] = \sigma_\phi^2\alpha[\alpha + \beta] \),

\[
(\alpha + \beta)\lambda < (\alpha + \beta)\frac{\sigma_\phi^2\alpha\phi}{\ell(\phi, \alpha)} < (\alpha + \beta)\frac{\sigma_\phi^2\alpha(\phi)\phi}{\sigma_\phi^2\alpha[\alpha + \beta]} = \phi.
\]
Using this latter bound in (A.16) leads to $\mathbb{E}[P_i] > \mu / 3$ whenever $\mu \neq 0$, as $r > 0$.

(iii) Quasiconvexity of $\alpha$. To prove this property, it is more useful to solve the last two equations in the system (A.6) for $\lambda$ and $\beta$, namely,

$$\lambda(\phi, \alpha) = -\frac{(\alpha - 1)(\kappa + r + \phi)(\kappa + \alpha(-\kappa + r + 3\phi) + r + \phi)}{\alpha^3(r + 2\phi)}, \quad (A.17)$$

$$\beta(\phi, \alpha) = -\frac{\alpha^2(r + 2\phi)}{\kappa + \alpha(-\kappa + r + 3\phi) + r + \phi}. \quad (A.18)$$

Substituting both expressions into (7) that defines $\lambda$, and recalling $s := \sigma^2 / \sigma^2_\theta$, we obtain an alternate locus $(\phi, \alpha(\phi))$ that satisfies

$$\bar{A}(\phi, \alpha) := \frac{(\alpha - 1)(\kappa + r + \phi)(\kappa + \alpha(-\kappa + r + 3\phi) + r + \phi)}{\alpha^3(r + 2\phi)} + \frac{\alpha(\kappa - \alpha(\kappa + r) + r + \phi)}{\alpha^3 + \kappa s(\kappa - \alpha r + r + \phi)} = 0. \quad (A.19)$$

Observe that $\bar{A}$ is increasing in $\alpha$ whenever $\bar{A}(\phi, \alpha) = 0$. In fact, since Proposition 1 establishes the uniqueness of an equilibrium, there is a unique $\alpha(\phi) \in [0, 1]$ solving $\bar{A}(\phi, \alpha) = 0$. In addition, $\bar{A}(\phi, 1) = \phi / [\kappa s(\kappa + \phi) + 1] > 0$. Thus, $\bar{A}(\phi, \cdot)$ must cross zero from below.

Now, the second partial derivative

$$\frac{\partial^2 \bar{A}(\phi, \alpha)}{(\partial \phi)^2} = -\frac{2(\alpha - 1)^2(2\kappa + r)^2}{(r + 2\phi)^3} = \frac{2\alpha^5 \kappa s(\alpha^2 + \kappa^2 s)}{(\alpha^3 + \kappa s(\kappa - \alpha r + r + \phi))^3}$$

is strictly negative because, by inspection, the first term is nonpositive and the second term is strictly negative. Furthermore, $\phi \mapsto \alpha(\phi)$ is twice continuously differentiable. Combined with the fact that $\bar{A}$ is increasing in its second argument whenever $\bar{A} = 0$, the Implicit Function Theorem implies that $\alpha''(\phi) > 0$ at any critical point $\alpha'(\phi) = 0$.

(iv) Effect of noise terms $\sigma_\xi / \sigma_\theta$. To show that $\phi \mapsto \alpha(\phi)$ is increasing in $\sigma_\xi / \sigma_\theta$ point-wise, consider again the locus $\bar{A}(\phi, \alpha) = 0$ in (A.19), and differentiate with respect to $s := \sigma^2_\xi / \sigma^2_\theta$.

We obtain

$$\frac{\partial \bar{A}}{\partial s} = -\frac{\alpha^4 \kappa ((1 - \alpha)(\kappa + r) + \phi)(\kappa + (1 - \alpha)r + \phi)}{(\alpha^3 + \kappa s(\kappa + (1 - \alpha)r + \phi))^2} < 0.$$ 

Because $\bar{A}$ is increasing in $\alpha$ at $(\phi, \alpha(\phi))$, we conclude that $\alpha$ is increasing in $s$.

Finally, using the three equations (A.17)–(A.19), the derivative of the expected price

$$\alpha'(\phi) = \frac{1 - \alpha(\phi) - \alpha(\phi)H_\phi(\phi, \alpha(\phi))}{r + \kappa + \phi + H(\phi, \alpha(\phi)) + \alpha(\phi)H_\alpha(\phi, \alpha(\phi))},$$

with $H$ as in (A.10) in Lemma A.3, and the right-hand side of the previous equality being continuously differentiable in $\phi$. 

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\[ [\alpha + \beta + \delta] \mu \text{ with respect to } \alpha \text{ can be written as} \]

\[
\mu \frac{\alpha(r + \phi)(r + 2\phi)(\kappa + r + \phi)(2(\kappa + r + \phi) - \alpha(2\kappa + r))}{[\alpha^2(\kappa^2 + r(\kappa + 2r) + 5r\phi + 3\phi^2) - \alpha(2\kappa + r)(\kappa + r + \phi) + (\kappa + r + \phi)^2]^2} > 0.
\]

Furthermore, when using (A.17)–(A.19), the expected price does not depend on \( s \) directly. Using that \( \phi \mapsto \alpha(\phi) \) is increasing in \( \sigma_\xi/\sigma_\theta \), therefore, the expected price is also increasing in \( s \). This concludes the proof. \( \square \)

**Proofs for Section 5**

The following results are used in the subsequent analysis, and their proofs can be found in the Supplementary Appendix (section S.2.5).

