Supplementary Appendix to "Consumer Scores and Price Discrimination"

Alessandro Bonatti and Gonzalo Cisternas

July 4, 2019

Contents

S.1 Online Display Advertising **2**

S.2 Omitted Proofs **7**

S.2.1 Section 4: Equilibrium Analysis . . . 7

S.2.2 Section 5: Equilibrium Learning . . . 11

S.2.3 Section 6: Welfare Analysis . . . 14

S.2.4 Section 7: Hidden Scores . . . 17

S.2.5 Appendix A: Omitted Proofs . . . 32

S.3 Noiseless Case **36**

S.3.1 Equilibrium Objects and Outcomes when $\sigma_\xi = 0$ . . . 36

S.3.2 Equilibrium Analysis when $\sigma_\xi = 0$ . . . 37

S.3.3 Continuous Differentiability of the Model at $\sigma_\xi = 0$ . . . 39

S.4 Discretized Model and Limit Demand Sensitivity **42**

S.5 Convexity Parameter in the Consumer's Cost Function **49**

S.5.1 Main Results . . . 49

S.5.2 Proofs . . . 51

S.6 Sum of Two Scores **55**

S.1 Online Display Advertising

In this Appendix, we describe a B2B setting in which sophisticated buyers strategically reduce their demand and sellers use the available information to mitigate the ratchet effect. Specifically, the market for online display (banner) advertising is economically relevant yet sufficiently simple to apply our model's insights. In this market, any publisher or website owner can be a seller of advertising space; the demand for space comes from advertisers who wish to reach final consumers with targeted messages. Worldwide spending on online display ads totaled $53 billion in 2018 and made up 20% of all digital advertising revenues.[1]

This market shares the following key economic properties with our model:

- The buyers' problem consists of choosing a flow volume of advertising at given prices.
- Sellers are able to price discriminate, both across buyers and over time, based on partial information about buyers' past purchases.
- Buyers are sophisticated and strategically manipulate their demand.
- Sellers are aware of the ratchet effect and take measures to mitigate it.

As such, this market can serve as a useful benchmark for our analysis of strategic consumer behavior. In what follows, we provide some background on this market, before turning to the more specific connections with our model and our results.

Background Most publishers of online content avail themselves of a Supply-Side Platform (SSP)—a technology platform that enables them to manage their advertising space. Examples of SSPs are the Rubicon Project, Oath Publisher Solutions (formerly Yahoo!), and Google Ad Manager (formerly Doubleclick Ad Exchange). In most cases, when an Internet user loads a page, the relevant SSP runs an auction to sell the advertising space in real time.[2] This auction has three main features: (i) unlike in the case of sponsored-search auctions, content is priced per impression (CPM, i.e., cost for a thousand impressions) rather than pay-per-click; (ii) the second-price auction is by far the most widely adopted format; (iii) many auctions have very few bidders, and hence reserve prices play a key role.

The last point is especially important for our model: in any display advertising auction, the SSP has an incentive to match the user's interests to the content of the ad. Furthermore,

[1] Source: Statista DMO 2019, available at https://www.statista.com/outlook/216/100/digital-advertising/worldwide.

[2] Some advertising space is contracted on ahead of time ("guaranteed direct buying"), in which case the forces in our model apply even more closely.

in many online market places, buyers have access to rich dynamic contextual information that predicts the consumers' willingness to pay for their products (Golrezaei, Javanmard, and Mirrokni, 2018). Consequently, SSPs allow advertisers to target a highly selected audience, i.e., to condition their bids on all available characteristics of the user (e.g., through cookies) and the webpage itself. Due to the extreme degree of *targeting*, many auctions have fewer bidders than available advertising space slots. For example, a retailer targeting a user who recently visited their website may be the sole bidder for that specific user.[3] However, while some alternative (negotiations-based) mechanisms have been suggested (Celis, Lewis, Mobius, and Nazerzadeh, 2011), most of the ad space volume is still sold through auctions.

In such thin markets, the role of reserve prices is then crucial for revenue generation (Milgrom, 2004). Indeed, because the second highest bid (if any) is often below the reserve price, bidders can often purchase advertising space at *de facto* fixed prices. Consistent with this interpretation, Ostrovsky and Schwarz (2016) quantify the impact of reserve prices through a field experiment at Yahoo! and show that reserve price optimization alone was responsible for raising *the entire company's* revenues by 11% year-on-year. Overall, given the large number of auctions run in a short amount of time, we can think of any bidder's problem as choosing what quantity to buy at the current reserve price. Crucially for our model, reserve prices are personalized for different buyers and dynamically adjusted.

Dynamic (Reserve) Price Discrimination The large amount of data available to SSPs allows sellers of advertising space to personalize the reserve prices at the individual bidder level. For example, Google Ad Manager provides explicit resources for personalization. In particular, it allows sellers to "[...] identify specific buyers, advertisers, or brands and associate a minimum CPM or target CPM with them. [The seller] can apply multiple pricing rules for a given ad [auction]."[4] Theoretical and computational issues around the design of personalized reserves are discussed, for example, in Paes Leme, Pal, and Vassilvitskii (2016). In particular, setting the optimal (static) reserve prices requires knowledge of the underlying distribution of bidders' values. Not surprisingly then, the dynamic adjustment of reserve prices as the sellers attempt to learn the bidders' demands has attracted considerable attention from researchers and practitioners alike.[5]

Sellers who participate in a SSP receive information that allows them to tailor reserve

[3] In the context of search advertising, Goldfarb and Tucker (2011) report that only in 6.5% of the auctions in their sample were there sufficiently many bidders. A more recent paper by Beyhaghi, Golrezaei, Paes Leme, Pal, and Siva (2018) suggests that the market is dominated by a small number of large bidders.

[4] See https://support.google.com/admanager/answer/2913506?hl=en.

[5] For example, according to the Rubicon Project (a leading SSP), "new floors are constantly set and re-set for the inventory in response to changes in demand." See https://rubiconproject.com/blog/using-dynamic-price-floors-to-protect-publisher-value/.

prices and to adjust them over time. As in our model, this information often consists of low-dimensional summary statistics about individual bidders' past behavior, rather than elaborate bid-history data. For example, Hummel (2018) notes that "If a publisher signs up for an online ad network such as the Google Display Network,[6] that publisher will be given reports about revenue and the number of ads shown that the publisher could then use in refining reserve prices. However, the publisher would not be told the individual bids that the advertisers had made in each auction." Consequently, Hummel (2018) assumes in his model that "the seller can only condition the reserve price on statistics about revenue and the fraction of the time that an advertiser's ad was shown, rather than allowing the seller to condition the reserve price on the entire history of bids."

The easiest protocol to update reserve prices exploits the dominant-strategy property of the second-price auction: the seller can simply estimate the distribution of values based on any available information about the buyers' bids and then set the optimal Myersonian reserve price. Of course, while this approach is tempting, bidding one's true valuation in a second-price auction is no longer a dominant strategy with a dynamic reserve price. In particular, a bidder's dynamic incentives depend on the updating process for their personalized reserve price. A large body of recent work in computer science and operations research focuses on diagnosing and solving this problem. For example, Amin, Rostamizadeh, and Syed (2014) worry that bidders will shade down their bids "forgoing short-term surplus in order to trick the algorithm into setting better prices in the future," and Golrezaei, Lin, Mirrokni, and Nazerzadeh (2017) claim, "[...] mitigating the negative impact of strategic behavior of bidders, even in the second-price auction, is still an open problem."

This is not just a matter of theory: there is ample evidence of buyers' strategic behavior in online advertising auctions, which we turn to next.

Buyers' Strategic Behavior Online advertisers are sophisticated players who act strategically. This was first shown by Edelman and Ostrovsky (2007), who documented Edgeworth cycles in (what were at the time) first-price auctions for keyword search results. More recently, an empirical study by Yuan, Wang, Chen, Mason, and Seljan (2014) on reserve price optimization shows that bidders understand the mechanism, as static reserve prices in second-price auctions do not affect their behavior. However, advertisers do shade their bids when reserve prices adjust dynamically. Thus, the ratchet effect is a first-order concern for sellers in these markets.

The potential for manipulation has only become more salient over time, as advertisers

[6]A collection of websites where Google places display advertising that includes proprietary as well as external partner sites.

employ automated bidding strategies. These strategies can be developed in-house or delegated to a demand-side platform (DSP) that manages several advertisers' campaigns to optimize their returns. Using machine-learning techniques to test different bids on different "impressions" (combinations of consumers and web pages with certain characteristics), the algorithms employed by major bidders and DSPs are quickly able to detect intertemporal links across prices and deviate from truthful bidding.

Recent research in this area (Lahaie, Munoz Medina, Sivan, and Vassilvitskii, 2018) develops tests based on bid perturbations that a buyer can use to identify the relationship, if any, between past bids and future reserve prices. Among practitioners, the French demand-side platform Criteo advertises its ability to reduce bids when the (static) second-price auction dynamics are manipulated by the seller, "When the engine anticipates manipulation of second-price auction mechanics, Criteo will reduce bid prices across all bids until [these mechanics] are restored."[7] These demand-shading techniques are explicitly described in the paper by Abeille, Calauzènes, Karoui, Nedelec, and Perchet (2018).

Sellers' Equilibrium Responses In response to ratcheting concerns, a large body of work focuses on the problem of incentive-aware learning (Golrezaei, Javanmard, and Mirrokni, 2018), where the seller is trying to both learn both the distribution of values and optimize reserve prices in real time against strategic bidders. Some of the suggested solutions to the problem include the "bank accounts" mechanism, which allows a bidder to trade off current losses (relative to the optimal amount of bid shading) with lower future reserves (Mirrokni, Paes Leme, Tang, and Zuo, 2016). Other solutions consist of conditioning reserve prices on coarser histories, e.g., average bids only. All of these are instances of scoresâ namely, scalars used for learning and pricing.

Kanoria and Nazerzadeh (2017) suggest a different solution, i.e., to use other bidders' behavior and to exploit the correlation structure in their underlying valuations when setting personalized reserves. One may think this clever solution completely solves the incentives problem in general. Recent evidence of coordinated bids, however, seems to undermine this argument (Decarolis, Goldmanis, and Penta, 2017). In particular, the use of common DSPs implies that several advertisers' algorithms will internalize some of the externalities involved in competitive bidding and optimally coordinate instead.

Closest to the message of our paper, Hummel (2018) studies repeated auctions in which the seller tries to optimize future reserve prices by learning the distribution of bidders' valuations. He shows conditions under which the auctioneer benefits by "giving up" on dynamic

[7] See the Rubicon Project's white paper "Maintaining the Equilibrium: How Dynamic Price Floors Preserve the Integrity of the Automated Advertising Ecosystem," available from the authors upon request.

floors, setting reserve prices according to the worst-case distribution of values instead. This result is in line with the optimality of uninformative scores (Proposition 7 in the paper), in which the seller can do no worse in a static game than pricing without information, but giving up on price discrimination can be profitable because it eliminates the ratchet effect.

This is also not just a theoretical result: some SSPs have already adopted this policy. In particular, the Rubicon Project no longer uses discriminatory reserve prices that condition on buyer-level information (i.e., characteristics or past bids). Specifically, the "Rubicon Project's Dynamic Price Floor algorithm applies Myerson's framework and seeks to respect all of the rules and requirements of a second-price auction. [...] It attempts to maximize seller revenue without impacting a buyer's incentive to bid their true value [...] As such, DSP, buyer and advertiser data are not dimensions used in Rubicon Project's Dynamic Price Floor algorithm. This is a key differentiator between Rubicon Project's algorithm and algorithms employed by other exchanges."

Therefore, different SSPs are heterogeneous in their approach to mitigating the ratchet effect. Consistent with our model, some sellers view the value of information as sufficiently high to encourage dynamic personalized pricing, while others prefer to limit the value of strategic behavior. What is more striking is that the Rubicon Project came to this decision explicitly acknowledging the possibility of manipulation: "Using buy-side dimensions when setting price floors [results in] buyer distrust. The impact of buyer distrust may not be immediate, but in the long term, this approach will result in changes in bidding behavior and reduced liquidity."[8]

[8] See the Rubicon Project's white paper "Maintaining the Equilibrium: How Dynamic Price Floors Preserve the Integrity of the Automated Advertising Ecosystem," available from the authors upon request.

S.2 Omitted Proofs

S.2.1 Section 4: Equilibrium Analysis

Theorem 1. The proof is completed after determining the rest of the coefficients and verifying standard transversality conditions.

1. Determination of the remaining coefficients. From the three matching coefficient conditions (system (A.3) in the paper), v_2, v_3 and v_5 are determined using δ, α and β as follows:

$$v_2 = \frac{2\delta\mu}{\lambda},\ v_3 = \frac{\alpha + 2\beta}{2\lambda} > 0, \text{ and } v_5 = \frac{\alpha - 1}{\lambda} < 0.$$

As for v_1 and v_4 (corresponding to θ and θ^2 in the value function) these can be obtained by differentiating the HJB equation with respect to θ. Specifically,

$$(r+\kappa)[v_1 + 2v_4\theta + v_5 M] = (\delta\mu + \alpha\theta + \beta M)[1 + v_5\lambda] - v_5\phi\left[M - \mu + \lambda\bar{Y}\right] - 2v_4\kappa(\theta - \mu)$$

leads to the additional equations

$$\begin{aligned} 2(r+\kappa)v_4 &= \alpha \cdot \underbrace{[1+\lambda v_5]}_{=\alpha;\text{ system }(A.3)} - 2v_4\kappa \Rightarrow v_4 = \frac{\alpha^2}{2(r+2\kappa)}, \quad \text{and,} \\ (r+\kappa)v_1 &= \delta\mu\alpha + v_5\phi(\mu - \lambda\bar{Y}) \Rightarrow v_1 = \frac{\delta\mu\alpha}{r+\kappa} + \frac{\phi(\mu - \lambda\bar{Y})\alpha\beta}{(r+\kappa+\phi)(r+\kappa)}. \end{aligned}$$

The coefficient v_0 can be found by equating the constant terms in the HJB equation—since the value function is quadratic, there is no constraint on this coefficient.

2. Transversality Conditions and Admissibility of the Candidate Equilibrium Strategy. Recall that the candidate value function is of the form

$$V(\theta, M) = v_0 + v_1\theta + v_2 M + v_3 M^2 + v_4\theta^2 + v_5\theta M.$$

Let $X_t := (\theta_t, M_t)$, $t \geq 0$. While the initial condition of $X := (X_t)_{t\geq 0}$ in the game is random, verifying the optimality of consumer's strategy requires evaluating payoffs at all histories of X_t, $t \geq 0$, i.e., at all possible realizations $X_t = x$, where $x \in \mathbb{R}^2$ is deterministic. Thus, let $(X_t^{Q,x})_{t\geq 0}$ denote the dynamic of X under an admissible strategy $Q := (Q_t)_{t\geq 0}$ when the initial condition is $x = (\vartheta, m) \in \mathbb{R}^2$. In this proof, the expectation operator $\mathbb{E}_0[\cdot]$ conditions on this realized value, and the corresponding variance and covariance operators are also indexed by 0.

By Theorem 3.5.3 in Pham (2009), the transversality conditions to verify are:

1. For every $x \in \mathbb{R}^2$, and any admissible strategy Q, $\limsup_{t\to\infty} e^{-rt}\mathbb{E}_0[V(X_t^{Q,x})] \geq 0$.

2. For every $x \in \mathbb{R}^2$, $\liminf_{t\to\infty} e^{-rt}\mathbb{E}_0[V(X_t^{\hat{Q},x})] \leq 0$ where $\hat{Q}_t = \delta\mu + \alpha\theta_t^{\vartheta} + \beta M_t^{\hat{Q},x}$, $t \geq 0$.

We proceed in two lemmas.