**Lemma A.5.** Suppose \( \alpha > 0 \) and \( \beta < 0 \) satisfy \( \nu(\alpha, \beta) > 0 \), where \( \nu(\alpha, \beta) \) is defined in (19). Then, \( \phi \mapsto G(\phi, \alpha, \beta) \) has a unique maximizer located at \( \phi = \nu(\alpha, \beta) \). Moreover,

(i) \( \Lambda_\phi(\nu(\alpha, \beta), \alpha, \beta) = \Lambda(\nu(\alpha, \beta), \alpha, \beta)/(\nu + \kappa) \), where \( \Lambda_\phi(\phi, \alpha, \beta) \) denotes the partial derivative of \( \Lambda(\phi, \alpha, \beta) \) with respect to \( \phi \), and,

(ii) \( \Lambda(\nu(\alpha, \beta), \alpha, \beta) = \alpha \gamma(\alpha)/\sigma_\xi^2 \), where \( \gamma(\alpha) = \sigma_\xi^2[(\kappa^2 + \alpha^2 \sigma_\theta^2/\sigma_\xi^2)^{1/2} - \kappa]/\alpha^2 \) is the posterior belief’s stationary variance when the histories \( \xi^t, t \geq 0, \) are public.

**Lemma A.6.** \( \kappa < \arg \min \alpha < \infty \), and \( [\alpha + \beta]'(\phi) < 0, \phi \in [\kappa, \arg \min \alpha] \); hence, \( \alpha + \beta \) is strictly decreasing at any point satisfying (20). If \( r > \kappa \), \( [\alpha + \beta]'(\phi) < 0 \) for all \( \phi \in [0, \arg \min \alpha] \).

**Proof of Proposition 4.** Part (i) is proved in the Supplementary Appendix as part of Lemma A.5 already introduced. As for (ii), suppose that \( \xi^t := (\xi_s : 0 \leq s < t) \) is observed by firm \( t \), and let \( M_t^* := \mathbb{E}[\theta_t | \mathcal{F}_t^\xi], t \geq 0 \), where \( (\mathcal{F}_t^\xi)_{t \geq 0} \) denotes the filtration generated by \( (\xi_t)_{t \geq 0} \). When the quantity demanded follows \( Q_t = \delta \mu + \alpha \theta_t + \beta M_t^* \), recorded purchases obey

\[
d\xi_t = (\delta \mu + \alpha \theta_t + \beta M_t^*)dt + \sigma_\xi dZ_t^\xi,
\]

where \( (M_t^*)_{t \geq 0} \) satisfies the filtering equation

\[
dM_t^* = -\kappa(M_t^* - \mu)dt + \frac{\alpha \gamma(\alpha)}{\sigma_\xi^2} [d\xi_t - (\delta \mu + [\alpha + \beta]M_t^*dt)].
\]
In this SDE, $\gamma(\alpha)$ is the unique positive solution to $x \mapsto -2\kappa x + \sigma_0^2 - (\alpha x/\sigma_\xi)^2 = 0$ (Theorem 12.1 in Liptser and Shiryaev 1977). As a function of $(Z_t^0, Z_t^\xi)_{t \geq 0}$, therefore,

$$dM_t^* = \left(-\kappa + \alpha^2 \gamma(\alpha)/\sigma_\xi^2\right) M_t^* + \kappa \mu + \frac{\alpha^2 \gamma(\alpha)}{\sigma_\xi^2} \theta_t \, dt + \frac{\alpha \gamma(\alpha)}{\sigma_\xi} dZ_t^\xi.$$

Now, let

$$\lambda^* = \frac{\alpha \gamma(\alpha)}{\sigma_\xi^2} \text{ and } \rho^* = \frac{1}{\nu(\alpha, \beta)} \left(\kappa \mu - \frac{\alpha \gamma(\alpha) \delta \mu}{\sigma_\xi^2}\right) \quad (A.20)$$

where $\nu(\alpha, \beta)$ satisfies (19), i.e., $\nu(\alpha, \beta) := \kappa + \alpha \gamma(\alpha)[\alpha + \beta]/\sigma_\xi^2 > 0$. In particular, observe that $dM_t^* = -\kappa + \lambda^* \alpha) M_t^* + \kappa \mu + \lambda^* \alpha \theta_t] dt + \lambda^* \sigma_\xi dZ_t^\xi$.

With this in hand, consider $(Y_t)_{t \geq 0}$ evolving according to

$$dY_t = -\nu(\alpha, \beta) Y_t + \delta \mu + \beta \rho^* + \alpha \theta_t + \beta \lambda^* Y_t] dt + \sigma_\xi dZ_t^\xi.$$  

From the proof of Proposition A.1, if $(Y_0, \theta_0)$ is independent of $(Z_t^0, Z_t^\xi)_{t \geq 0}$ and

$$\mathbb{E}[Y_0] = \frac{\delta \mu + \beta \rho^* + \alpha \mu}{\nu(\alpha, \beta) - \beta \lambda^*}, \quad \text{Var}[Y_0] = \frac{1}{2(\nu(\alpha, \beta) - \beta \lambda^*)} \left[\sigma_\xi^2 + \frac{\alpha^2 \sigma_\theta^2}{\kappa(\nu(\alpha, \beta) - \beta \lambda^* + \kappa)}\right]$$

and

$$\text{Cov}[\theta_0, Y_0] = \frac{\alpha \sigma_\theta^2}{2(\nu(\alpha, \beta) - \beta \lambda^* + \kappa)},$$

the pair $(\theta_t, Y_t)_{t \geq 0}$ is stationary Gaussian, as $\phi - \beta \lambda^* = \nu(\alpha, \beta) - \beta \alpha \gamma(\alpha)/\sigma_\xi^2 = \kappa + \alpha^2 \gamma(\alpha)/\sigma_\xi^2 > 0$. Denote the previous process by $(Y^{\nu(\alpha, \beta)})_{t \geq 0}$, and note that

$$Y_t = e^{-\nu(\alpha, \beta)t} Y_0 + \int_0^t e^{-\nu(\alpha, \beta)(t-s)} d\xi_s, \quad t \geq 0.$$  

Defining $X_t = \rho^* + \lambda^* Y_t^{\nu(\alpha, \beta)}$, it is easy to verify that

$$dX_t = [\lambda^* (\delta \mu + \alpha \theta_t + \beta X_t) - \nu(\alpha, \beta)[X_t - \rho^*]] + \lambda^* \sigma_\xi dZ_t^\xi$$

$$= [-(\kappa + \lambda^* \alpha) X_t + \kappa \mu + \lambda^* \alpha \theta_t] dt + \lambda^* \sigma_\xi dZ_t^\xi,$$

where in the last equality we used that $\nu(\alpha, \beta) = \kappa + \lambda^* (\alpha + \beta)$ and that $\lambda^* \delta \mu + \nu(\alpha, \beta) = \mu \kappa$.