Lemma 1. *For any admissible strategy Q,*

$$\lim_{t\to\infty} e^{-rt}\mathbb{E}_0[M_t^{Q,x}] = \lim_{t\to\infty} e^{-rt}\mathbb{E}_0[\theta_t^{\vartheta}] = \lim_{t\to\infty} e^{-rt}\mathbb{E}_0[(\theta_t^{\vartheta})^2] = 0.$$

Also, $\limsup_{t\to\infty} e^{-rt}\mathbb{E}_0[(M_t^{Q,x})^2] < 0$.

Proof: Let $x = (\vartheta, m) \in \mathbb{R}^2$. That $\lim_{t\to\infty} e^{-rt}\mathbb{E}_0[\theta_t^{\vartheta}] = \lim_{t\to\infty} e^{-rt}\mathbb{E}_0[(\theta_t^{\vartheta})^2] = 0$ follows directly from $(\theta_t^{\vartheta})_{t\geq 0}$ being mean-reverting, as this implies that both the mean and variance of θ_t are bounded. To see that $\lim_{t\to\infty} e^{-rt}\mathbb{E}_0[M_t^{Q,x}] = 0$, observe first that

$$M_t^{Q,x} = me^{-\phi t} + [\mu - \phi\lambda\bar{Y}][1 - e^{-\phi t}] + \int_0^t e^{-\phi(t-s)}\lambda Q_s ds + \int_0^t e^{-\phi(t-s)}\sigma_\xi dZ_t^{\xi},\ t \geq 0.$$

Thus, it suffices to show that the transversality condition holds for $J_t := \int_0^t e^{-\phi(t-s)}\lambda Q_s ds$, as the rest of the terms trivially vanish in the limit. By Cauchy-Schwarz, however,

$$\begin{aligned} e^{-rt}\mathbb{E}_0[J_t] &\leq \left(e^{-rt}\int_0^t e^{-2\phi(t-s)}\lambda^2 ds\right)^{1/2}\left(e^{-rt}\int_0^t \mathbb{E}_0[Q_s^2]ds\right)^{1/2} \\ &\leq \underbrace{\left(e^{-rt}\int_0^t e^{-2\phi(t-s)}\lambda^2 ds\right)^{1/2}}_{\to 0 \text{ as } t\to\infty}\underbrace{\left(\mathbb{E}_0\left[\int_0^\infty e^{-rs}Q_s^2 ds\right]\right)^{1/2}}_{C(Q):=}. \end{aligned}$$

We claim that $C(Q) < \infty$ for any admissible strategy. Let V^Q the corresponding payoff. By (iii) in the notion of admissibility, we can separate this payoff into

$$V^Q = \mathbb{E}_0\left[\int_0^\infty e^{-rt}[\theta_t Q_t - Q_t^2/2]dt\right] - \mathbb{E}_0\left[\int_0^\infty e^{-rt}P_t Q_t dt\right],$$

which are both finite. Also, since the first term is integrable, Fubini's Theorem applies, and thus ,

$$\mathbb{E}_0\left[\int_0^\infty e^{-rt}[\theta_t Q_t - Q_t^2/2]dt\right] = \int_0^\infty e^{-rt}[\mathbb{E}_0[\theta_t Q_t] - \mathbb{E}_0[Q_t^2]/2]dt.$$

Moreover, by Tonelli, $C(Q) = \int_0^\infty e^{-rt}\mathbb{E}_0[Q_t^2]dt$. But since $\mathbb{E}_0[\int_0^T Q_t^2 dt] < +\infty$ for all $T > 0$ (part (ii) in the definition of an admissible strategy), $C(Q) = +\infty$ implies that $\mathbb{E}_0[Q_t^2] = O(e^{\rho t})$ for $\rho \geq r$ for large t. Using that $\mathbb{E}_0[\theta_t Q_t] < (\mathbb{E}_0[\theta_t^2])^{1/2}(\mathbb{E}_0[Q_t^2])^{1/2}$, and that $\mathbb{E}_0[\theta_t^2]$ is bounded due to mean reversion, we deduce that the tail $\int_T^\infty e^{-rt}[\mathbb{E}_0[\theta_t Q_t] - \mathbb{E}_0[Q_t^2]/2]dt$ cannot converge, and so the consumer's payoff is $-\infty$, a contradiction. It follows that $\lim_{t\to\infty} e^{-rt}\mathbb{E}_0[J_t] = 0$, and hence, that $\lim_{t\to\infty} e^{-rt}\mathbb{E}_0[M_t^{Q,x}] = 0$.

To conclude, observe that in $(M_t^{Q,x})^2$ the only non-trivial terms are

$$K_t := \left(\int_0^t e^{-\phi(t-s)}Q_s ds\right)^2 \quad \text{and} \quad L_t := \int_0^t e^{-\phi(t-s)}Q_s ds \int_0^t e^{-\phi(t-s)}dZ_t^\xi.$$

However,

$$e^{-rt}\mathbb{E}_0[K_t] \leq e^{-rt}\int_0^t e^{-2\phi(t-s)}ds\mathbb{E}_0\left[\int_0^t Q_s^2 ds\right] \leq \frac{1-e^{-2\phi t}}{2\phi}C(Q) < \frac{C(Q)}{2\phi}.$$

Also, by the same logic

$$e^{-rt}[L_t] \leq (e^{-rt}\mathbb{E}_0[K_t])^{1/2}\left(e^{-rt}\mathbb{E}_0\left[\left(\int_0^t e^{-\phi(t-s)}dZ_t\right)^2\right]\right)^{1/2} \leq \left(\frac{C(Q)}{2\phi}\frac{e^{-rt}[1-e^{-2\phi t}]}{2\phi}\right)^{1/2}.$$

Thus, $\lim_{t\to\infty} e^{-rt}\mathbb{E}_0[L_t] = 0$, from where $\limsup_{t\to\infty} e^{-rt}\mathbb{E}_0[(M_t^{Q,x})^2] < \infty$.

Lemma 2. *(a) Under any admissible strategy Q, $\limsup_{t\to\infty} \mathbb{E}_0[V(X_t^{Q,x})] \geq 0$. (b) Under the candidate equilibrium strategy $\hat{Q}$, $\lim_{t\to\infty} e^{-rt}\mathbb{E}_0[V(X_t^{\hat{Q},x})] = 0$. (c) $(\hat{Q}_t)_{t\geq 0}$ is admissible.*

Proof. To prove (a), we first show that $\lim_{t\to\infty} e^{-rt}\mathbb{E}_0[v_5\theta_t^x M_t^{Q,x}] = 0$ for any admissible Q. Observe first that, by Cauchy-Schwarz,

$$|e^{-rt}\mathbb{E}_0[\theta_t^x M_t^{Q,x}]| \leq \underbrace{(e^{-rt}\mathbb{E}_0[(\theta_t^x)^2])^{1/2}}_{f(t):=}\underbrace{(e^{-rt}\mathbb{E}_0[(M_t^{Q,x})^2])^{1/2}}_{g(t):=}.$$

Since $f, g > 0$, we have that $0 \leq \limsup_{t\to\infty} fg \leq \limsup_{t\to\infty} f \limsup_{t\to\infty} g$. But $\limsup_{t\to\infty} g < \infty$ and $\limsup_{t\to\infty} f = \lim_{t\to\infty} f = 0$, and so $\limsup_{t\to\infty} |e^{-rt}\mathbb{E}_0[\theta_t^x M_t^{Q,x}]| = 0$. However, it is easy to see that $\lim_{t\to\infty} e^{-rt}\mathbb{E}_0[\theta_t^x M_t^{Q,x}] = 0$ if and only if $\limsup_{t\to\infty} |e^{-rt}\mathbb{E}_0[\theta_t^x M_t^{Q,x}]| = 0$.

With this in hand, and using the previous lemma,

$$\lim_{t\to\infty} e^{-rt}\mathbb{E}_0[v_0 + v_1\theta_t^\vartheta + v_2 M_t^{Q,x} + v_4(\theta_t^\vartheta)^2 + v_5\theta_t^\vartheta M_t^{Q,x}] = 0.$$

Thus, $\limsup_{t\to\infty} e^{-rt}\mathbb{E}_0[V(X_t^x)] = \limsup_{t\to\infty} e^{-rt}\mathbb{E}_0[v_3(M_t^{Q,x})^2]$. However, the last term is non-negative due to $v_3 = (\alpha+2\beta)/2\lambda > 0$.

To show (b), observe that under the candidate equilibrium strategy,

$$dM_t = [-\phi M_t + \rho + \lambda(\alpha\theta_t + \beta M_t)]dt + \lambda\sigma_\xi dZ_t^\xi,\ t \geq 0,$$

where $\rho = \phi[\mu - \lambda\bar{Y}] + \lambda\delta\mu$. Thus, the dynamics of $(\theta_t, M_t)_{t\geq 0}$ admit a solution given by

$$\begin{aligned}
\theta_t^\vartheta &= e^{-\kappa t}\vartheta + \mu[1-e^{-\kappa t}] + \sigma_\theta^2\int_0^t e^{-\kappa(t-s)}dZ_s, \quad \text{and} \\
M_t^{\hat{Q},x} &= e^{-(\phi-\beta\lambda)t}m + \rho\frac{1-e^{(\phi-\beta\lambda)t}}{\phi-\beta\lambda} + \lambda\alpha\int_0^t e^{-(\phi-\beta\lambda)(t-s)}\theta_s^\vartheta ds + \sigma_\xi\int_0^t e^{-(\phi-\beta\lambda)(t-s)}dZ_t^\xi.
\end{aligned}$$

Since $\mathbb{E}_0[\theta_t]$ is bounded over $t\in\mathbb{R}_+$, so is $\mathbb{E}_0[M_t^{\hat{Q},x}]$. Also, $\mathrm{Var}_0[\theta_t] = \sigma_\theta^2[1-e^{-2\kappa t}]/2\kappa$. We conclude that $\lim_{t\to\infty} e^{-rt}\mathbb{E}_0[\Psi_t] = 0$ for $\Psi_t \in \{\theta_t, M_t^{\hat{Q},x}, \theta_t^2\}$. Furthermore,

$$\mathrm{Cov}_0[\theta_t^\vartheta, M_t^{\hat{Q},x}] = \lambda\alpha\int_0^t e^{-(\phi-\beta\lambda)(t-s)}\mathrm{Cov}_0[\theta_t^\vartheta, \theta_s^\vartheta]ds$$

where $\mathrm{Cov}_0[\theta_t^\vartheta,\theta_s^\vartheta] = \frac{\sigma_\theta^2}{2\kappa}[e^{-\kappa(t-s)} - e^{-\kappa(t+s)}]$, $t\geq s$. Thus, $\lim_{t\to\infty} e^{-rt}\mathbb{E}_0[\theta_t^\vartheta M_t^{\hat{Q},x}] = 0$.

Now, $\mathrm{Var}_0\left[\int_0^t e^{-(\phi-\beta\lambda)(t-s)}dZ_s^\xi\right] = [1-e^{-2(\phi-\beta\lambda)t}]/[2(\phi-\beta\lambda)]$, which is bounded. Also,

$$\begin{aligned}
\mathrm{Var}_0\left[\int_0^t e^{-(\phi-\beta\lambda)(t-s)}\theta_s^\vartheta ds\right] &= \mathrm{Cov}_0\left[\int_0^t e^{-(\phi-\beta\lambda)(t-s)}\theta_s^\vartheta ds, \int_0^t e^{-(\phi-\beta\lambda)(t-s)}\theta_s^\vartheta ds\right] \\
&= e^{-2(\phi-\beta\lambda)t}\int_0^t\int_0^t e^{(\phi-\beta\lambda)(u+v)}\mathrm{Cov}_0[\theta_u^\vartheta,\theta_v^\vartheta]dudv,
\end{aligned}$$

where the last equality follows from integrability and Fubini's Theorem. Since

$$\mathrm{Cov}_0[\theta_t^\vartheta,\theta_s^\vartheta] = \frac{\sigma_\theta^2}{2\kappa}[e^{-\kappa(\max\{u,v\}-\min\{u,v\})} - e^{-\kappa(u+v)}],$$

it is easy to verify that $\mathrm{Var}_0\left[\int_0^t e^{-(\phi-\beta\lambda)(t-s)}\theta_s^\vartheta ds\right]$ is also bounded, and so $\mathrm{Var}_0[M_t^{Q,x}]$ is bounded. We conclude that $\lim_{t\to\infty} e^{-rt}\mathrm{Var}_0[M_t^{\hat{Q},x}] = \lim_{t\to\infty} e^{-rt}\mathbb{E}_0[(M_t^{\hat{Q},x})^2] = 0$.

Finally, to show (c), observe that $(\theta_t^\vartheta - \hat{P}_t)\hat{Q}_t - (\hat{Q}_t)^2/2$, where $\hat{P}_t := \delta\mu + (\alpha+\beta)M_t^{\hat{Q},x}$, $t\geq 0$, is quadratic in $(\theta_t^\vartheta, M_t^{\hat{Q},x})$. Thus, there is $C>0$ large enough such that $|(\theta_t^\vartheta\hat{Q}_t - (\hat{Q}_t)^2/2| + |\hat{P}_t\hat{Q}_t| \leq C[1 + (\theta_t^\vartheta)^2 + (M_t^{\hat{Q},x})^2]$. But from the previous arguments, the second moments $\mathbb{E}_0[(\theta_t^\vartheta)^2]$ and $\mathbb{E}_0[(M_t^{\hat{Q},x})^2]$ are bounded over $\mathbb{R}_+$, and hence, (iii) in the admissibility requirement holds. It is easy to see that (ii) holds via an identical argument, and observe that we already showed that the controlled dynamics admit a solution under the candidate equilibrium strategy. This concludes the proof. □

S.2.2 Section 5: Equilibrium Learning

Proof of Proposition 3. Part (i) is proved in section S.2.5 as part of the proof of Lemma A.5 stated in the paper. As for (ii), suppose that $\xi^t := (\xi_s : 0 \leq s < t)$ is observed by firm t, and let $M_t^* := \mathbb{E}[\theta_t|\mathcal{F}_t^\xi]$, $t \geq 0$, where $(\mathcal{F}_t^\xi)_{t\geq 0}$ denotes the filtration generated by $(\xi_t)_{t\geq 0}$. When the quantity demanded follows $Q_t = \delta\mu + \alpha\theta_t + \beta M_t^*$, recorded purchases obey

$$d\xi_t = (\delta\mu + \alpha\theta_t + \beta M_t^*)dt + \sigma_\xi dZ_t^\xi,$$

where $(M_t^*)_{t\geq 0}$ satisfies the filtering equation

$$dM_t^* = -\kappa(M_t^* - \mu)dt + \frac{\alpha\gamma(\alpha)}{\sigma_\xi^2}[d\xi_t - (\delta\mu + [\alpha + \beta]M_t^* dt)].$$

In this SDE, $\gamma(\alpha)$ is the unique positive solution to $x \mapsto -2\kappa x + \sigma_\theta^2 - (\alpha x/\sigma_\xi)^2 = 0$ (Theorem 12.1 in Liptser and Shiryaev 1977). As a function of $(Z_t^\theta, Z_t^\xi)_{t\geq 0}$, therefore,

$$dM_t^* = \left(-\left[\kappa + \frac{\alpha^2\gamma(\alpha)}{\sigma_\xi^2}\right] M_t^* + \kappa\mu + \frac{\alpha^2\gamma(\alpha)}{\sigma_\xi^2}\theta_t\right) dt + \frac{\alpha\gamma(\alpha)}{\sigma_\xi} dZ_t^\xi.$$

Now, let

$$\lambda^* = \frac{\alpha\gamma(\alpha)}{\sigma_\xi^2} \quad \text{and} \quad \rho^* = \frac{1}{\nu(\alpha,\beta)}\left(\kappa\mu - \frac{\alpha\gamma(\alpha)\delta\mu}{\sigma_\xi^2}\right) \tag{S.1}$$

where $\nu(\alpha, \beta)$ satisfies (18) in the paper, i.e., $\nu(\alpha, \beta) := \kappa + \alpha\gamma(\alpha)[\alpha + \beta]/\sigma_\xi^2 > 0$. In particular, observe that $dM_t^* = [-(\kappa + \lambda^*\alpha)M_t^* + \kappa\mu + \lambda^*\alpha\theta_t]dt + \lambda\sigma_\xi dZ_t^\xi$.