We conclude that $M_t^* - X_t$ satisfies $d[M_t^* - X_t] = -(\kappa + \lambda^* \alpha)[M_t^* - X_t] dt$, and therefore that $M_t^* - X_t = [M_0 - X_0] e^{-(\kappa + \lambda^* \alpha)t}$ for all $t \geq 0$.

Notice, however, that since $(X_t)_{t \geq 0}$ is stationary, stationarity of $(M_t^*)_{t \geq 0}$ implies that
Proposition 5. We first show that \( a'(\phi) < 0 \) at any \( \phi \) satisfying (20), i.e., \( \phi = \nu(\alpha(\phi), \beta(\phi)) \). To this end, recall that \( \alpha(\phi) \) is the only value in \((0, 1)\) satisfying \((r + \kappa + \phi)(\alpha(\phi) - 1) + \alpha(\phi)H(\phi, \alpha(\phi)) = 0\), where

\[
H(\phi, \alpha) := -\Lambda(\phi, \alpha, B(\phi, \alpha))B(\phi, \alpha) = \sqrt{\ell^2(\phi, \alpha) - 4\kappa(\sigma_{\xi} \sigma_{\theta})^2} \mathcal{B}(\phi, \alpha) \phi - \ell(\phi, \alpha)
\]

and \( \ell(\phi, \alpha) := \sigma_{\theta}^2 \alpha[\alpha + B(\phi, \alpha)] + \kappa \sigma_{\xi}^2 [\phi + \kappa] \). Also, recall from the proof of Lemma A.3 in the proof of Proposition 1 that \( \alpha \mapsto H(\phi, \alpha) \) is strictly increasing over \([0, 1]\).
Thus, denoting the partial derivatives with subindices,

$$\alpha'(\phi) [r + \kappa + \phi + H(\phi, \alpha(\phi)) + \alpha(\phi)H_\phi(\phi, \alpha(\phi))] = 1 - \alpha(\phi) - \alpha(\phi)H_\phi(\phi, \alpha(\phi)).$$

Consequently, because $H > 0$, we conclude that the sign of $\alpha'$ is always determined by the sign of the right-hand side of the previous expression. We now show that the latter side is negative at any point $\phi$ s.t. $\phi = \nu(\alpha(\phi), \beta(\phi)) = \kappa + \alpha(\phi)\gamma(\alpha(\phi))[\alpha(\phi) + \beta(\phi)]/\sigma_\xi^2$.

To simplify notation, let $\Delta(\phi, \alpha) := \sqrt{\ell^2(\phi, \alpha) - 4\kappa(\sigma_\xi\sigma_\theta)^2\alpha B(\phi, \alpha)\phi}$. Omitting the dependence on $(\phi, \alpha(\phi))$ of $H, \Delta, \ell, B$, and of their respective partial derivatives,

$$H_\phi = \frac{1}{2\kappa\sigma_\xi^2} \left[ \frac{\ell\ell - 2\kappa(\sigma_\xi\sigma_\theta)^2\alpha[\phi B_\phi + B]}{\Delta} - \ell \phi \right].$$

Moreover, since $\ell \phi = \sigma_\theta^2 \alpha B_\phi + \kappa \sigma_\xi^2$ we can write

$$H_\phi = \frac{\kappa\sigma_\xi^2[\ell - \Delta] - 2\kappa(\sigma_\xi\sigma_\theta)^2\alpha B}{2\kappa\sigma_\xi^2\Delta} + \frac{\sigma_\theta^2 \alpha B_\phi[\ell - \Delta] - 2\kappa(\sigma_\xi\sigma_\theta)^2\alpha \phi B_\phi}{2\kappa\sigma_\xi^2\Delta}.$$

Consider now the first term of the previous expression. In fact,

$$\frac{\kappa\sigma_\xi^2[\ell - \Delta] - 2\kappa(\sigma_\xi\sigma_\theta)^2\alpha B}{2\kappa\sigma_\xi^2\Delta} = -B \frac{\partial \Lambda}{\partial \phi}(\phi, \alpha, B).$$

From Lemma A.5, moreover, $\Lambda_\phi(\nu(\alpha, \beta), \alpha, \beta) = \Lambda(\nu(\alpha, \beta), \alpha, \beta)/[\nu(\alpha, \beta) + \kappa]$; therefore, this equality must holds at any $\phi$ such that $(\phi, \alpha(\phi), \beta(\phi)) = (\nu(\alpha(\phi), \beta(\phi)), \alpha(\phi), \beta(\phi))$.

On the other hand, the second term of $H_\phi$ can be written as

$$\frac{\sigma_\theta^2 \alpha B_\phi}{\Delta} \left[ \frac{\ell - \Delta}{2\kappa\sigma_\xi^2} - \phi \alpha \right] = \frac{\sigma_\theta^2 \alpha B_\phi}{\Delta} [(r + \kappa + \phi)(\alpha - 1) - \phi \alpha],$$

where we used that $\alpha H = \alpha(\Delta - \ell)/2\kappa\sigma_\xi^2$. We deduce that, at the point of interest,

$$1 - \alpha - \alpha H_\phi = 1 - \alpha + \frac{\lambda \alpha \beta}{\phi + \kappa} - \frac{\sigma_\theta^2 \alpha B_\phi}{\Delta} [(r + \kappa + \phi)(\alpha - 1) - \phi \alpha] \quad (A.21)$$

Straightforward differentiation shows that

$$B_\phi = \frac{\partial}{\partial \phi} \left( \frac{-\alpha^2(r + 2\phi)}{2(r + 2\phi)\alpha - (r + \kappa + \phi)(\alpha - 1)} \right) = \frac{\alpha^2(\alpha - 1)(r + 2\kappa)}{[2(r + 2\phi)\alpha - (r + \kappa + \phi)(\alpha - 1)]^2} < 0,$$

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so $K_2 > 0$. As for the other term, $(r + \kappa + \phi)(\alpha - 1) - \lambda\alpha\beta = 0$ yields

$$K_1 = \frac{(\phi + \kappa)(1 - \alpha) + \lambda\alpha\beta}{\phi + \kappa} = \frac{r(\alpha - 1)}{\phi + \kappa} < 0.$$ 

We conclude that $\alpha'(\phi) < 0$ at any point satisfying (20), provided any such point exists.