With this in hand, consider $(Y_t)_{t\geq 0}$ evolving according to

$$dY_t = [-\nu(\alpha, \beta)Y_t + \delta\mu + \beta\rho^* + \alpha\theta_t + \beta\lambda^* Y_t]dt + \sigma_\xi dZ_t^\xi.$$

From the proof of Lemma A.1 in the paper, if (Y_0, θ_0) is independent of $(Z_t^\theta, Z_t^\xi)_{t\geq 0}$ and

$$\mathbb{E}[Y_0] = \frac{\delta\mu + \beta\rho^* + \alpha\mu}{\nu(\alpha,\beta) - \beta\lambda^*},\ \text{Var}[Y_0] = \frac{1}{2(\nu(\alpha,\beta) - \beta\lambda^*)}\left[\sigma_\xi^2 + \frac{\alpha^2\sigma_\theta^2}{\kappa(\nu(\alpha,\beta) - \beta\lambda^* + \kappa)}\right] \text{ and}$$
$$\text{Cov}[\theta_0, Y_0] = \frac{\alpha\sigma_\theta^2}{2\kappa(\nu(\alpha,\beta) - \beta\lambda^* + \kappa)},$$

the pair $(\theta_t, Y_t)_{t\geq 0}$ is stationary Gaussian, as $\phi - \beta\lambda^* = \nu(\alpha,\beta) - \beta\alpha\gamma(\alpha)/\sigma_\xi^2 = \kappa + \alpha^2\gamma(\alpha)/\sigma_\xi^2 > 0$. Denote the previous process by $(Y^{\nu(\alpha,\beta)})_{t\geq 0}$, and note that

$$Y_t = e^{-\nu(\alpha,\beta)t}Y_0 + \int_0^t e^{-\nu(\alpha,\beta)(t-s)}d\xi_s\ t \geq 0.$$

Defining $X_t = \rho^* + \lambda^* Y_t^{\nu(\alpha,\beta)}$, it is easy to verify that

$$\begin{aligned} dX_t &= [\lambda^*(\delta\mu + \alpha\theta_t + \beta X_t) - \nu(\alpha,\beta)[X_t - \rho^*]] + \lambda^*\sigma_\xi dZ_t^\xi \\ &= [-(\kappa + \lambda^*\alpha)X_t + \kappa\mu + \lambda^*\alpha\theta_t]dt + \lambda^*\sigma_\xi dZ_t^\xi, \end{aligned}$$

where in the last equality we used that $\nu(\alpha,\beta) = \kappa + \lambda^*(\alpha+\beta)$ and that $\lambda^*\delta\mu + \nu(\alpha,\beta) = \mu\kappa$. We conclude that $M_t^* - X_t$ satisfies $d[M_t^* - X_t] = -(\kappa + \lambda^*\alpha)[M_t^* - X_t]dt$, and therefore that $M_t^* - X_t = [M_0 - X_0]e^{-(\kappa+\lambda^*\alpha)t}$ for all $t \geq 0$.

Notice, however, that since $(X_t)_{t\geq 0}$ is stationary, stationarity of $(M_t^*)_{t\geq 0}$ implies that $M_0^* - X_0 \equiv 0$ a.s. To see this, notice first that $M_0^* - X_0$ cannot be random: otherwise, the constraint that $\text{Var}[M_t]$ must be independent of time becomes

$$\underbrace{\text{Var}[X_t]}_{\text{independent of } t} + e^{-2[\kappa+\lambda^*\alpha]t}\text{Var}[M_0^* - X_0] + 2e^{-[\kappa+\lambda^*\alpha]t}\underbrace{\text{Cov}[X_t, M_0^* - X_0]}_{\text{independent of } t} = \text{constant},$$

which cannot hold for all $t \geq 0$. Thus, $M_0 - X_0 = C \in \mathbb{R}$. From here, however, $C = 0$, as the requirement that $\mathbb{E}[M_t^*]$ is independent of time would be violated otherwise. Consequently, if beliefs are stationary,

$$M_t^* = X_t = \rho^* + \lambda^* Y_t^{\nu(\alpha,\beta)} = \left[\frac{1}{\nu(\alpha,\beta)}\left(\kappa\mu - \frac{\alpha\gamma(\alpha)\delta\mu}{\sigma_\xi^2}\right)\right] + \frac{\alpha\gamma(\alpha)}{\sigma_\xi^2}Y_t^{\nu(\alpha,\beta)}, \text{ for all } t \geq 0.$$

To prove the converse, consider $(\theta_t, Y_t)_{t\geq 0}$ as in Lemma A.1 with $\phi = \nu(\alpha,\beta)$. We aim to show that $(M_t)_{t\geq 0}$ coincides with $(M_t^*)_{t\geq 0}$ path-by-path of $(Y_t)_{t\geq 0}$. In fact, because $M_t = \mu + \lambda[Y_t - \bar{Y}]$, with λ and $\bar{Y}$ as in Lemma A.1, the task reduces to showing that, $\lambda = \lambda^*$ and $\mu - \lambda\bar{Y} = \rho^*$ when $\phi = \nu(\alpha,\beta)$, where λ^* and ρ^* are defined in (S.1).

With $\alpha > 0$, inspection of (7) in the paper reveals that the stationarity condition $\phi - \beta\lambda > 0$ implies that $\lambda > 0$. Thus, $\lambda = \Lambda(\phi, \alpha, \beta)$, where the right-hand side is defined in (A.8) in the paper. The equality $\lambda = \lambda^*$ then follows directly from (ii) in Lemma A.5 in the paper (the proof of which can be found in Section S.2.5 in this Appendix).

To show the second equality, recall that $\bar{Y} = \mu[\alpha + \beta + \delta]/\phi$. Thus, when $\phi = \nu(\alpha, \beta) = \kappa + \lambda^*(\alpha + \beta)$ and $\lambda = \lambda^*$,

$$\mu - \lambda\bar{Y} = \frac{\mu\nu(\alpha,\beta) - \lambda\mu[\alpha + \beta + \delta]}{\nu(\alpha,\beta)} = \frac{\mu\kappa - \lambda^*\delta\mu}{\nu(\alpha,\beta)} = \rho^*.$$

This concludes the proof. □

Now we establish a stronger result than (iv) in Proposition 4 (Equilibrium Learning) in section 5 in the paper.

Effect of noise—public histories case. $\sigma_\xi \mapsto G(\phi^*(\sigma_\xi); \sigma_\xi)$ is decreasing for $\sigma_\xi \in (0, \infty)$.

Proof: Fix $\sigma_\xi > 0$, and recall that $\phi^*(\sigma_\xi)$ is the non-concealing score defined as a solution to (19) in the paper. Using that

$$\beta = -\frac{\alpha^2(r + 2\phi)}{2(r + 2\phi)\alpha - (r + \kappa + \phi)(\alpha - 1)} \quad \text{and} \quad \gamma(\alpha) = \frac{\alpha^2(\sqrt{\kappa^2 + \alpha^2\sigma_\theta^2/\sigma_\xi^2} - \kappa)}{\sigma_\xi^2}$$

we obtain that $\nu := \kappa + \frac{\alpha\gamma(\alpha)(\alpha+\beta)}{\sigma_\xi^2}$ can be written as

$$\nu = \kappa - \frac{\left(\kappa\sqrt{s} - \sqrt{\alpha^2 + \kappa^2 s}\right)(-\alpha\kappa + \alpha\phi + \kappa + r + \phi)}{\sqrt{s}(\kappa + \alpha(-\kappa + r + 3\phi) + r + \phi)},$$

where $s := \sigma_\xi^2/\sigma_\theta^2$. We then solve the fixed point equation $\nu = \phi$ for s, and we obtain

$$s^*(\alpha, \phi) = \frac{\alpha^2(-\alpha\kappa + \alpha\phi + \kappa + r + \phi)^2}{(\phi - \kappa)(\kappa + \alpha(-\kappa + r + 3\phi) + r + \phi)(\alpha(\phi - \kappa)(\kappa + r + 3\phi) + (\kappa + \phi)(\kappa + r + \phi))}.$$

Thus, a given pair (α, ϕ) characterizes the equilibrium with publicly observable signals for $s = s^*$. We now compare s^* with the expression that can be obtained by solving for s in the equilibrium condition (A.19) in the paper, i.e.,

$$s = -\frac{\alpha^3\left(\alpha^2\left(-\kappa(\kappa + r) + 2r\phi + 3\phi^2\right) + \alpha(2\kappa + r)(\kappa + r + \phi) - (\kappa + r + \phi)^2\right)}{(\alpha - 1)\kappa(\kappa + r + \phi)(\kappa - \alpha r + r + \phi)(\kappa + \alpha(-\kappa + r + 3\phi) + r + \phi)}.$$

By setting $s = s^*$ and solving for α and selecting the only root between 0 and 1, we obtain an expression for the equilibrium coefficient α in the case of public signals in terms of the persistence level ϕ^* only:

$$\alpha^* = \frac{2(\rho + \phi^* + 1)}{\sqrt{(\rho + 2\phi^*)(\rho + 6\phi^* - 4)} + \rho + 2}.$$

Finally,using that $G(\phi, \alpha, \beta) = \tilde{G}(\phi, \alpha)$ with $\tilde{G}$ as in (A.22), and letting $\rho := r/\kappa$, we obtain

$$G(\phi^*) = \frac{\sqrt{(\rho + 2\phi^*)(\rho + 6\phi^* - 4)} - \rho - 2}{\sqrt{(\rho + 2\phi^*)(\rho + 6\phi^* - 4)} - \rho + 2},$$

which is increasing in ϕ^*. Finally, it is easy to show that ϕ^* is decreasing in σ_ξ: this is obtained by substituting $\gamma = (1 - G)\sigma_\theta^2/2\kappa$ and previous expression for β into the fixed point condition $\nu = \phi$, and showing that the left-hand side is decreasing in both ϕ and σ_ξ. Therefore, G is itself decreasing in σ_ξ. □

S.2.3 Section 6: Welfare Analysis

Expression for Consumer Surplus. In this section, expectation (and hence, variance and covariance) operators are with respect to the prior distribution of (θ_0, Y_0).

Recall that, in equilibrium,

$$\mathbb{E}[\theta_t] = \mu, \ \text{Var}[\theta_t] = \frac{\sigma_\theta^2}{2\kappa}, \ M_t = \mu + \lambda[Y_t - \bar{Y}], \ \lambda := \frac{\text{Cov}[\theta_t, Y_t]}{\text{Var}[Y_t]}, \text{ and } G(\phi) = \frac{\text{Cov}[\theta_t, Y_t]^2}{\text{Var}[Y_t]\text{Var}[\theta_t]}.$$

Thus,

$$\mathbb{E}[\theta_t M_t] = \mathbb{E}[M_t^2] = \text{Var}[M_t] + \mu^2 = \frac{\text{Cov}[\theta_t, Y_t]^2}{\text{Var}[Y_t]} + \mu^2 = \text{Var}[\theta_t]G(\phi) + \mu^2.$$

Recall that consumer surplus is defined as the consumer's ex ante equilibrium payoff normalized by the discount rate, i.e., $CS(\phi) := \mathbb{E}[Q_t(\theta_t - P_t - Q_t/2)]$, where $P_t = \delta(\phi)\mu + [\alpha(\phi) + \beta(\phi)]M_t$ and $Q_t = \delta(\phi)\mu + \alpha(\phi)\theta_t + \beta(\phi)M_t$. Omitting the dependence on ϕ of all equilibrium coefficients, therefore,

$$\begin{aligned} CS(\phi) = \quad & \delta\mu \left[\mathbb{E}[\theta_t]\left(1-\frac{\alpha}{2}\right) - \left(\alpha + \frac{3\beta}{2}\right)\mathbb{E}[M_t] - \frac{3\delta\mu}{2}\right] \\ + \quad & \alpha \left[\mathbb{E}[\theta_t^2]\left(1-\frac{\alpha}{2}\right) - \left(\alpha + \frac{3\beta}{2}\right)\mathbb{E}[M_t\theta_t] - \frac{3\delta\mu}{2}\mathbb{E}[\theta_t]\right] \\ + \quad & \beta \left[\mathbb{E}[\theta_t M_t]\left(1-\frac{\alpha}{2}\right) - \left(\alpha + \frac{3\beta}{2}\right)\mathbb{E}[M_t^2] - \frac{3\delta\mu}{2}\mathbb{E}[M_t]\right]. \end{aligned}$$

Using the above expressions for the first two moments of (θ_t, M_t),

$$\begin{aligned} CS(\phi) = \quad & \frac{\sigma_\theta^2}{2\kappa}G(\phi)\underbrace{\left[-\alpha\left(\alpha+\frac{3\beta}{2}\right)+\beta\left(1-\frac{\alpha}{2}\right)-\beta\left(\alpha+\frac{3\beta}{2}\right)\right]}_{=-3(\alpha+\beta)^2/2+\alpha^2/2+\beta} + \frac{\sigma_\theta^2}{2\kappa}\left[\alpha\left(1-\frac{\alpha}{2}\right)\right] \\ & +\mu^2\underbrace{\left[-\alpha\left(\alpha+\frac{3\beta}{2}\right)+\beta\left(1-\frac{\alpha}{2}\right)-\beta\left(\alpha+\frac{3\beta}{2}\right)+\alpha\left(1-\frac{\alpha}{2}\right)\right]}_{=-3(\alpha+\beta)^2/2+\alpha+\beta} \\ & \underbrace{+\,\delta\mu^2\left[\left(1-\frac{3(\alpha+\beta)}{2}-\frac{3\alpha}{2}-\frac{3\beta}{2}\right)-\frac{3\delta}{2}\right]}_{=\mu^2\delta[(1-3(\alpha+\beta))-3\delta/2]}. \end{aligned}$$

Collecting terms in the last two lines yields

$$\begin{aligned} & \mu^2\left[-3(\alpha+\beta)^2/2+\alpha+\beta+\delta-3\delta(\alpha+\beta)-\frac{3\delta^2}{2}\right] \\ = \quad & \mu^2\left[\alpha+\beta+\delta-\frac{3}{2}\underbrace{\left\{(\alpha+\beta)^2+2\delta(\alpha+\beta)+\delta^2\right\}}_{(\alpha+\beta+\delta)^2}\right] \\ = \quad & \mu^2(\alpha+\beta+\delta)\left[1-\frac{3}{2}(\alpha+\beta+\delta)\right], \\ = \quad & \mathbb{E}[P_t]\left(\mu-\frac{3}{2}\mathbb{E}[P_t]\right). \end{aligned}$$

Since $\text{Var}[P_t] = (\alpha+\beta)^2\text{Var}[M] = (\alpha+\beta)^2\frac{\sigma_\theta^2}{2\kappa}G(\phi)$, we can write the first term in $CS(\phi)$ as $\{\frac{\alpha^2/2+\beta}{(\alpha+\beta)^2} - 3/2\}\text{Var}[P]$, from where we obtain the desired expression for consumer surplus, i.e., (21) in the paper.