For existence, let $\eta(\phi) := \phi - \nu(\alpha(\phi), \beta(\phi))$, where $\nu(\alpha, \beta) = \kappa + \alpha\gamma(\alpha)[\alpha + \beta]/\sigma_\xi^2$, and

$$\gamma(\alpha) := \frac{\sigma_\xi^2}{\alpha^2} \left[ \sqrt{\kappa^2 + \alpha^2 \frac{\sigma_\theta^2}{\sigma_\xi^2}} - \kappa \right],$$

(i.e., $\gamma(\alpha)$ is the unique positive solution of $0 = \sigma_\theta^2 - 2\kappa \gamma - (\alpha\gamma/\sigma_\xi)^2$). Since $\alpha \in (1/2, 1)$, $\gamma$ is bounded, and so $\eta(\phi) > 0$ for $\phi$ large. Also, using that $\lim_{\phi \to 0}(\alpha(\phi), \beta(\phi)) = (1, -1/2)$, we have that $\lim_{\phi \to 0}\eta(\phi) < 0$. The existence of $\phi$ s.t. $\eta(\phi) = 0$ follows from the continuity of $\eta(\cdot)$.

We now turn to uniqueness. Observe first that $\nu(\alpha(\phi), \beta(\phi)) > \kappa$ for all $\phi > 0$. Also, since $\alpha$ is quasiconvex, $\alpha(\phi) \in (1/2, 1)$ if $\phi > 0$, and $\lim_{\phi \to 0}\alpha(\phi) = 1$, we have that $\alpha$ is decreasing in $[0, \arg\min\alpha)$, and non-decreasing thereafter. Since $\kappa < \arg\min\alpha$ (Lemma A.6) and $\alpha$ is strictly decreasing at any point satisfying $\phi = \nu(\alpha(\phi), \beta(\phi))$, we conclude that any such point must lie in $[\kappa, \arg\min\alpha]$.

Equipped with this observation, it then suffices show that $[\nu(\alpha(\phi), \beta(\phi))]' < 0$ over $[\kappa, \arg\min\alpha]$. In fact, because the identity function is increasing, the existence of two such points would imply the existence of an intermediate third point at which $\phi = \nu(\alpha(\phi), \beta(\phi))$ and $[\nu(\alpha(\phi), \beta(\phi))]' > 0$, yielding a contradiction. To this end, write

$$[\nu(\alpha(\phi), \beta(\phi))]' = \frac{d}{d\phi} \left( \frac{\alpha(\phi)\gamma(\alpha(\phi))}{\sigma_\xi^2} \right) (\alpha(\phi) + \beta(\phi)) + \left( \frac{\alpha(\phi)\gamma(\alpha(\phi))}{\sigma_\xi^2} \right) \frac{d(\alpha(\phi) + \beta(\phi))}{d\phi}.$$

From Lemma A.6, $\alpha(\phi) + \beta(\phi)$ is strictly decreasing over $[\kappa, \arg\min\alpha]$. Since $\alpha + \beta > 0$ and $\alpha\gamma(\alpha(\phi)) > 0$ is sufficient to show that $[\alpha(\phi)\gamma(\alpha(\phi))]' < 0$ over the same region. However,

$$\frac{\alpha\gamma(\alpha)}{\sigma_\xi^2} = \frac{1}{\alpha} \left[ \sqrt{\kappa^2 + \alpha^2 \frac{\sigma_\theta^2}{\sigma_\xi^2}} - \kappa \right] = \frac{\sigma_\theta^2}{\sigma_\xi^2} \left( \sqrt{\frac{\kappa^2}{\alpha} + \frac{\sigma_\theta^2}{\sigma_\xi^2}} + \frac{\kappa}{\alpha} \right)^{-1},$$

which is strictly increasing in $\alpha$. We conclude by using that $\alpha' < 0$ over $[\kappa, \arg\min\alpha]$. 

\textbf{Proof of Proposition 6.} For part (i), recall that $G(\phi) = \alpha(\phi)\lambda(\phi)/[\phi + \kappa - \beta(\phi)\lambda(\phi)] \geq 0$ for all $\phi \geq 0$, where $\lambda(\cdot) = \Lambda(\cdot, \alpha(\cdot), \beta(\cdot))$. Since $\lambda(\phi)$ is bounded (second bound in (A.15)), $\lim_{\phi \to \infty} G(\phi) = 0$. Also $G(0) = 0$. By continuity, $G$ has a global optimum that is interior.
From the definition of $\nu(\alpha, \beta)$,

$$G(\phi) := G(\phi, \alpha(\phi), \beta(\phi)) \leq G(\nu(\alpha(\phi), \beta(\phi)), \alpha(\phi), \beta(\phi)),$$

with equality only at $\phi^*$. Also, from Lemma A.5, $A(\nu(\alpha, \beta), \alpha, \beta) = \alpha \gamma(\alpha)/\sigma^2_\xi$. Thus, letting $\nu(\phi) := \nu(\alpha(\phi), \beta(\phi))$,

$$G(\nu(\phi), \alpha(\phi), \beta(\phi)) = \frac{\alpha(\phi)A(\nu(\phi), \alpha(\phi), \beta(\phi))}{\nu(\phi) + \kappa - \beta(\phi)A(\nu(\phi), \alpha(\phi), \beta(\phi))} = \frac{\alpha^2(\phi)\gamma(\alpha(\phi))}{\alpha^2(\phi)\gamma(\alpha(\phi)) + 2\kappa \sigma^2_\xi}, \quad (A.22)$$

where we used that $\nu(\phi) = \kappa + \alpha(\phi)\gamma(\alpha(\phi))[(\alpha(\phi) + \beta(\phi))/\sigma^2_\xi]$.

However, by definition of $\gamma(\alpha)$, $\alpha^2(\phi)\gamma(\alpha(\phi)) = \sigma^2_\xi[(\kappa^2 + \alpha^2(\phi)\sigma^2_\alpha/\sigma^2_\xi)^{1/2} - \kappa]$; thus, from (A.22), $G(\nu(\phi), \alpha(\phi), \beta(\phi))$ is decreasing when $\alpha(\phi)$ is decreasing. Since $G(\phi)$ is bounded from above by a decreasing function of $\phi$ on $[\phi^*, \arg\min \alpha]$, $G(\phi^*) > G(\phi)$ over the same interval.

We now show that $G(\phi)$ is decreasing when $\alpha(\phi)$ is increasing, i.e., over $(\arg\min \alpha(\phi), \infty)$. Using that $G(\phi, \alpha, \beta) := \alpha A(\phi, \alpha, \beta)/[\phi + \kappa - \beta A(\phi, \alpha, \beta)]$, and equations (A.17) and (A.18) to substitute for $\lambda$ and $\beta$, we obtain that $G(\phi) = \tilde{G}(\phi, \alpha(\phi))$ where

$$\tilde{G}(\alpha, \phi) := (1 - \alpha) \frac{(\kappa + r + \phi)}{(\kappa + (1 - \alpha)r + \phi)} \frac{(\kappa + \alpha(-\kappa + r + 3\phi) + r + \phi)}{\alpha(r + 2\phi)}. \quad (A.23)$$

Using that $\alpha \in (0, 1)$, it is easy to verify that the two fractions on the right-hand side are strictly positive and strictly decreasing in $\phi$. Thus, $\partial \tilde{G}/\partial \phi < 0$. Also, up to a positive multiplicative term,

$$\frac{\partial \tilde{G}}{\partial \alpha} = -((\rho + f + 1) - \alpha \rho)^2 + \alpha^2(f + 1)(2\rho + 3f - 1),$$

where $\rho := r/\kappa$ and $f := \phi/\kappa$. However, $-((\rho + f + 1) - \alpha \rho)^2 + \alpha^2(f + 1)(2\rho + 3f - 1) < -(f + 1)(\alpha^2(2\rho + 3f - 1) + f + 1)$ and the latter term is strictly negative for $\alpha \in [0, 1]$. The quasiconvexity of $\alpha$ then allows us to conclude that

$$\frac{d \tilde{G}}{d \phi} = \frac{\partial \tilde{G}}{\partial \alpha} \alpha'(\phi) + \frac{\partial \tilde{G}}{\partial \phi} < 0, \quad \phi \in (\arg\min \alpha, \infty).$$

Part (ii) (i.e., $G$ is decreasing in $\sigma_\xi$) follows immediately from the fact that $\tilde{G}$ is decreasing in $\alpha$ for a fixed $\phi$, because $\alpha$ is itself increasing in $\sigma_\xi$ (Proposition 3).
Proofs for Section 6

Proof of Proposition 7. We start with (i) for profits. From (A.16), \( \alpha + \beta + \delta > 1/3 \). Thus, omitting the dependence of the equilibrium coefficients on \( \phi \),

\[
\Pi(\phi) := \mu^2[\alpha + \beta + \delta]^2 + \frac{\sigma^2}{2\kappa}(\alpha + \beta)^2G(\phi) \geq \frac{\mu^2}{9} + \frac{\sigma^2}{2\kappa}(\alpha + \beta)^2G(\phi), \quad \text{for all } \phi > 0.
\]

On the other hand, from the proof of Proposition 6, \( \lim_{\phi \to 0, \infty} G(\phi) = 0 \). Moreover, \( \alpha + \beta \) is bounded. Therefore, \( \lim_{\phi \to 0, \infty} (\alpha + \beta)^2G(\phi) = 0 \), and so \( \lim_{\phi \to 0, \infty} \Pi(\phi) = \mu^2/4 \). Thus, if

\[
\frac{\mu^2}{9} + \frac{\sigma^2}{2\kappa}(\alpha + \beta)^2G(\phi) \geq \frac{\mu^2}{4} \iff \mu^2 \leq \frac{18\sigma^2}{5\kappa}(\alpha + \beta)^2G(\phi),
\]

it follows that \( \Pi(\phi) > \mu^2/4 \). Since \( \phi \mapsto [\alpha(\phi) + \beta(\phi)]^2G(\phi) \) is continuous, strictly positive, and converges to 0 as \( \phi \to 0 \) and \( +\infty \), it has a global maximum; denote it by \( \phi^\dagger \). Letting

\[
\mu_f := \left[ \frac{18\sigma^2}{5\kappa}(\alpha(\phi^\dagger) + \beta(\phi^\dagger))^2G(\phi^\dagger) \right]^{1/2} > 0
\]

the result follows.

As for the consumer, let \( CS_\mu(\phi) \) denote her surplus for a given \( \mu \geq 0 \) and observe that

\[
CS_\mu(\phi) = CS_0(\phi) + \mu^2R(\phi),
\]

where \( R(\phi) := [\alpha(\phi) + \beta(\phi) + \delta(\phi)]\left(1 - \frac{3}{2}[\alpha(\phi) + \beta(\phi) + \delta(\phi)]\right) \) and

\[
CS_0(\phi) = \frac{\sigma^2}{2\kappa}G(\phi)L(\phi) + \frac{\sigma^2}{2\kappa}\left[\alpha(\phi) - \frac{(\alpha(\phi))^2}{2}\right].
\]

Importantly since \( \alpha(\phi) + \beta(\phi) + \delta(\phi) \to 1/2 \) as \( \phi \to 0 \) and \( +\infty \), we have that \( \lim_{\phi \to 0} R(\phi) = \lim_{\phi \to +\infty} R(\phi) = 1/8 \). In addition we know that \( 1/3 < \alpha(\phi) + \beta(\phi) + \delta(\phi) < 1/2 \) for all \( \phi > 0 \). Because the function \( x \mapsto x - 3x^2/2 \) is strictly decreasing in \([1/3, 1/2]\), we have that \( R(\phi) > 1/8 \), for all \( \phi > 0 \).