Finally, notice that we can write the term that multiplies $\sigma_\theta^2 G(\phi)/2\kappa$ as

$$-\frac{3(\alpha+\beta)^2}{2}+\frac{\alpha^2}{2}+\beta = \underbrace{-\alpha[\alpha+2\beta]}_{<0}+\underbrace{\beta[1-\alpha]}_{<0}-\frac{3\beta^2}{2} < 0.$$

On the other hand, since $-1/2 < \beta < 0$ and $\alpha > 0$, and $0 < \alpha + \beta < 1$,

$$-\frac{3}{2}(\alpha(\phi) + \beta(\phi))^2 + \frac{\alpha(\phi)^2}{2} + \beta(\phi) > 0 - \frac{1}{2} - \frac{3}{2} = -2.$$

This concludes the proof. $\square$

The value of information $\mathrm{Var}[P_t]$ **is maximized to the left of** ϕ^*. Observe that $\mathrm{Var}[P_t] = [\alpha(\phi) + \beta(\phi)]^2 \mathrm{Var}[\theta_t] G(\phi)$. From the proof of Proposition 4, $\lim_{\phi \to 0,\infty} G(\phi) = 0$. Also, $\alpha + \beta$ is bounded. By continuity, we conclude that $\mathrm{Var}[P_t]$ has a global optimum that is interior.

From (iii) in Proposition 4, however, $G(\phi)$ is maximized to the left of ϕ^*. Also, from Lemma A.6, $\alpha'(\phi) + \beta'(\phi) < 0$ over $[\kappa, \arg\min \alpha]$. Since $\kappa < \phi^* < \arg\min \alpha$, $\mathrm{Var}[P_t]$ cannot attain a maximum in $[\phi^*, \arg\min \alpha]$. One can then verify that the total derivative of $\mathrm{Var}[P_t]$ with respect to ϕ is negative over $[\arg\min \alpha, +\infty)$ for all parameter values $(r, \kappa, \sigma_\theta, \sigma_\xi)$. This is done in `scores.nb` posted on our websites. $\square$

Proof of Proposition 5. The final step for proving (i) and (ii) requires demonstrating (A.23), i.e.,

$$\lim_{\phi \to 0,+\infty} \frac{[\alpha(\phi) - 1]^2}{R(\phi)} = 0, \quad \text{and} \quad \lim_{\phi \to 0,+\infty} \frac{G(\phi)}{R(\phi) - 1/8} > 0,$$

where $R(\phi) := [\alpha(\phi) + \beta(\phi) + \delta(\phi)] \left(1 - \frac{3}{2}[\alpha(\phi) + \beta(\phi) + \delta(\phi)]\right)$.

To this end, we can use the expressions (A.7) and (A.12) in Appendix A for β and δ, respectively, to obtain,

$$\begin{aligned} \frac{[\alpha(\phi) - 1]^2}{R(\phi) - 1/8} &= -\frac{8(\alpha - 1)}{(\kappa + r + \phi)(-\alpha\kappa + \alpha\phi + \kappa + r + \phi)} \times \\ &\quad \frac{[\alpha^2(\kappa^2 + r(\kappa + 2r) + 5r\phi + 3\phi^2) - \alpha(2\kappa + r)(\kappa + r + \phi) + (\kappa + r + \phi)^2]^2}{\alpha^2(-\kappa^2 + 4r^2 - \kappa r + 13r\phi + 9\phi^2) + \alpha(2\kappa + r)(\kappa + r + \phi) - (\kappa + r + \phi)^2}. \end{aligned}$$

In addition, we can use expression (A.22) for G to obtain,

$$\begin{aligned} \frac{G(\phi)}{R(\phi) - 1/8} &= \frac{8[\kappa + \alpha(-\kappa + r + 3\phi) + r + \phi]}{\alpha(r + 2\phi)(\kappa - \alpha r + r + \phi)(-\alpha\kappa + \alpha\phi + \kappa + r + \phi)} \times \\ &\quad \frac{[\alpha^2(\kappa^2 + r(\kappa + 2r) + 5r\phi + 3\phi^2) - \alpha(2\kappa + r)(\kappa + r + \phi) + (\kappa + r + \phi)^2]^2}{\alpha^2(-\kappa^2 + 4r^2 - \kappa r + 13r\phi + 9\phi^2) + \alpha(2\kappa + r)(\kappa + r + \phi) - (\kappa + r + \phi)^2}, \end{aligned}$$

where we have omitted the dependence of α on ϕ. Consider now the first expression. To examine the case $\phi \to +\infty$, write $\alpha - 1 = \lambda\alpha\beta/[r + \kappa + \phi]$ in the first ratio. Recalling that $(\alpha, \beta, \lambda) \to (1, -1/2, \sigma_\theta^2/[\kappa\sigma_\xi^2])$ (proof of Proposition 3 in the paper), we have that the numerator is of $O(\phi^4)$ for ϕ large. In contrast, it is easy to see that the denominator is of $O(\phi^5)$. Thus, the first expression converges to zero as $\phi \to +\infty$. Regarding the case $\phi \to 0$, it is easy to see that since $\alpha \to 1$, the denominator converges to $4(\kappa + r)r^3 > 0$, while the numerator converges to zero. Thus, the first expression attains the same value as $\phi \to 0$.

Turning to the second expression, using that $\kappa, r > 0$ and $\alpha \to 1$ as $\phi \to +\infty$, the numerator is $O(\phi^5)$ for ϕ large, where the associated constant is $8 \times 4 \times 16$. Similarly, the denominator is also $O(\phi^5)$ for ϕ large, with constant $2 \times 1 \times 2 \times 8$. Thus, the limit is 16. Finally, when $\phi \to 0$, the denominator converges to $4r^4\kappa$. Instead, the numerator converges to $64r^5$. Thus, the limit is $16r/\kappa > 0$. This concludes the proof. □

S.2.4 Section 7: Hidden Scores

Proof of Proposition 7. The proof parallels the steps followed for proving Theorem 1. If $(\theta_t, Y_t)_{t\geq 0}$ is as in Proposition A.1 (Appendix A in the paper) with $Q_t = \delta^h\mu/2 + \alpha^h\theta_t + \beta^h M_t$ and $\beta^h = -\alpha^h/2$, we have that

$$\lambda^h = \frac{\alpha^h\sigma_\theta^2(\phi - \beta^h\lambda^h)}{(\alpha^h)^2\sigma_\theta^2 + \kappa\sigma_\xi^2(\phi + \kappa - \beta^h\lambda^h)} \quad \text{and} \quad \bar{Y}^h = \frac{\mu[\delta^h/2 + \alpha^h + \beta^h]}{\phi}.$$

In addition, (ii) in the same proposition becomes $\phi - \beta^h\lambda^h = \phi + \lambda^h\alpha^h/2 > 0$.

Observe that $\alpha^h \neq 0$ in equilibrium as well: otherwise, using that $\phi - \beta^h\lambda^h > 0$ in the equation for λ^h implies that $\lambda^h = 0$, and so price is constant—but this leads to a demand with unit weight on the type. Using that $\alpha^h \neq 0$ and $\phi - \beta^h\lambda^h > 0$, the same equation implies that $\lambda^h \neq 0$.

Recalling that $P_t = -\mathbb{E}[Q_t|Y_t]/\zeta^h$, and using that $M_t = \mu + \lambda^h[Y_t - \bar{Y}^h]$,

$$\begin{aligned} dP_t = -\frac{\alpha^h\lambda^h}{2\zeta^h}dY_t &= -\frac{\alpha^h\lambda^h}{2\zeta^h}[(Q_t - \phi Y_t)dt + \sigma_\xi dZ_t^\xi] \\ &= \left[-\frac{\alpha^h\lambda^h}{2\zeta^h}Q_t - \phi\left(P_t + \frac{\delta^h\mu + \alpha^h\rho^h\mu}{2\zeta^h}\right)\right]dt - \frac{\alpha^h\lambda^h}{2\zeta^h}\sigma_\xi dZ_t^\xi, \quad \text{(S.2)} \end{aligned}$$

where $\rho^h := 1 - \lambda^h[\delta^h/2 + \alpha^h + \beta^h]/\phi$. Thus, the consumer's problem is to maximize her utility subject to (S.2) and the law of motion of her type.

We guess a value function $V = v_0 + v_1\theta + v_2P + v_3P^2 + v_4\theta^2 + v_5\theta P$, which gives the first-order condition

$$q = \theta - P - \frac{\alpha^h\lambda^h}{2\zeta^h}\underbrace{[v_2 + 2v_3P + v_5\theta]}_{\partial V/\partial P} = -\frac{\alpha^h\lambda^h}{2\zeta^h}v_2 + \left[1 - \frac{\alpha^h\lambda^h}{2\zeta^h}v_5\right]\theta + \left[-1 - \frac{\alpha^h\lambda^h}{\zeta^h}v_3\right]P.$$

As a result, we obtain the matching-coefficients conditions

$$\delta^h\mu = -\frac{\alpha^h\lambda^h}{2\zeta^h}v_2,\ \ \alpha^h = 1-\frac{\alpha^h\lambda^h}{2\zeta^h}v_5\ \text{ and } \zeta^h = -1-\frac{\alpha^h\lambda^h}{\zeta^h}v_3. \tag{S.3}$$

Moreover, by the Envelope Theorem,

$$(r+\phi)[v_2+2v_3P+v_5\theta] = q\left[-1-v_3\frac{\alpha^h\lambda^h}{\zeta^h}\right]-2v_3\phi\left[P+\frac{\delta^h\mu+\alpha^h\rho^h\mu}{2\zeta^h}\right]-\kappa v_5(\theta-\mu),$$

which leads to the system

$$\begin{aligned}(r+\phi)v_2 &= \delta^h\mu\left[-1-v_3\frac{\alpha^h\lambda^h}{\zeta^h}\right]-2v_3\phi\frac{\delta^h\mu+\alpha^h\rho^h\mu}{2\zeta^h}+\kappa\mu v_5\\ 2(r+\phi)v_3 &= \zeta^h\left[-1-v_3\frac{\alpha^h\lambda^h}{\zeta^h}\right]-2v_3\phi\\ (r+\phi)v_5 &= \alpha^h\left[-1-v_3\frac{\alpha^h\lambda^h}{\zeta^h}\right]-\kappa v_5.\end{aligned}$$

Using that v_2, v_3 and v_5 can be written as a function of $\delta^h\mu, \alpha^h$ and ζ^h, respectively, and dividing by ζ^h in each equation, we obtain the following system

$$\begin{cases}-(r+\phi)\frac{2\delta^h\mu}{\alpha^h\lambda^h} = \delta^h\mu+2\phi\frac{\zeta^h+1}{\lambda^h\alpha^h}\frac{\delta^h\mu+\alpha^h\rho^h\mu}{2\zeta^h}+\kappa\mu\frac{2(1-\alpha^h)}{\alpha^h\lambda^h}\\ -2(r+2\phi)\frac{\zeta^h+1}{\alpha^h\lambda^h} = \zeta^h\\ (r+\phi+\kappa)\frac{2(1-\alpha^h)}{\alpha^h\lambda^h} = \alpha^h.\end{cases} \tag{S.4}$$

Observe that the last equation is independent of the other two. Also, the second equation is linear in ζ^h given α^h, while the first equation is linear in δ^h given ζ^h and α^h. Thus, we can solve for α^h, ζ^h, and δ^h sequentially. We proceed by finding α^h first.

It is immediate that $\alpha^h \in (0,1)$: (i) if $\alpha^h < 0$, the last equation in system (S.4) reads

$$\phi-\beta^h\lambda^h = \phi+\frac{\alpha^h}{2}\lambda^h = (r+\kappa)\left(\frac{1}{\alpha^h}-1\right)+\frac{\alpha^h}{\phi} < 0,$$

which contradicts stationarity; (ii) if $\alpha^h = 1$, the equation for λ^h implies that $\lambda^h > 0$, and so the last equation in (S.4) yields that $\alpha^h = 0$, a contradiction; (iii) and if $\alpha^h > 1$, it follows that $\lambda^h < 0$ from the same equation, but the equation for λ^h yields $\lambda^h > 0$ in a stationary linear Markov equilibrium. Since we already know that $\alpha^h \neq 0$, it follows that $\alpha^h \in (0,1)$, from where we conclude that $\lambda^h > 0$ using again the equation for λ^h.

Since $\alpha^h > 0$ and $\beta^h = -\alpha^h/2 < 0$, the unique possible value for $\lambda^h > 0$ is given by

$\Lambda(\phi, \alpha^h, -\alpha^h/2)$ as defined in (A.8) in the paper. The last equation in (S.4) then reads $A^h(\phi, \alpha^h) = 0$, where

$$A^h(\phi, x) := (r + \kappa + \phi)(x - 1) - x\Lambda(\phi, x, -x/2)\left[-\frac{x}{2}\right],\ (\phi, x) \in (0, \infty) \times [0, 1].$$

The final steps of the proof are as follows:

1. Existence and uniqueness of solution to $A^h(\phi, x) = 0,\ x \in [0, 1]$. Lemma A.3 in Appendix A of the paper (i.e., the existence and uniqueness of $\alpha \in (0, 1)$ s.t. $A(\phi, \alpha) = 0$ in the observable-score case), carries over to this setting. To see this, recall that $A(\phi, x) = (r+\kappa+\phi)(x-1)-x\Lambda(\phi, x, B(\phi, x))B(\phi, x)$, where $B(\phi, x) \in (-x/2, 0)$. It is then easy to verify that the steps that showed that $x \in [0, 1] \mapsto H(\phi, x) := -\Lambda(\phi, x, B(\phi, x))B(\phi, x)$ is strictly increasing also imply that $x \in [0, 1] \mapsto H^h(\phi, x) := -\Lambda(\phi, x, -x/2)[-x/2]$ is strictly increasing. This is because, when $B(\phi, x)$ is replaced by $-x/2$ in H: $B_\alpha(\phi, \alpha)\alpha + B(\phi, \alpha) < 0$ becomes $-\alpha < 0$; $\ell_\alpha(\phi, \alpha) > 0$ becomes $\sigma_\theta^2 \alpha > 0$; and $-\alpha B_\alpha(\phi, \alpha) + B(\phi, \alpha) \geq 0$ becomes $0 \geq 0$, which were the critical steps to prove that H was increasing. That α^h solving $A^h(\phi, \sigma_\xi^2, \alpha) = 0$, $(\phi, \sigma_\xi^2) \in (0, \infty)^2$ (the dependence on σ_ξ^2 made explicit) is of class C^1 follows from an identical argument.

2. Determination of the rest of the coefficients. Returning to ζ^h, it is easy to see from the second equation in (S.4) that

$$\zeta^h = -\frac{2(r + 2\phi)}{\lambda^h \alpha^h + 2(r + 2\phi)} \in (-1, 0),$$

   where the bounds follow from $\alpha^h \lambda^h > 0$.

   Regarding δ^h, observe that the first equation in system (S.4) is trivially satisfied if $\mu = 0$; in this case, the constant term in the demand function is simply zero. When $\mu \neq 0$, we can eliminate μ on both sides to obtain

$$-(r + \phi)\frac{2\delta^h}{\alpha^h \lambda^h} = \delta^h + \phi\frac{\zeta^h + 1}{\zeta^h \lambda^h \alpha^h}[\delta^h + \alpha^h \rho^h] + \kappa\mu\frac{2(1 - \alpha^h)}{\lambda^h \alpha^h}$$

   where $\rho^h = 1 - \lambda^h[\delta^h/2 + \alpha^h + \beta^h]/\phi$. Also, from the second equation in (S.4), $(\zeta^h + 1)/(\zeta^h \lambda^h \alpha^h) = -1/[2(r + 2\phi)]$. Thus, the coefficient that multiplies δ^h in the previous equation is given by

$$\frac{1}{\alpha^h \lambda^h}\left[-2(r + \phi) - \lambda^h \alpha^h + \frac{\phi}{2(r + 2\phi)}\alpha^h \lambda^h - \frac{\phi}{4(r + 2\phi)}(\alpha^h \lambda^h)^2\right].$$

But observe that $\phi\alpha^h\lambda^h/[2(r+2\phi)] \in (0, \alpha^h\lambda^h/4)$, and so the second term dominates the third. We conclude that the previous expression is strictly negative, which implies that the equation for δ^h admits a solution for all parameters.