Fix any \( \hat{\phi} > 0 \). Then, using that \( CS_\mu(0) = \mu^2/8 \),

\[
CS_\mu(\hat{\phi}) - CS_\mu(0) = \mu^2 \left[ R(\hat{\phi}) - \frac{1}{8} \right] + \frac{\sigma^2}{2\kappa} \left[ G(\hat{\phi})L(\hat{\phi}) + \alpha(\hat{\phi}) - \frac{(\alpha(\hat{\phi}))^2}{2} - \frac{1}{2} \right] =: K(\hat{\phi})
\]

Observe that \( K(\cdot) \) and \( R(\cdot) \) are independent of \( \mu \), we can choose \( \mu \) arbitrarily large such that
the right-hand side is strictly positive. Since \( CS_\mu(0) = \lim_{\phi \to \infty} CS_\mu(0) \), the consumer’s global maximum \( \phi^* \) must be interior.

We now turn to (ii), starting with profits. Towards a contradiction, suppose that there are sequences \( \mu_n \nearrow \infty \) and \( \phi_n > 0, n \in \mathbb{N} \), such that \( \Pi_{\mu_n}(\phi_n) \geq \Pi_{\mu_n}(0) = \Pi_{\mu_n}(+\infty) \). Then,

\[
\Pi_{\mu_n}(\phi_n) \geq \Pi_{\mu_n}(0) \iff \frac{\sigma^2}{2\kappa}[\alpha(\phi_n) + \beta(\phi_n)]^2 G(\phi_n) \geq \mu_n^2 \left[ \frac{1}{4} - [\alpha(\phi_n) + \beta(\phi_n) + \delta(\phi_n)]^2 \right]
\]

Observe first that \((\phi_n)_{n \in \mathbb{N}}\) cannot have a cluster point different from zero. Otherwise, along such subsequence, say \((\phi_{n_k})_{k \in \mathbb{N}}\), both \([\alpha(\phi_{n_k}) + \beta(\phi_{n_k})]^2 G(\phi_{n_k})\) and \(1/4 - [\alpha(\phi_{n_k}) + \beta(\phi_{n_k}) + \delta(\phi_{n_k})]^2\) converge to strictly positive numbers; the inequality is then violated for large \( k \).

Suppose now that there is a subsequence \((\phi_{n_k})_{k \in \mathbb{N}}\) that diverges. Using that \(\alpha + \beta + \delta = (r + \phi)/[2(r + \phi) + \lambda(\alpha + \beta)]\) and \(G = \alpha \lambda/[\phi + \kappa - \beta \lambda]\), we obtain

\[
\Pi_{\mu_{n_k}}(\phi_{n_k}) \geq \Pi_{\mu_n}(0) \iff \frac{\sigma^2}{2\kappa} \frac{4(\alpha + \beta)^2 \alpha \lambda [2(r + \phi) + \lambda(\alpha + \beta)]^2}{[4(r + \phi)(\alpha + \beta)\lambda + (\alpha + \beta)^2 \lambda^2](\phi + \kappa - \beta \lambda)} \geq \mu_{n_k}.
\]

However, because \(\alpha, \beta, \lambda\) are all bounded and \((\alpha, \beta, \lambda) \to (1, -1/2, \sigma^2/\kappa \sigma^2_\xi)\) as \(\phi \to +\infty\), both the numerator and denominator are \(O(\phi^2)\) for large \(\phi\), so the limit of the left-hand side of the second inequality exists. The inequality is then violated for large \( k \), a contradiction.

From the previous argument, the only remaining possibility is that \((\phi_n)_{n \in \mathbb{N}}\) converges to zero. However, from Proposition 3, \(\lim_{\phi \to 0} (\alpha(\phi), \beta(\phi), \lambda(\phi)) = (1, -1/2, 0)\), and so

\[
\frac{4(\alpha + \beta)^2 \alpha \lambda [2(r + \phi) + \lambda(\alpha + \beta)]^2}{[4(r + \phi)(\alpha + \beta)\lambda + (\alpha + \beta)^2 \lambda^2](\phi + \kappa - \beta \lambda)} \to \frac{2r}{\kappa}, \quad \text{as} \quad \phi \to \infty,
\]

and so the same inequality is again violated, a contradiction.

The case for the consumer is proved in an analogous fashion. Namely, towards a contradiction, assume that there are \((\mu_n)_{n \in \mathbb{N}}\) decreasing towards zero and \((\phi_n)_{n \in \mathbb{N}}\) strictly positive such that \(CS_{\mu_n}(\phi_n) \geq CS_{\mu_n}(0)\). Straightforward algebraic manipulation shows that

\[
CS_{\mu_n}(\phi_n) \geq CS_{\mu_n}(0) \iff \frac{1}{\text{Var}[\theta_0]} \left( \frac{(\alpha(\phi_n)-1)^2}{R(\phi_n)-1/8} - 2L(\phi_n) \frac{G(\phi_n)}{R(\phi_n)-1/8} \right) \geq \frac{1}{\mu_n},
\]

with \(R(\phi)\) defined in part (i) of the proof and \(L(\phi) := \frac{\alpha(\phi)^2}{2} + \beta(\phi) - \frac{3}{2}(\alpha(\phi) + \beta(\phi))^2 < 0.\) As in the firms’ case, there can’t be a subsequence of \((\phi_n)_{n \in \mathbb{N}}\) converging to a value different from zero; otherwise, the left-hand side of the inequality on the right converges, but the
right-hand side converges. In the Supplementary Appendix (section S.2.3) we show that

\[
\lim_{\phi \to 0, +\infty} \frac{\alpha(\phi) - 1}{R(\phi)} = 0, \quad \text{and} \quad \lim_{\phi \to 0, +\infty} \frac{G(\phi)}{R(\phi) - 1/8} > 0.
\] (A.24)

But since \( \lim_{\phi \to 0, +\infty} L(\phi) = -1/8 \), the left-hand side of the same inequality is again violated, a contradiction. Thus, there must exist \( \mu_c > 0 \) such that for all \( \mu < \mu_c \), \( CS(0) > CS(\phi) \) for all \( \phi \in (0, \infty) \).