The rest of the unknowns are determined as follows. First, v_2, v_3 and v_5 are determined from the matching-coefficient conditions (S.3) using δ^h, α^h and ζ^h; it is easy to see that all these equations admit a solution. The coefficients v_1 and v_4 can in turn be obtained via the Envelope Theorem. Specifically,

$$\begin{aligned}(r+\kappa)[v_1 + 2v_4\theta + v_5 P] &= (\delta^h\mu + \alpha^h\theta + \zeta^h P)\left[1 - v_5\frac{\alpha^h\lambda^h}{2\zeta^h}\right] \\ &\quad - v_5\phi\left[P + \frac{\delta^h\mu + \alpha^h\rho^h\mu}{2\zeta^h}\right] - 2v_4\kappa(\theta - \mu)\end{aligned}$$

yields the additional equations

$$\begin{aligned}2(r+\kappa)v_4 &= \alpha^h \underbrace{\left[1 - v_5\frac{\alpha^h\lambda^h}{2\zeta^h}\right]}_{=\alpha^h \text{ from (S.3)}} - 2v_4\kappa \Rightarrow v_4 = \frac{(\alpha^h)^2}{2(r+2\kappa)} \quad \text{and} \\ (r+\kappa)v_1 &= \delta^h\alpha^h\mu - v_5\phi\frac{\delta^h\mu + \alpha^h\rho^h\mu}{2\zeta^h} \Rightarrow v_1 = \frac{\delta^h\alpha^h\mu}{r+\kappa} - \frac{v_5\phi(\delta^h\mu + \alpha^h\rho^h\mu)}{2\zeta^h(r+\kappa)}.\end{aligned}$$

The coefficient v_0 in turn corresponds to

$$v_0 = \frac{1}{r}\left[-(\delta^h\mu)^2 + v_2\left(\delta^h\mu\frac{\alpha^h\lambda^h}{2\zeta^h} - \phi\frac{\delta^h\mu + \alpha^h\rho^h\mu}{2\zeta^h}\right) + v_1\kappa\mu + \sigma_\theta^2 v_3 + \left(\frac{\alpha^h\lambda^h\sigma_\xi}{2\zeta^h}\right)^2 v_4\right],$$

which is obtained by equating the constant terms in the HJB equation.

3. Transversality conditions and admissibility of the candidate equilibrium strategy.

   Recall that under any admissible strategy,

$$dP_t = \left[-\frac{\alpha^h\lambda^h}{2\zeta^h}Q_t - \phi\left(P_t + \frac{\delta^h\mu + \alpha^h\rho^h\mu}{2\zeta^h}\right)\right]dt - \frac{\alpha^h\lambda^h}{2\zeta^h}\sigma_\xi dZ_t^\xi,$$

   whereas under the candidate equilibrium strategy, $Q_t = \delta^h\mu + \alpha^h\theta_t + \zeta^h P_t$,

$$dP_t = \left[-\left(\phi + \frac{\alpha^h\lambda^h}{2}\right)P_t - \frac{(\alpha^h)^2\lambda^h}{2\zeta^h}\theta_t - \mu\left(\frac{\delta^h\alpha^h\lambda^h}{2\zeta^h} + \phi\frac{\delta^h + \alpha^h\rho^h}{2\zeta^h}\right)\right]dt - \frac{\alpha^h\lambda^h\sigma_\xi}{2\zeta^h}dZ_t^\xi.$$

   Since ϕ and $\phi + \alpha^h\lambda^h/2$ are strictly positive, both dynamics have the exact same structure as the corresponding ones for $(M_t)_{t\geq 0}$ in the observable-scores case. Moreover, from (S.3), the coefficient on P^2 in the value function is $v_3 = -(\zeta^h+1)\zeta^h/[\alpha^h\lambda^h] > 0$, where the last inequality follows from $\zeta^h \in (-1,0)$. It can be easily seen that these facts imply that all the arguments used to prove Lemmas 1 and 2 in Section S.2.1 of this supplementary Appendix also apply to the hidden-scores case.

4. Upper bound for ζ^h. To conclude, we derive the upper bound for ζ^h. Using the last equation in the system (S.4),

$$\lambda^h\alpha^h = \frac{(r+\kappa+\phi)2(1-\alpha^h)}{\alpha^h}.$$

Consequently,

$$-\zeta^h = \frac{(r+2\phi)\alpha^h}{(r+2\phi)\alpha^h + (r+\kappa+\phi)(1-\alpha^h)} = \frac{1}{1+\frac{(r+\kappa+\phi)(1-\alpha^h)}{(r+2\phi)\alpha^h}}.$$

However, it is easy to see from (A.14)–(A.15) (proof of (ii) in Proposition 3 in Appendix A) that the lower bound $\alpha^h \geq [r+\kappa+\phi]/[r+\kappa+2\phi]$ also holds in the hidden case. Thus,

$$\frac{1-\alpha^h}{\alpha^h} \leq \frac{\phi}{r+\kappa+\phi} \Rightarrow -\zeta^h \geq \frac{1}{1+\phi/[r+2\phi]} = \frac{r+2\phi}{r+3\phi}.$$

□

Proof of Proposition 8. We begin by proving (ii). A simple rearrangement of terms in $A(\phi,\alpha^o)=0$ and $A^h(\phi,\alpha^h)=0$ shows that α^o and α^h are defined solutions to

$$\begin{aligned}\tilde{A}(\phi,\alpha) &:= -2(r+\kappa+\phi)+\alpha\,(2r+\kappa+\phi)+\alpha h(B(\phi,\alpha),\alpha) &=& \;0, \;\text{ and}\\ \tilde{A}^h(\phi,\alpha) &:= -2(r+\phi+\kappa)+\alpha(2r+\phi+\kappa)+\alpha h(-\alpha/2,\alpha) &=& \;0,\end{aligned}$$

respectively, where $B(\phi,\alpha)\in(-\alpha/2,0)$ and

$$h(y,\alpha) := \left[\left(\frac{\sigma_\theta^2\alpha[\alpha+y]}{\kappa\sigma_\xi^2}+\phi+\kappa\right)^2 - \frac{4\sigma_\theta^2\alpha y\phi}{\kappa\sigma_\xi^2}\right]^{1/2} - \frac{\sigma_\theta^2\alpha[\alpha+y]}{\kappa\sigma_\xi^2}.$$

We now show that, given $\alpha>0$, $y\mapsto h(y,\alpha)$ is strictly decreasing over $\mathbb{R}_-$. In fact, observe that $\partial h(y,\alpha)/\partial y < 0$ if and only if

$$\left(\frac{\sigma_\theta^2\alpha[\alpha+y]}{\kappa\sigma_\xi^2}+\phi+\kappa\right)\frac{\sigma_\theta^2\alpha}{\kappa\sigma_\xi^2}-\frac{2\sigma_\theta^2\alpha\phi}{\kappa\sigma_\xi^2} < \frac{\sigma_\theta^2\alpha}{\kappa\sigma_\xi^2}\left[\left(\frac{\sigma_\theta^2\alpha[\alpha+y]}{\kappa\sigma_\xi^2}+\phi+\kappa\right)^2-\frac{4\sigma_\theta^2\alpha y}{\kappa\sigma_\xi^2}\right]^{1/2}$$

$$\Leftrightarrow \quad \left(\frac{\sigma_\theta^2\alpha[\alpha+y]}{\kappa\sigma_\xi^2}+\phi+\kappa\right)-2\phi < \left[\left(\frac{\sigma_\theta^2\alpha[\alpha+y]}{\kappa\sigma_\xi^2}+\phi+\kappa\right)^2-\frac{4\sigma_\theta^2\alpha y}{\kappa\sigma_\xi^2}\right]^{1/2}.$$

If the left-hand side is negative, the result follows immediately. Suppose to the contrary that $\left(\sigma_\theta^2\alpha[\alpha+y]/\kappa\sigma_\xi^2+\phi+\kappa\right)-2\phi > 0$. Squaring both sides of the inequality under study yields

$$-4\left(\frac{\sigma_\theta^2\alpha[\alpha+y]}{\kappa\sigma_\xi^2}+\phi+\kappa\right)\phi+4\phi^2 < -\frac{4\sigma_\theta^2\alpha y\phi}{\kappa\sigma_\xi^2} \Leftrightarrow 0 < \frac{\sigma_\theta^2\alpha^2}{\kappa\sigma_\xi^2}+\kappa,$$

which is always true. Because $-\alpha/2 < B(\phi,\alpha) < 0$, we conclude that $\tilde{A}(\phi,\alpha) < \tilde{A}^h(\phi,\alpha)$ for all $\alpha \in [0,1]$. But since $\tilde{A}(\phi,\alpha) = 2A(\phi,\alpha)$ and $\tilde{A}^h(\phi,\alpha) = 2A^h(\phi,\alpha)$, and both $\alpha \mapsto A(\phi,\alpha)$ and $\alpha \mapsto A^h(\phi,\alpha)$ are increasing (proofs of Theorem 1 and Proposition 9), it follows that $\alpha^o(\phi) > \alpha^h(\phi)$.

We now turn to (i) and (ii). Recall that in the observable case, $Q_t = \delta\mu + \alpha\theta_t + \beta M_t$ and $P_t = \delta\mu + (\alpha+\beta)M_t$. Thus, omitting the dependence of all equilibrium coefficients on ϕ,

$$\alpha^o = \alpha \;\text{ and }\; \pi_1^o = (\alpha+\beta)\lambda,$$

where $\lambda = \Lambda(\phi,\alpha,B(\phi,\alpha))$.

To show (ii), we use the second equation in (A.6) to obtain

$$\begin{aligned}\pi_1^o = (\alpha+\beta)\lambda = \frac{[\lambda\beta]^2}{r+2\phi}-\beta\lambda \;&=\; \frac{4[\lambda\beta]^2-4\beta\lambda(r+2\phi)}{4(r+2\phi)}\\ &\;\frac{[-2\lambda\beta+(r+2\phi)]^2-(r+2\phi)^2}{4(r+2\phi)}.\end{aligned}$$

However, using the expression for ζ^h in the hidden-scores case,

$$\pi_1^h := -\frac{\alpha^h\lambda^h}{2\zeta^h} = \frac{[\alpha^h\lambda^h]^2+2(r+2\phi)\alpha^h\lambda^h}{4(r+2\phi)} = \frac{[\alpha^h\lambda^h+(r+2\phi)]^2-(r+2\phi)^2}{4(r+2\phi)}.$$

Thus, we must compare $-2\lambda\beta$ with $\alpha^h\lambda^h$. However, from the last equation in (A.6) in

Appendix A, and the last equation in (S.4) in this Appendix,

$$(r+\kappa+\phi)\frac{2(1-\alpha)}{\alpha}=-2\lambda\beta, \text{ and } (r+\kappa+\phi)\frac{2(1-\alpha^h)}{\alpha^h}=\alpha^h\lambda^h,$$

But since $1>\alpha^o>\alpha^h>0$, we have $0<-2\lambda\beta<\alpha^h\lambda^h$.[9] It follows that $0<\pi_1^o<\pi_1^h$.

We conclude by proving (iii). To find the expected price and quantities in the hidden case, observe that the equation for δ^h is given by the first equation in (S.4) in this Appendix:

$$-(r+\phi)\frac{2\delta^h\mu}{\alpha^h\lambda^h}=\delta^h\mu+\phi\frac{\zeta^h+1}{\alpha^h\lambda^h\zeta^h}[\delta^h\mu+\alpha^h\rho^h\mu]+\kappa\mu\frac{2(1-\alpha^h)}{\alpha^h\lambda^h},$$

where $\rho^h:=1-\lambda^h[\delta^h+\alpha^h]/[2\phi]$. Also, from the second and third equations in (S.4) again,

$$\frac{\zeta^h+1}{\alpha^h\lambda^h\zeta^h}=-\frac{1}{2(r+2\phi)} \text{ and } \kappa\frac{2(1-\alpha^h)}{\alpha^h\lambda^h}=\alpha^h-(r+\phi)\frac{2(1-\alpha^h)}{\alpha^h\lambda^h}.$$

Consequently, we can write

$$\frac{2(r+\phi)}{\alpha^h\lambda^h}[\mu-(\delta^h\mu+\alpha^h\mu)]=\delta^h\mu+\alpha^h\mu-\frac{\phi}{2(r+2\phi)}[\delta^h\mu+\alpha^h\rho^h\mu]. \tag{S.5}$$

On the other hand, on-path prices and quantities in the hidden case take the form $P_t^h=-[\delta^h\mu+\alpha^h M_t]/2\zeta^h$ and $Q_t^h=\delta^h\mu+\alpha^h\theta_t+\zeta^h P_t^h$. Hence,

$$\mathrm{P}^h(\phi):=\mathbb{E}[P_t^h]=-\mu\frac{\delta^h+\alpha^h}{2\zeta^h} \text{ and } \mathrm{Q}^h(\phi):=\mathbb{E}[Q_t^h]=\mu\frac{\delta^h+\alpha^h}{2}=-\zeta^h\mathrm{P}^h(\phi).$$

From here, $\mathrm{P}^h(\phi)=\mathrm{Q}^h(\phi)$ if and only if $\mu=0$. In this case, expected prices and quantities in the observable and hidden case all coincide, and their common value is zero. We assume $\mu>0$ in what follows; in particular, $\mathrm{P}^h(\phi)>\mathrm{Q}^h(\phi)$ due to $\zeta^h\in(-1,0)$.

Let $\bar{\mathrm{P}}^h(\phi)=\mathrm{P}^h(\phi)/\mu$ and observe that $-\zeta^h\bar{\mathrm{P}}^h(\phi)=(\alpha^h+\delta^h)/2$. In addition,

$$\rho^h:=1-\lambda^h\frac{\delta^h+\alpha^h}{2\phi}\Rightarrow\delta^h\mu+\alpha^h\rho^h\mu=\mu(\alpha^h+\delta^h)-\lambda^h\alpha^h\mu\frac{\delta^h+\alpha^h}{2\phi}=-\mu\zeta^h\bar{\mathrm{P}}^h(\phi)\left[2-\frac{\lambda^h\alpha^h}{\phi}\right].$$

[9] As a corollary, $-1<\zeta^o<\zeta^h<0$. To see this, observe that $\zeta^o=\frac{\beta}{\alpha+\beta}$. Now, from the second equation for the system (A.6) that defines (δ,α,β) in the observable case, $(r+2\phi)\frac{\alpha+2\beta}{\lambda}=\beta^2\Rightarrow\zeta^o=\frac{\beta}{\beta+\alpha}=\frac{-2(r+2\phi)}{2(r+2\phi)-2\lambda\beta}$. On the other hand, in the hidden-scores case, $\zeta^h=-\frac{2(r+2\phi)}{2(r+2\phi)+\alpha^h\lambda^h}$.