For (iii), by pointwise convergence (section S.3.3 in the Supplementary appendix), we can directly consider the case of noiseless signals (\( \sigma_\xi = 0 \)). In this case, it is possible to show that both surplus levels are decreasing as \( \phi \to \infty \). Up to a positive multiplicative constant, the Mathematica file scores.nb shows that

\[
\lim_{\phi \to \infty} \phi^2 \Pi'(\phi) = \lim_{\phi \to \infty} \phi^2 CS'(\phi) < 0.
\]

On the other hand, \( \phi^* \to \infty \) as \( \sigma_\xi^2 \to 0 \) (section S.3.2 in the Supplementary appendix). Therefore, expected profits in the noiseless case converge to their limiting value from above, which means that the firm-optimal score satisfies \( \phi^f < \infty \) for all values of \( \mu \geq 0 \).

Finally, for part (iv), differentiating \( CS \) and \( \Pi \) with respect to \( \phi \) and setting equal to zero yields two loci \( \mu^f(\phi) \) and \( \mu^c(\phi) \) that describe the critical points of the surplus levels. The expressions for these loci are in the Mathematica file scores.nb posted on the authors’ websites. In the Mathematica file, we also establish the following properties: (a) \( \mu^f(\phi) \) is strictly quasiconcave, and \( \mu^c(\phi) \) is strictly quasiconvex; (b) if \( r/\kappa > \frac{1}{16} \sqrt{337 - 7} \approx 0.71 \), then \( \mu^f(0) < 0 \); and (c) if \( \rho := r/\kappa \) satisfies \( \rho(26 - \rho(24\rho + 31)) + 25 < 0 \) (i.e., if \( \rho < \tilde{\rho} \approx 0.96 \)), then \( \mu^c(0) > 0 \). Therefore, when \( \rho \) lies in the (approximate) range \([0.71, 0.96]\), the expected surplus levels admit at most one critical point for each \( \mu \). Because both surplus levels are decreasing at \( \phi = \infty \), this critical point is a local maximum. Furthermore, because the inverses \( \mu^f \) and \( \mu^c \) are respectively quasiconcave and quasiconvex, as well as decreasing and increasing at \( \phi = 0 \), these loci are also monotone.

Finally, at \( \phi = 0 \), we have \( \Pi'(0) \propto r/\kappa^2 - \mu^2 \) and \( CS'(0) \propto \mu^2 - 3r/\kappa^2 \). Therefore, the conditions in part (iv) of the statement describe the values of \( \mu \) for which the expected surplus levels admit an interior maximum.

\[ \Box \]

**Proof of Proposition 8.** We begin with consumer surplus. Fix \( \phi > 0 \). By the continuity of the equilibrium variables at \( \sigma_\xi = 0 \) (Supplementary Appendix section S.3.3), we can evaluate at \( \sigma_\xi = 0 \) to study the vanishing noise case.
The expected price level with strategic consumers when \( \sigma_\xi = 0 \) is below the naive benchmark for all \( \phi \). Thus, it suffices to show that consumer surplus in the strategic case exceeds the naive level when \( \mu = 0 \). We therefore compare the following expression in the two cases:

\[
\text{Var}[\theta] \left( \alpha - \frac{\alpha^2}{2} \right) + \text{Var}[\theta] \left( \frac{\alpha^2}{2} - \frac{3}{2} (\alpha + \beta)^2 + \beta \right) \frac{\alpha \lambda}{-\beta \lambda + \phi + \kappa}.
\]

In both cases, we let \( \lambda = \phi / (\alpha + \beta) \). In the naive case, we further impose \( \alpha = 1 \) and \( \beta = -1/2 \), while in the strategic case \( \beta \) satisfies (A.18). Solving for \( s \) from (A.19), and imposing that the solution is positive, we obtain the following restriction on the equilibrium values of \( \alpha \) and \( \phi \):

\[
\alpha^2 (\rho (2f - 1) + 3f^2 - 1) + \alpha (\rho + 2)(\rho + f + 1) - (\rho + f + 1)^2 > 0,
\]

where \( \rho := r / \kappa \) and \( f := \phi / \kappa \). We then show (in the Mathematica file scores.nb) that for no triple \( (\alpha, f, \rho) \in [1/2, 1] \times \mathbb{R}_+ \times \mathbb{R}_+ \) such that the above condition is verified does consumer surplus with naive consumers exceed the equilibrium level with strategic consumers. \( \square \)

**Proofs for Section 7**

Refer to the Supplementary Appendix, section S.2.4.

**Appendix B: Hidden Scores**

We briefly illustrate some key arguments used in the proofs of the observable-scores model that have a direct analog in the hidden case.

**B.1 Existence and uniqueness of stationary linear Markov equilibria.** A key step in the proof of Theorem 1 is Lemma A.3, which establishes the existence and uniqueness of a solution to (11). From part 1. in the proof of Proposition 9 in the Supplementary Appendix (Section S.2.4 in that file), replacing \( B(\phi, \alpha) \in (-\alpha/2, 0) \) as defined in (A.7) by \(-\alpha/2\) in (11) leads \( \alpha \mapsto H^h(\phi, \alpha) := -\Lambda(\phi, \alpha, -\alpha/2)[-\alpha/2] \) to be strictly increasing in \((0, 1)\), which was the main property driving existence and uniqueness in the observable counterpart.