Plugging these expressions into (S.5), and multiplying the resulting equation by $\mu/2\zeta^h$ yields

$$\begin{aligned}
\frac{2(r+\phi)}{\alpha^h\lambda^h}\left[\frac{1}{2\zeta^h}+\bar{\mathrm{P}}^h(\phi)\right] &= -\bar{\mathrm{P}}^h(\phi)+\bar{\mathrm{P}}^h(\phi)\frac{\phi}{2(r+2\phi)}\left[1-\frac{\alpha^h\lambda^h}{2\phi}\right] \\
\Rightarrow \bar{\mathrm{P}}^h(\phi) &= \frac{\frac{r+\phi}{\alpha^h\lambda^h\zeta^h}}{-1+\frac{\phi}{2(r+2\phi)}-\frac{\alpha^h\lambda^h}{4(r+2\phi)}-\frac{2(r+\phi)}{\alpha^h\lambda^h}} \\
&= \frac{2(r+\phi)[2(r+2\phi)+\alpha^h\lambda^h]}{8(r+\phi)(r+2\phi)+(\alpha^h\lambda^h)^2-2\phi\alpha^h\lambda^h+4\alpha^h\lambda^h(r+2\phi)}, \quad \text{(S.6)}
\end{aligned}$$

where in the last equality we used that $\zeta^h=-2(r+2\phi)/[2(r+2\phi)+\alpha^h\lambda^h]$. The expression is well-defined because $8(r+\phi)(r+2\phi)+(\alpha^h\lambda^h)^2-2\phi\alpha^h\lambda^h+4\alpha^h\lambda^h(r+2\phi)=4(r+\phi)[\alpha^h\lambda^h+2(r+2\phi)]+(\alpha^h\lambda^h)^2+2\phi\alpha^h\lambda^h>0$.

We first prove $\bar{\mathrm{Q}}^o(\phi)>\bar{\mathrm{Q}}^h(\phi)$. Since $\bar{\mathrm{Q}}^h=-\zeta^h\bar{\mathrm{P}}^h$, it follows that

$$\bar{\mathrm{Q}}^h=\frac{1}{2+\frac{(\alpha^h\lambda^h)^2-2\phi\alpha^h\lambda^h+4\alpha^h\lambda^h(r+2\phi)}{4(r+\phi)(r+2\phi)}}=\frac{1}{2+\frac{\alpha^h\lambda^h}{2(r+\phi)}\frac{\alpha^h\lambda^h+4r+6\phi}{2(r+2\phi)}}.$$

Also, from (A.16) in the proof of Proposition 3 (Appendix A in the paper),

$$\bar{\mathrm{P}}^o(\phi):=\frac{\mathrm{P}^o(\phi)}{\mu}=\frac{r+\phi}{2(r+\phi)+\lambda(\alpha+\beta)}. \quad \text{(S.7)}$$

Since $\bar{\mathrm{Q}}^o(\phi)=\bar{\mathrm{P}}^o(\phi)$, the desired inequality holds if and only if

$$\frac{\alpha^h\lambda^h}{2(r+\phi)}\frac{\alpha^h\lambda^h+4r+6\phi}{2(r+2\phi)}>\frac{\lambda(\alpha+\beta)}{r+\phi}\Leftrightarrow\frac{\alpha^h\lambda^h}{2}\frac{\alpha^h\lambda^h+4r+6\phi}{2(r+2\phi)}\underbrace{>}_{(*)}\lambda(\alpha+\beta).$$

Now, using the equations that define α and α^h we obtain

$$\lambda=\frac{(r+\kappa+\phi)(\alpha-1)}{\alpha\beta}\quad\text{and}\quad\frac{\alpha^h\lambda^h}{2}=\frac{(r+\kappa+\phi)(1-\alpha^h)}{\alpha^h}.$$

Also, using that $\beta=B(\phi,\alpha)=-\alpha^2(r+2\phi)/[2(r+2\phi)\alpha-(r+\kappa+\phi)(\alpha-1)]$, it is easy to see that

$$\frac{\alpha+\beta}{\alpha\beta}=\frac{\alpha(r+2\phi)-(r+\kappa+\phi)(\alpha-1)}{-\alpha^2(r+2\phi)}.$$

The inequality (*) then holds if and only if

$$\frac{(1-\alpha^h)[2(r+\kappa+\phi)(1-\alpha^h)+(4r+6\phi)\alpha^h]}{2(\alpha^h)^2(r+2\phi)}>\frac{(1-\alpha)[(r+\kappa+\phi)(1-\alpha)+\alpha(r+2\phi)]}{\alpha^2(r+2\phi)}.$$

Since $\alpha > \alpha^h$, we have that $2(1-\alpha^h)^2\alpha^2[r+\kappa+\phi] > 2(1-\alpha)^2(\alpha^h)^2[r+\kappa+\phi]$. It therefore suffices to show that

$$\begin{aligned}
(1-\alpha^h)\alpha^2(4r+6\phi)\alpha^h &> (1-\alpha)(\alpha^h)^2\alpha(2r+4\phi) \\
\Leftrightarrow [1-\alpha^h]\alpha^2(2r+4\phi)\alpha^h + 2[1-\alpha^h]\alpha^2(r+\phi)\alpha^h &> (1-\alpha)(\alpha^h)^2\alpha(2r+4\phi) \\
\Leftrightarrow \alpha\alpha^h[\alpha-\alpha^h] + 2[1-\alpha^h]\alpha^2(r+\phi)\alpha^h &> 0,
\end{aligned}$$

which is always true.

To establish the ranking of prices, we introduce the following:

Lemma 3. *A sufficient condition for* $\bar{\mathrm{P}}^h(\phi) > \bar{\mathrm{P}}^o(\phi)$ *is*

$$\alpha^h > \frac{\alpha^2(r+2\phi)}{\alpha(r+2\phi)+(1-\alpha)^2(r+\kappa+\phi)} \tag{S.8}$$

Proof. From (S.6) in this Appendix, and using that $8(r+\phi)(r+2\phi)+(\alpha^h\lambda^h)^2-2\phi\alpha^h\lambda^h+4\alpha^h\lambda^h(r+2\phi) = 4(r+\phi)[\alpha^h\lambda^h+2(r+2\phi)]+(\alpha^h\lambda^h)^2+2\phi\alpha^h\lambda^h > 0$, we can write

$$\bar{\mathrm{P}}^h(\phi) = \frac{1}{2+\frac{\alpha^h\lambda^h[\alpha^h\lambda^h+2\phi]}{2(r+\phi)[2(r+2\phi)+\alpha^h\lambda^h]}}.$$

As a result, using (S.7),

$$\bar{\mathrm{P}}^o(\phi) < \bar{\mathrm{P}}^h(\phi) \Leftrightarrow \frac{\alpha^h\lambda^h}{2}\frac{\alpha^h\lambda^h+2\phi}{2(r+2\phi)+\alpha^h\lambda^h} < \lambda(\alpha+\beta).$$

Using again that $\alpha^h\lambda^h = 2(r+\kappa+\phi)(1-\alpha^h)/\alpha^h$, $\lambda = (r+\kappa+\phi)(\alpha-1)/[\alpha\beta]$, and

$$\frac{\alpha+\beta}{\alpha\beta} = \frac{\alpha(r+2\phi)-(r+\kappa+\phi)(\alpha-1)}{-\alpha^2(r+2\phi)},$$

The inequality of interest becomes

$$\begin{aligned}
\alpha^2(r+2\phi)(1-\alpha^h)[\alpha^h\lambda^h+2\phi] &< (1-\alpha)\alpha^h[\alpha^h\lambda^h+2(r+2\phi)] \\
&\quad \times[\alpha(r+2\phi)-(r+\kappa+\phi)(\alpha-1)] \\
\Leftrightarrow \alpha^2(r+2\phi)[\alpha^h\lambda^h+2\phi] &< (1-\alpha)^2\alpha^h[\alpha^h\lambda^h+2(r+2\phi)](r+\kappa+\phi) \\
&\quad -2\alpha^2\alpha^h(r+\phi)(r+2\phi) \\
&\quad +\alpha^h\alpha(r+2\phi)[\alpha^h\lambda^h+2(r+2\phi)] \\
\Leftrightarrow \alpha^2(r+2\phi)[\alpha^h\lambda^h+2\phi+2\alpha^h(r+\phi)] &< (1-\alpha)^2\alpha^h[\alpha^h\lambda^h+2(r+2\phi)](r+\kappa+\phi) \\
&\quad +\alpha^h\alpha(r+2\phi)[\alpha^h\lambda^h+2(r+2\phi)].
\end{aligned}$$

But since $\alpha^h < 1$, the left-hand side is less than $\alpha^2(r+2\phi)[\alpha^h\lambda^h + 2(r+2\phi)]$. Inserting the latter expression on the left-hand side and dividing by $\alpha^h\lambda^h + 2(r+2\phi) > 0$, we conclude that the desired inequality is equivalent to $\alpha^2(r+2\phi) < (1-\alpha)^2\alpha^h(r+\kappa+\phi) + \alpha^h\alpha(r+2\phi)$. This concludes the proof of the lemma. □

Let $s = \sigma_\xi^2/\sigma_\theta^2$. We first write the equilibrium condition for α as in (A.19) in the paper, and obtains an analogous expression for α^h in the hidden case. Specifically,

$$0 = \frac{(\alpha-1)(\kappa+r+\phi)(\kappa+\alpha(-\kappa+r+3\phi)+r+\phi)}{\alpha^3(r+2\phi)} + \frac{\alpha(\kappa-\alpha(\kappa+r)+r+\phi)}{\alpha^3+\kappa s(\kappa-\alpha r+r+\phi)},$$
$$0 = \frac{2\left(\alpha^h-1\right)(\kappa+r+\phi)}{(\alpha^h)^2} + \frac{\alpha^h\left(\kappa-\alpha^h(\kappa+r)+r+\phi\right)}{(\alpha^h)^3+\kappa s\left(\kappa-\alpha^h r+r+\phi\right)}.$$

Solving both equations for s, we obtain the following expressions

$$s = \underbrace{\frac{\alpha^3\left(\alpha^2\left(-\kappa(\kappa+r)+2r\phi+3\phi^2\right)+\alpha(2\kappa+r)(\kappa+r+\phi)-(\kappa+r+\phi)^2\right)}{(1-\alpha)\kappa(\kappa+r+\phi)(\kappa+r(1-\alpha)+\phi)(\kappa(1-\alpha)+\alpha(r+3\phi)+r+\phi)}}_{S^o(\alpha):=},$$
$$s = \underbrace{\frac{(\alpha^h)^3\left(\kappa-\alpha^h(\kappa+r+2\phi)+r+\phi\right)}{2\kappa\left(\alpha^h-1\right)(\kappa+r+\phi)\left(\kappa-\alpha^h r+r+\phi\right)}}_{S^h(\alpha^h):=}.$$

In particular, observe that since $\alpha^h > [r+\kappa+\phi]/[r+\kappa+2\phi]$, $S^h(\alpha^h)$ is increasing.

Now fix $\alpha = \alpha(\phi)$ and consider the difference $S^o(\alpha) - S^h(A^P(\alpha))$, where is the $A^P(\alpha)$ is the function defined by the right-hand side of (S.8). After simplifications, we obtain

$$\begin{aligned}
&S^o(\alpha) - S^h(A^P(\alpha)) \\
=&[(\kappa+r(1-\alpha)+\phi)(\kappa(1-\alpha)+\alpha(r+3\phi)+r+\phi]\frac{S^o(\alpha)}{2}\times \\
&\left[\frac{2}{(\kappa-\alpha r+r+\phi)(\kappa+\alpha(-\kappa+r+3\phi)+r+\phi)}\right. \\
&\left.-\frac{\left(\kappa+\alpha^2(\kappa+r+\phi)-\alpha(2\kappa+r)+r+\phi\right)^{-2}\alpha^3(r+2\phi)^3}{(-\alpha\kappa+\alpha\phi+\kappa+r+\phi)\left(\alpha^2\left((\kappa+\phi)^2+2\kappa r\right)-\alpha(2\kappa+r)(\kappa+r+\phi)+(\kappa+r+\phi)^2\right)}\right].
\end{aligned}$$

Finally, it can be verified that the term in parentheses is strictly positive for all $\alpha \in (0,1)$ and for all $(\kappa,\phi,r) \in \mathbb{R}^3_+$—see `scores.nb` on our websites. Because in equilibrium we must have $S^o(\alpha) = S^h(\alpha^h) = s$, it follows that $\alpha^h(\phi) > A^P(\alpha(\phi))$ for all ϕ. This concludes the proof of the proposition. □

Before turning to the proof of Proposition 9, we need expressions for consumer surplus and firms' profits that are analogous to the ones derived in the beginning of Section S.2.3:

Lemma 4. *For all $\phi \geq 0$ and $\sigma_\xi > 0$,*

$$
\begin{aligned}
CS^h(\phi) &= \mathrm{Var}[\theta_t]\alpha^h\left(1-\frac{\alpha^h}{2}\right) + (-\zeta^h\mathbb{E}[P_t])\left(\mu - \left[\frac{1}{2}-\frac{1}{\zeta^h}\right](-\zeta^h\mathbb{E}[P_t])\right)\\
&+ \underbrace{\alpha^h\left(\frac{\alpha^h}{4}\left[\frac{3}{2}+\frac{1}{\zeta^h}\right]-\frac{1}{2}\right)}_{<0}\mathrm{Var}[\theta_t]G^h(\phi)\\
\Pi^h(\phi) &= -\zeta^h\mathbb{E}[(P_t^h)^2]
\end{aligned}
$$

where $\mathbb{E}[P_t] = \mu[\delta^h + \alpha^h]/[-2\zeta^h]$ and $G^h(\phi) = \Lambda(\phi, \alpha^h(\phi), -\alpha^h(\phi)/2)$.

Proof: Consumer surplus always takes the form $\mathbb{E}[Q_t[\theta_t - P_t - Q_t/2]]$. Using that $\mathrm{Cov}[\theta_t, M_t] = \mathrm{Var}[M_t] = \mathrm{Var}[\theta_t]G^h(\phi)$, it is easy to verify that

$$
\begin{aligned}
\mathbb{E}[Q_t\theta_t] &= \frac{\delta^h\mu^2}{2} + \alpha^h\underbrace{[\mathrm{Var}[\theta_t]+\mu^2]}_{=\mathbb{E}[\theta_t^2]} - \frac{\alpha^h}{2}\underbrace{[\mathrm{Var}[\theta_t]G^h(\phi)+\mu^2]}_{\mathbb{E}[\theta_t M_t]}\\
&= \mu^2\left(\frac{\delta^h+\alpha^h}{2}\right) + \alpha^h\mathrm{Var}[\theta_t] - \frac{\alpha^h}{2}\mathrm{Var}[\theta_t]G^h(\phi)\\
\mathbb{E}[Q_t^2] &= \left(\frac{\delta^h\mu}{2}\right)^2 + [\alpha^h]^2[\mathrm{Var}[\theta_t]+\mu^2] + \left(\frac{\alpha^h}{2}\right)^2[\mathrm{Var}[\theta_t]G^h(\phi)+\mu^2]\\
&\quad + \delta^h\alpha^h\mu^2 - \frac{\delta^h\alpha^h}{2}\mu^2 - [\alpha^h]^2[\mathrm{Var}[\theta_t]G^h(\phi)+\mu^2]\\
&= \mu^2\left(\frac{\delta^h+\alpha^h}{2}\right)^2 + [\alpha^h]^2\mathrm{Var}[\theta_t] - \frac{3[\alpha^h]^2}{4}\mathrm{Var}[\theta_t]G^h(\phi)\\
\mathbb{E}[Q_tP_t] &= -\frac{1}{\zeta^h}\left(\frac{\delta^h\mu}{2}\right)^2 - \frac{1}{\zeta^h}\frac{\alpha^h\delta^h\mu^2}{2} - \frac{1}{\zeta^h}\left(\frac{\alpha^h}{2}\right)^2[\mathrm{Var}[\theta_t]G^h(\phi)+\mu^2]\\
&= -\frac{\mu^2}{\zeta^h}\left(\frac{\delta^h+\alpha^h}{2}\right)^2 - \frac{1}{\zeta^h}\left(\frac{\alpha^h}{2}\right)^2\mathrm{Var}[\theta_t]G^h(\phi)
\end{aligned}
$$

Collecting terms that accompany $\mathrm{Var}[\theta_t]$, $\mathrm{Var}[\theta_t]G^h(\phi)$ and μ^2 yields

$$
\begin{aligned}
CS^h(\phi) &= \mathrm{Var}[\theta_t]\alpha^h\left[1-\alpha^h/2\right] + \mathrm{Var}[\theta_t]G^h(\phi)\left[\frac{-\alpha^h}{2}+\frac{3[\alpha^h]^2}{8}+\frac{[\alpha^h]^2}{4\zeta^h}\right]\\
&\quad + \mu^2\left(\frac{\delta^h+\alpha^h}{2}\right)\left[1-\left(\frac{1}{2}-\frac{1}{\zeta^h}\right)\left(\frac{\delta^h+\alpha^h}{2}\right)\right]
\end{aligned}
$$

But since $\mathbb{E}[P_t] = \mu[\delta^h + \alpha^h]/[-2\zeta^h]$, we can write

$$\mu^2\left(\frac{\delta^h+\alpha^h}{2}\right)\left[1-\left(\frac{1}{2}-\frac{1}{\zeta^h}\right)\left(\frac{\delta^h+\alpha^h}{2}\right)\right] = (-\zeta^h\mathbb{E}[P_t])\left(\mu-\left[\frac{1}{2}-\frac{1}{\zeta^h}\right](-\zeta^h\mathbb{E}[P_t])\right).$$

As for the profits expression, it follows from $\Pi^h := \mathbb{E}[P_t^h Q_t^h]$ and $\mathbb{E}[Q_t^h|Y_t] = -\zeta^h P_t^h$. □

Proof of Proposition 9. From section S.3.3, the equilibrium variables of the observable and hidden cases are of class C^1 over $(\phi, \sigma_\xi^2) \in (0,\infty)\times[0,\infty)$. Thus, to study the noiseless limit case, we set $\sigma_\xi = 0$ in the *solutions* of the observable and hidden cases (the former literally understood as a noiseless case, while the latter understood as the C^1 extension of the equilibrium variables to $\sigma_\xi = 0$). Letting CS_μ^x denote the consumer surplus as a function of $\mu \geq 0$ in case $x \in \{o, h\}$ in this case, we have that

$$\begin{aligned} CS_\mu^o - CS_0^o &= \mathbb{E}[Q_t^o]\left(\mu - \frac{3}{2}\mathbb{E}[Q_t^o]\right) = \mu^2\frac{(r+\phi)(r+3\phi)}{2(2r+3\phi)^2}, \quad \text{and} \\ CS_\mu^h - CS_0^h &= \mathbb{E}[Q_t^h]\left(\mu - \left[\frac{1}{2}-\frac{1}{\zeta^h}\right]\mathbb{E}[Q_t^h]\right) = \mu^2\frac{(r+\phi)(r+2\phi)[r^2+5r\phi+8\phi^2]}{8[r^2+4r\phi+4\phi^2]^2} \end{aligned}$$

where the last equalities follow from the expressions for $\mathbb{E}[Q_t^o]$, $\mathbb{E}[Q_t^h]$ and ζ^h when $\sigma_\xi = 0$ (section S.3) After straightforward manipulation of terms, if $\mu > 0$,

$$\begin{aligned} & CS_\mu^o - CS_0^o > CS_\mu^h - CS_0^h \\ \Leftrightarrow \quad & 4(r+3\phi)[r^4+24r^2\phi^2+8r^3\phi+32r\phi^3+16\phi^4] \\ > \quad & (2r+3\phi)^2[r^3+6r^2\phi+18r\phi^2+16\phi^2]. \end{aligned}$$

It is then easy to verify that the strict inequality holds component-wise across all different exponents as long as $\phi > 0$, with equality at $\phi = 0$.

On the other hand, the comparison between the remaining components can be obtained by studying consumer surplus when $\mu = 0$, namely, $CS_0^o - CS_0^h$. One can show that

$$\begin{aligned} CS_0^o - CS_0^h = \; & \mathrm{Var}[\theta] f_1(\phi, r, \kappa)\left\{2\kappa^2(\kappa - r)r(\kappa + r)(\Delta - r)\right. \\ & +\phi\kappa[2\kappa r^2(20r-\Delta)+r^3(9r+\Delta)+\kappa^3(-6r+8\Delta)+\kappa^2 r(13r+17\Delta)] \\ & \left.+f_2(r,\phi,\kappa)\right\} \end{aligned}$$

where $\Delta := \sqrt{(2\phi+r)(6\phi+r)}$, and where f_1 and f_2 are strictly positive functions (as we show in the Mathematica file `scores.nb`). Thus, it suffices to show that

$$\begin{aligned} & \phi\kappa[40\kappa r^3 + r^3(9r+\Delta) + \kappa^3(-6r+8\Delta) + \kappa^2 r(13r+17\Delta)] \\ > \; & 2\kappa^2(r-\kappa)r(\kappa+r)(\Delta-r) + 2\phi\kappa^2 r^2\Delta. \end{aligned}$$

Since $\Delta > r$, the inequality is trivially satisfies when $\kappa \geq r$, as $2\kappa^2(r-\kappa)r(\kappa+r)(\Delta-r) < 0$ and $2\phi\kappa^2r^2\Delta < 17\phi\kappa^3 r\Delta$ in this case. Suppose now that $r > \kappa$. Using that $\Delta^2 - r^2 = \phi(12\phi + 8r)$ the inequality can be written as

$$\begin{aligned} & \kappa(\Delta+r)[40\kappa r^3 + r^3(9r+\Delta) + \kappa^3(-6r+8\Delta) + \kappa^2 r(13r+17\Delta)] \\ > \; & 2\kappa^2(r-\kappa)r(\kappa+r)(12\phi+8r) + 2\kappa^2r^2\Delta(\Delta+r). \end{aligned} \tag{S.9}$$

Notice first that,

$$40\kappa^2r^3(\Delta+r) > 40\kappa^2r^3[\sqrt{12}\phi + r] > 2\kappa^2r^3(12\phi+8r) > 2\kappa^2(r-\kappa)r(\kappa+r)(12\phi+8r).$$

On the other hand, we can write $\kappa^3(-6r+8\Delta) = 6\kappa^3(\Delta - r) + 2\kappa^3\Delta$, each term being positive. Thus,

$$\begin{aligned} \kappa(\Delta+r)[r^3\Delta + 2\kappa^3\Delta + 17\kappa^2 r\Delta] &> 2\kappa^2r^2\Delta(\Delta+r) \\ \Leftrightarrow \kappa(\Delta+r)\Delta[r^3 + 2\kappa^3 + 17\kappa^2 r - 2\kappa r^2] &> 0 \\ \Leftrightarrow \kappa(\Delta+r)\Delta[r(r^2 - 2\kappa r + 17\kappa^2) + 2\kappa^3] &> 0 \end{aligned}$$

which is clearly true. Since the remaining terms of the left-hand side of (S.9) are all positive, we deduce that $CS_0^o - CS_0^h > 0$ for all $\phi > 0$.

To prove (ii), set again $\sigma_\xi = 0$ in the equilibrium objects and outcomes of both models. We proceed in an analogous fashion, letting $\Pi_\mu^x(\phi)$ denote profits at $\phi > 0$ as a function of $\mu \geq 0$ when $x \in \{0, h\}$. Using the expression from section S.3 in this online appendix,

$$\begin{aligned} \underbrace{\Pi_\mu^o - \Pi_0^o}_{\mu^2-\text{component of } \Pi^o} &= \mu^2 \frac{(r+\phi)^2}{(2r+3\phi)^2}, \quad \text{and} \\ \underbrace{\Pi_\mu^h - \Pi_0^h}_{\mu^2-\text{component of } \Pi^h} &= \mu^2 \frac{(r+2\phi)[(r+\phi)(r+3\phi)]^2}{(r+3\phi)[2(r+\phi)(r+3\phi)+2\phi^2]^2} \end{aligned}$$

Straightforward manipulation then shows that, if $\mu > 0$,

$$\begin{aligned}\Pi_\mu^o - \Pi_0^o &> \Pi_\mu^h - \Pi_0^h \\ \Leftrightarrow 4[(r+\phi)(r+3\phi)+\phi^2]^2 &> (r+2\phi)(r+3\phi)(2r+3\phi)^2 \\ \Leftrightarrow 4(r+2\phi)^3 &> (r+3\phi)(2r+3\phi)^2,\end{aligned}$$

which is true for all $\phi > 0$.

We now show that $\Pi_0^o > \Pi_0^h$ for ϕ large enough. Indeed, using that in the observable case

$$\alpha = \frac{2(r+\kappa+\phi)}{r+2\kappa+\sqrt{(r+2\phi)(r+6\phi)}} \text{ and } \beta = -\frac{\alpha^2(r+2\phi)}{2(r+2\phi)\alpha-(r+\kappa+\phi)(\alpha-1)},$$

it is easy to see that $\lim_{\phi\to\infty}(\alpha,\beta) = \left(\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}(3+\sqrt{3})}\right)$. On the other hand, $\alpha^h = (r+\kappa+\phi)/(r+\kappa+2\phi) \to 1/2$ as $\phi\to\infty$. Thus,

$$\begin{aligned}\lim_{\phi\to\infty}\Pi_0^o(\phi) &= \mathrm{Var}[\theta]\frac{(\alpha+\beta)^2\phi\alpha}{\alpha\phi+\kappa(\alpha+\beta)} = \left(\frac{1+\sqrt{3}}{\sqrt{3}(3+\sqrt{3})}\right)^2, \text{ and} \\ \lim_{\phi\to\infty}\Pi_0^h(\phi) &= \mathrm{Var}[\theta]\frac{r+3\phi}{4(r+2\phi)}\frac{(\alpha^h)^2\phi}{\phi+\kappa/2} = \frac{3}{32}.\end{aligned}$$

Thus, $\lim_{\phi\to\infty}\Pi_0^o(\phi) - \Pi_0^h(\phi) > 0$, and hence, there exists $\bar{\phi}$ such that, for all $\mu \geq 0$, we have that $\Pi_\mu^o(\phi) > \Pi_\mu^h(\phi)$ for all $\phi > \bar{\phi}$ when $\sigma_\xi^2 = 0$. (Observe, moreover, that since the expressions involved are continuously differentiable functions of $(\phi, \sigma_\xi^2) \in (0,\infty)\times[0,\infty)$, uniform convergence of $[CS_\mu^o - CS_\mu^h](\sigma,\cdot)$ and $[\Pi_\mu^o - \Pi_\mu^h](\sigma,\cdot)$ as $\sigma_\xi \searrow 0$ holds over compact sets of levels of persistence—refer to Section S.3.3 for the details.)

Finally, we turn to (iii) and set $\sigma_\xi = 0$ again. Let us begin with the comparison of consumer surplus levels with the terms that do not depend on μ. For both the hidden and the naive case, we consider the following expression

$$\alpha - \frac{\alpha^2}{2} + \frac{\alpha\phi\left(\frac{1}{4}\alpha\left(\frac{1}{\zeta}+\frac{3}{2}\right)-\frac{1}{2}\right)}{\phi+\frac{\kappa}{2}},$$

where in the hidden case, $\alpha = \frac{\phi+\kappa+r}{2\phi+\kappa+r}$ and $\zeta = -\frac{2\phi+r}{3\phi+r}$, while in the naive case $\alpha = 1 = -\zeta$.

Letting $\rho := r/\kappa$, the difference in these terms is given by

$$CS_0^h - CS_0^{\text{naive}} = \frac{\phi^2(\rho(\phi-4)-4\phi-2)}{4(2\phi+1)(\rho+2\phi)(\rho+2\phi+1)^2},$$

which is negative for all ϕ if $\rho < 4$.

Now consider the terms proportional to μ^2. We have

$$(CS^h_\mu - CS^h_0) - (CS^{\text{naive}}_\mu - CS^{\text{naive}}_0) = \frac{\mu^2\rho\phi^2}{8(\rho+2\phi)^3} > 0 \quad \text{for all } \phi > 0.$$

Combining terms, we obtain

$$CS^h_\mu - CS^{\text{naive}}_\mu \propto \mu^2\rho + \frac{(\rho+2\phi)^2(\rho(\phi-4)-4\phi-2)}{(2\phi+1)(\rho+2\phi+1)^2}.$$

Evaluating this expression at $\phi = 0, +\infty$ we obtain two necessary conditions for the above expression to be negative, i.e., $\mu < \frac{\sqrt{4-\rho}}{\sqrt{2}\sqrt{\rho}}$ and $\mu < \frac{2\sqrt{\rho(2\rho+1)}}{\rho+1}$. The file `scores.nb` shows these conditions are also sufficient. □

Further properties.

1. α^h is decreasing at a non-concealing point. From the proof of Proposition 7, $\alpha^h \in (0,1)$ is defined via $(r+\kappa+\phi)(\alpha^h-1)+\alpha H(\phi,\alpha^h) = 0$ where $H(\phi,\alpha) := \Lambda(\phi,\alpha,-\alpha/2)[\alpha/2]$ satisfies $H_\alpha > 0$. Thus, the sign of $[\alpha^h]'$ is given by the sign of $1-\alpha^h(\phi)-H^h_\phi(\phi,\alpha^h(\phi))$, as in the proof of Proposition 5. In particular, replacing B_ϕ by 0 in (A.20) in the paper yields that, at any $\phi^{*,h}$ satisfying (19) in Section 5 (with $\beta = -\alpha/2$),

$$\text{sign}([1-\alpha^h(\phi)-\alpha^h(\phi)H_\phi(\phi,\alpha^h(\phi))]|_{\phi=\phi^{*,h}}) = \text{sign}\left(\left[1-\alpha^h+\frac{\lambda\alpha^h[-\alpha^h/2]}{\phi+\kappa}\right]_{\phi=\phi^{*,h}}\right) < 0.$$

2. The non-concealing score in the hidden case exists, is unique, and has more persistence. Existence and uniqueness follows from identical arguments as those used in the proof of (i)–(ii) in Proposition 5. Now, since $\alpha^o + B(\phi,\alpha^o) \geq \alpha^o/2$ and $\alpha^o > \alpha^h$, it follows that $\nu^o(\phi) \geq \nu^h(\phi)$; therefore, $\phi^{*,o} \geq \phi^{*,h}$, as each ν crosses the identity from above.

3. Quasiconvexity of α^h and $\arg\max_{\phi\geq 0} G^h(\phi) := G(\phi,\alpha^h(\phi),-\alpha^h(\phi)/2) < \phi^{*,h}$. They follow identical arguments as the ones used in the observable case (Lemma A.4 and Proposition 4 in the paper).

S.2.5 Appendix A: Omitted Proofs

Proof of Lemma A.5. We start by showing (ii). To this end, recall that when $\alpha > 0$ and $\beta < 0$, the quadratic in λ

$$\lambda = \frac{\alpha\sigma_\theta^2(\phi - \beta\lambda)}{\alpha^2\sigma_\theta^2 + \sigma_\xi^2\kappa(\phi - \beta\lambda + \kappa)}, \tag{S.10}$$

has a unique strictly positive root, which we denoted by $\Lambda(\phi, \alpha, \beta)$. It then suffices to show that $\alpha\gamma(\alpha)/\sigma_\xi^2 > 0$ solves the previous equation.