**B.2 \( \alpha^h \) is decreasing at any non-concealing point.** In the hidden case, \( \alpha^h(\phi) \) solves \((r + \kappa + \phi)(\alpha^h - 1) + \alpha H^h(\phi, \alpha^h) = 0 \). Moreover, from the previous paragraph \( \alpha \mapsto H^h(\phi, \alpha) \)

\[53\text{In fact, } \mathbb{E}[P_t] = \mu r + \frac{\phi}{2r} \in (\mu/3, \mu/2) \text{ in this case, which follows (A.16) when } \lambda = \frac{\phi}{\alpha + \beta} \text{ when } \sigma_\xi = 0.\]
is strictly increasing in \((0,1)\). Thus, the sign of \([\alpha^h]'\) is given by the sign of \(1 - \alpha^h(\phi) - H^h(\phi, \alpha^h(\phi))\), as in the proof of Proposition 5. In particular, replacing \(B_\phi\) by 0 in (A.21) in that proof yields that, at any \(\phi^{*,h}\) satisfying \(\phi^{*,h} = \nu(\alpha^h(\phi^{*,h}), -\alpha^h(\phi^{*,h})/2)\),

\[
\text{sign}([\alpha^h]'(\phi^{*,h})) = \text{sign}([1 - \alpha^h(\phi) - \alpha^h(\phi)H^h(\phi, \alpha^h(\phi))]|_{\phi = \phi^{*,h}}) = \text{sign} \left( \frac{1 - \alpha^h + \frac{\lambda \alpha^h - \alpha^h/2}{\phi + \kappa}}{\phi = \phi^{*,h}} \right) < 0.
\]

**B.3 Existence and uniqueness of a non-concealing score.** Since \(\alpha^h + \beta^h = \alpha^h/2\) in the hidden case,

\[
\nu^h(\phi) := \nu(\alpha^h(\phi), -\alpha^h(\phi)/2) = \kappa + \frac{[\alpha^h(\phi)]^2 \gamma(\alpha^h(\phi))}{2\sigma^2}. 
\]

Thus, the existence of \(\phi^{*,h}\) and the corresponding bounds follow directly from the arguments in the proof of (ii) of Proposition 5. Moreover, using the definition of \(\gamma(\alpha)\),

\[
\nu(\phi) = \kappa + \frac{1}{2} \left[ \sqrt{\kappa^2 + [\alpha^h(\phi)]^2 \frac{\sigma^2}{\sigma^2} - \kappa} \right],
\]

so \(\text{sign}(\nu^h(\phi)) = \text{sign}(\alpha'(\phi))\). Since \([\alpha^h]'(\phi) < 0\) over \([\kappa, \arg \min \alpha^h]\), we conclude that there is only one such a point.

**B.4 The non-concealing score in the hidden case has more persistence.** Since \(\alpha^o + B(\phi, \alpha^o) \geq \alpha^o/2\) in the observable case, and \(\alpha^o > \alpha^h\), it follows that \(\nu^o(\phi) \geq \nu^h(\phi)\); therefore, \(\phi^{*,o} \geq \phi^{*,h}\), as each \(\nu\) crosses the identity from above.

**B.5 Quasiconvexity of \(\alpha^h\) and \(\arg \max_{\phi \geq 0} G^h(\phi) := G(\phi, \alpha^h(\phi), -\alpha^h(\phi)/2) < \phi^{*,h}\).** They follow identical arguments as the ones used in the observable case (Propositions 3 and 6).

**Appendix C: Noiseless Scores**

We briefly elaborate on some continuity and limit properties of the model at \(\sigma_\xi = 0\), which allows us to directly evaluate at this value when examining the noiseless limit \(\sigma_\xi \searrow 0\), and to extend some properties to \(\sigma^2_\xi > 0\) by continuity; for further details on the solution of this case, we refer the reader to Section S.3 in the Supplementary Appendix.
C.1 In both the observable and hidden cases, the variables \((\alpha, \beta, \delta, \lambda, G)\) are continuously differentiable over \((\phi, \sigma_\xi^2) \in (0, \infty) \times [0, \infty)\). This is established in section S.3.3 of the Supplementary Appendix, and it allows us to extend any point-wise comparison of ex ante payoffs in the noiseless to a neighborhood of \(\sigma_\xi = 0\) (in the sense of Section 7).

C.2 The hidden-scores model is “discontinuous” at \(\sigma_\xi = 0\), while the observable-scores model is not. When \(\sigma_\xi = 0\) the score is effectively observed by the consumer in any stationary linear Markov equilibrium, as the observation of \((Y_0, (Q_t)_{t \geq 0})\) allows the consumer to perfectly track the score—one cannot then talk about hidden-scores model when \(\sigma_\xi = 0\), as the observable and hidden models coincide. In this line, two observations are in order. First, (C.1) above implies that the equilibrium objects and outcomes of the (proper) hidden model admit a continuously differentiable extension to \(\sigma_\xi = 0\); any statement about the “hidden-score model at \(\sigma_\xi = 0\)” then refers to the value of such extension. By contrast, there is no discrepancy between these two notions when the scores are observed.

Second, because the sensitivity of demand is determined differently in each case, directly evaluating each model solution at \(\sigma_\xi = 0\) yields different outcomes. In particular, for all \(\phi < 0\),

\[
\alpha^h(0; \phi) \leq \frac{r + \kappa + \phi}{r + \kappa + 2\phi} < \frac{2(r + \kappa + \phi)}{r + 2\kappa + \sqrt{(r + 2\phi)(r + 6\phi)}} =: \alpha^o(0; \phi),
\]

thus reflecting that the relative strength of the ratcheting in the hidden-score model does not vanish in the limit as \(\sigma_\xi \downarrow 0\).

C.3 \(\phi^* \to \infty\) as \(\sigma_\xi^2 \downarrow 0\) in both settings. The intuition is that a score that decays infinitely fast essentially reveals the “last purchase,” and the latter is informative because the signaling coefficient is bounded away from zero uniformly for all \(\phi > 0\).
References