To this end, we omit the dependence of ν on (α, β) and of γ on α in what follows. Rewrite (S.10) at $\phi = \nu$ as $-\kappa\sigma_\xi^2\beta\lambda^2 + \lambda[\alpha^2\sigma_\theta^2 + \kappa\sigma_\xi^2(\nu + \kappa) + \alpha\sigma_\theta^2\beta] - \alpha\sigma_\theta^2\nu = 0$. However, using that

$$\lambda\kappa\sigma_\xi^2(\nu + \kappa) = \lambda\kappa\sigma_\xi^2\left(2\kappa + \frac{\alpha\gamma}{\sigma_\xi^2}[\alpha + \beta]\right) \quad \text{and} \quad \alpha\sigma_\theta^2\nu = \alpha\sigma_\theta^2\left(\kappa + \frac{\alpha\gamma}{\sigma_\xi^2}[\alpha + \beta]\right),$$

we obtain

$$\begin{aligned} 0 &= \lambda\alpha^2\sigma_\theta^2 + 2\kappa^2\sigma_\xi^2\lambda + 2\kappa\lambda\alpha^2\gamma + \kappa\lambda\alpha\gamma\beta - \kappa\lambda^2\sigma_\xi^2\beta - \alpha\sigma_\theta^2\kappa - \frac{\alpha^2\gamma\sigma_\theta^2}{\sigma_\xi^2}[\alpha + \beta] + \alpha\sigma_\theta^2\beta\lambda \\ &= \lambda\alpha^2\sigma_\theta^2 + 2\kappa^2\sigma_\xi^2\lambda + \kappa\lambda\alpha^2\gamma - \alpha\sigma_\theta^2\kappa - \frac{\alpha^3\gamma\sigma_\theta^2}{\sigma_\xi^2} + \beta\left[\kappa\lambda\alpha\gamma - \kappa\lambda^2\sigma_\xi^2 - \frac{\alpha^2\gamma\sigma_\theta^2}{\sigma_\xi^2} + \alpha\sigma_\theta^2\lambda\right]. \end{aligned}$$

Setting $\lambda = \alpha\gamma/\sigma_\xi^2$, the first and last term of the first line in the second equality cancel out, and the last bracket vanishes. Thus, we are left with

$$0 = 2\kappa\alpha\underbrace{\left[2\kappa\gamma + \frac{\alpha^2\gamma^2}{\sigma_\xi^2} - \sigma_\theta^2\right]}_{\equiv 0},$$

which is true by definition of γ.

We now prove that $\nu(\alpha, \beta)$ is an extreme point of $\phi \mapsto G(\phi, \alpha, \beta)$, and verify (i) in the process. For notational simplicity, we again omit any dependence on variables unless it is strictly necessary. Recall that

$$G = \frac{\alpha\Lambda}{\phi + \kappa - \beta\Lambda}.$$

Thus, $G_\phi = 0$ if and only if $\Lambda_\phi(\phi + \kappa) = \Lambda$. We first check that this equality is satisfied at $(\nu(\alpha, \beta), \alpha, \beta)$.

From (ii), $\Lambda = \alpha\gamma/\sigma_\xi^2$ at the point of interest; hence, the claim reduces to showing that

$\Lambda_\phi(\nu(\alpha,\beta),\alpha,\beta) = \alpha\gamma/\sigma_\xi^2(\nu+\kappa)$. However, it is easy to check that

$$\Lambda_\phi = \frac{\alpha\sigma_\theta^2[1-\beta\Lambda_\phi][\alpha^2\sigma_\theta^2+\kappa^2\sigma_\xi^2]}{[\alpha^2\sigma_\theta^2+\kappa\sigma_\xi^2(\phi+\kappa-\beta\Lambda)]^2}.$$

Also, $\nu+\kappa-\beta\Lambda(\nu(\alpha,\beta),\alpha,\beta) = 2\kappa+\alpha^2\gamma/\sigma_\xi^2 = \sigma_\theta^2/\gamma$, where the last equality comes from the definition of γ. Thus,

$$\begin{aligned} [\alpha^2\sigma_\theta^2+\kappa\sigma_\xi^2(\phi+\kappa-\beta\Lambda)]^2\Big|_{\phi=\nu} = \frac{\sigma_\theta^4[\alpha^2\gamma+\kappa\sigma_\xi^2]^2}{\gamma^2} &= \frac{\sigma_\theta^4[\alpha^2\overbrace{(\alpha^2\gamma^2+2\kappa\gamma\sigma_\xi^2)}^{=\sigma_\theta^2\sigma_\xi^2,\text{ by def. of }\gamma}+\kappa^2\sigma_\xi^4]}{\gamma^2} \\ &= \frac{\sigma_\theta^4\sigma_\xi^2[\alpha^2\sigma_\theta^2+\kappa^2\sigma_\xi^2]}{\gamma^2}. \end{aligned}$$

We conclude that at $(\nu(\alpha,\beta),\alpha,\beta)$,

$$\Lambda_\phi = \frac{\gamma^2\alpha}{\sigma_\theta^2\sigma_\xi^2}[1-\beta\Lambda_\phi] \Rightarrow \Lambda_\phi \underbrace{[\sigma_\theta^2\sigma_\xi^2+\gamma^2\alpha\beta]}_{=2\kappa\gamma\sigma_\xi^2+\gamma^2\alpha^2+\gamma^2\alpha\beta} = \gamma^2\alpha \Rightarrow \Lambda_\phi = \underbrace{\frac{\gamma\alpha}{\sigma_\xi^2}}_{\Lambda(\nu(\alpha,\beta),\alpha,\beta)} \times \underbrace{\frac{1}{2\kappa+\frac{\alpha\gamma[\alpha+\beta]}{\sigma_\xi^2}}}_{1/(\nu+\kappa)},$$

which shows that ν is an extreme point of $\phi \mapsto G(\phi,\alpha,\beta)$.

On the other hand, it is easy to verify that at an extreme point ϕ,

$$G_{\phi\phi} = \frac{\alpha\sigma_\theta^2}{2\kappa}\frac{\Lambda_{\phi\phi}(\phi+\kappa)}{[\phi+\kappa-\beta\lambda]^2}.$$

Since $\alpha>0$, the sign of $G_{\phi\phi}$ is determined by $\Lambda_{\phi\phi}$ at that point. We now show that $\Lambda_{\phi\phi}(\phi,\alpha,\beta)<0$ for all $\phi>0$, $\alpha>0$ and $\beta<0$, and hence, that any extreme point of $\phi\mapsto G(\phi,\alpha,\beta)$ must be a strict local maximum. But this is enough to guarantee that $\phi\mapsto G(\phi,\alpha,\beta)$ has a unique extreme point, and hence that $\nu(\alpha,\beta)$ is a global maximum.

Recall that $\Lambda(\phi,\alpha,\beta) = [\sqrt{\ell^2(\phi,\alpha,\beta)-4\kappa(\sigma_\xi\sigma_\theta)^2\beta\alpha\phi} - \ell(\phi,\alpha,\beta)]/[-2\kappa\sigma_\xi^2\beta]$ where $\ell(\phi,\alpha,\beta) := \alpha\sigma_\theta^2[\alpha+\beta]+\kappa\sigma_\xi^2(\phi+\kappa)$. Thus,

$$\begin{aligned} \Lambda_\phi &= \underbrace{\frac{1}{[-2\kappa\sigma_\xi^2\beta]}}_{=:K_1}\left[\frac{\kappa\sigma_\xi^2\ell(\phi,\alpha,\beta)-2\kappa(\sigma_\xi\sigma_\theta)^2\beta\alpha}{\sqrt{\ell^2(\phi,\alpha,\beta)-4\kappa(\sigma_\xi\sigma_\theta)^2\beta\alpha\phi}}-\kappa\sigma_\xi^2\right], \text{ and so} \\ \Lambda_{\phi\phi} &= K_2(\phi)\underbrace{\left\{(\kappa\sigma_\xi^2)^2(\ell^2(\phi,\alpha,\beta)-4\kappa(\sigma_\xi\sigma_\theta)^2\beta\alpha\phi)-(\kappa\sigma_\xi^2\ell(\phi,\alpha,\beta)-2\kappa(\sigma_\xi\sigma_\theta)^2\beta\alpha)^2\right\}}_{J(\phi):=}, \end{aligned}$$

where $K_2(\phi) := K_1/[\ell^2(\phi,\alpha,\beta)-4\kappa(\sigma_\xi\sigma_\theta)^2\beta\alpha\phi]^{3/2}$. Since $\beta<0$, we have that $K_1>0$, from where $K_2 > 0$. Moreover,

$$\begin{aligned} J(\phi) &= -4\kappa^3\sigma_\xi^6\sigma_\theta^2\beta\alpha\phi + 4\kappa^2\sigma_\xi^4\sigma_\theta^2\beta\alpha\ell(\phi,\alpha,\beta) - 4\kappa^2(\sigma_\xi\sigma_\theta)^4\beta^2\alpha^2 \\ &= \underbrace{-4\kappa^2\sigma_\xi^4\sigma_\theta^2\alpha\beta}_{>0,\text{ as }\beta<0}\underbrace{[\kappa\sigma_\xi^2\phi - \ell(\phi,\alpha,\beta) + \sigma_\theta^2\beta\alpha]}_{=-\alpha^2\sigma_\theta^2-\kappa^2\sigma_\xi^2\text{ by def. of }\ell(\phi,\alpha,\beta)} < 0, \end{aligned}$$

concluding the proof. □

Proof of Lemma A.6. Recall from Proposition 5 and its proof show (i) α is decreasing at any point satisfying $\phi = \nu(\alpha(\phi), \beta(\phi))$, and (ii) a point like that is shown to exist via a simple application of the Intermediate Value Theorem. Importantly, neither step (nor the derivation of the lower bound κ for ϕ^*) relies on knowledge of $[\alpha+\beta]'$, which is used to establish the uniqueness part of the proposition only.

That $\arg\min\alpha > \kappa$ follows from the proof of the same proposition. Also, $\arg\min\alpha < +\infty$ follows directly from $\alpha \in [1/2, 1]$, $\lim\limits_{\phi\to 0,+\infty} \alpha = 1$, and α being continuous.

To prove the last two parts, omit the dependence of α and β on ϕ and write

$$\alpha + \beta = \alpha\left[1 - \frac{\alpha(r+2\phi)}{2(r+2\phi)\alpha - (r+\kappa+\phi)(\alpha-1)}\right] =: \alpha h(\phi,\alpha).$$

Thus, $[\alpha+\beta]'(\phi) = \alpha'[h + \alpha h_\alpha] + \alpha h_\phi$, where

$$h_\phi(\alpha,\phi) = \frac{\alpha(\alpha-1)(r+2\kappa)}{[2(r+2\phi)\alpha - (r+\kappa+\phi)(\alpha-1)]^2} < 0.$$

We will show that $h + \alpha h_\alpha > 0$ over $[\kappa, \infty)$, which implies that $\alpha + \beta$ must be decreasing over $[\kappa, \arg\min\alpha]$, and hence, at any point satisfying $\phi = \nu(\alpha(\phi), \beta(\phi))$.

To this end, notice that

$$\begin{aligned} h + \alpha h_\alpha > 0 \quad\Leftrightarrow\quad & [2(r+2\phi)\alpha - (r+\kappa+\phi)(\alpha-1)][(r+2\phi)\alpha - (r+\kappa+\phi)(\alpha-1)] \\ & -\alpha(r+2\phi)[2(r+2\phi)\alpha - (r+\kappa+\phi)(\alpha-1)] \\ & +\alpha^2(r+2\phi)[2(r+2\phi) - (r+\kappa+\phi)] > 0 \\ \Leftrightarrow\quad & 2(r+2\phi)^2\alpha^2 + (r+\kappa+\phi)^2(\alpha-1)^2 - \alpha(3\alpha-2)(r+2\phi)(r+\kappa+\phi) > 0. \end{aligned}$$

If $\phi \geq \kappa$, however,

$$
\begin{aligned}
& 2(r+2\phi)^2\alpha^2 - \alpha(3\alpha-2)(r+2\phi)(r+\kappa+\phi) + \underbrace{(r+\kappa+\phi)^2(\alpha-1)^2}_{>0} \\
\geq \; & \alpha(r+2\phi)[2\underbrace{(r+2\phi)}_{>r+\kappa+\phi}\alpha - 3\alpha(r+\kappa+\phi) + 2(r+\kappa+\phi)] \\
\geq \; & \alpha(r+2\phi)(r+\kappa+\phi)[2-\alpha] > 0,
\end{aligned}
$$

and the result follows. As a final step, observe that since $\phi > 0$ and $\alpha < 1$,

$$
\begin{aligned}
& 2(r+2\phi)^2\alpha^2 - \alpha(3\alpha-2)(r+2\phi)(r+\kappa+\phi) + \underbrace{(r+\kappa+\phi)^2(\alpha-1)^2}_{>0} \\
\geq \; & \alpha(r+2\phi)[\alpha(\phi-r-3\kappa) + 2(r+\kappa+\phi)] \\
\geq \; & \alpha(r+2\phi)[-r-3\kappa+2(r+\kappa)] = \alpha(r+2\phi)[r-\kappa].
\end{aligned}
$$

which is non-negative when $r \geq \kappa$. This concludes the proof. $\square$

S.3 Noiseless Case

In this section we state the expressions that the equilibrium objects and outcomes take when we set $\sigma_\xi = 0$ in the solutions of the observable and hidden cases (section S.3.1). After this, we explain how the expressions in each case can be obtained with minimal modifications of the arguments used in the corresponding $\sigma_\xi > 0$ counterpart (section S.3.2). Finally, we establish regularity properties of all the equilibrium variables at $\sigma_\xi = 0$ (section S.3.3). Importantly, recall that the solution of the hidden-scores model when $\sigma_\xi = 0$ should be understood as a vanishing-noise limit of the hidden case with non-trivial noise: the reason is that, when $\sigma_\xi = 0$, the consumer can always keep track of her score (so this one is effectively observable).

S.3.1 Equilibrium Objects and Outcomes when $\sigma_\xi = 0$

From an operational perspective, the only substantial change is that the equation that λ satisfies ((7) in the paper) is greatly simplified. This, in turn, allows us to find analytic expressions for the signaling coefficient α in both the observable and hidden case. We state below all the relevant expressions used in the paper; their derivation is straightforward.

Observable scores.

$$
\begin{aligned}
\alpha &= \frac{2(r+\kappa+\phi)}{r+2\kappa+\sqrt{(r+2\phi)(r+6\phi)}} \in \left(\frac{1}{\sqrt{3}}, 1\right) \\
\beta &= -\frac{\alpha^2(r+2\phi)}{2(r+2\phi)\alpha-(r+\kappa+\phi)(\alpha-1)} \in \left(-\frac{\alpha}{2}, 0\right) \\
\lambda &= \frac{\phi}{\alpha+\beta} > 0 \\
\mathbb{E}[P] &= \mu\frac{r+\phi}{2r+3\phi} \quad \left(= \frac{r+\phi}{2(r+\phi)+\lambda(\alpha+\beta)}\right) \in \left(\frac{\mu}{3}, \frac{\mu}{2}\right) \\
\mathbb{E}[Q] &= \mathbb{E}[P] \\
G &= \frac{\phi\alpha}{\phi\alpha+\kappa(\alpha+\beta)} \quad \left(= \frac{\alpha\lambda}{\phi+\kappa-\beta\lambda}\right) \in (0,1) \\
\Pi &= \mu^2\frac{(r+\phi)^2}{(2r+3\phi)^2} + \mathrm{Var}[\theta](\alpha+\beta)^2\frac{\phi\alpha}{\alpha\phi+\kappa(\alpha+\beta)} \quad \left(= \mathbb{E}[P^2]+\mathrm{Var}[P]\right) \\
CS &= \mathrm{Var}[\theta]\alpha\left(1-\frac{\alpha}{2}\right) + \mathbb{E}[P]\left(\mu-\frac{3}{2}\mathbb{E}[P]\right) + \mathrm{Var}[\theta]\left[\frac{\alpha^2}{2}+\beta-\frac{3}{2}(\alpha+\beta)^2\right]G
\end{aligned}
$$

Hidden scores.

$$
\begin{aligned}
\alpha^h &= \frac{r+\kappa+\phi}{r+\kappa+2\phi}(<\alpha) \in \left(\frac{1}{2},1\right) \\
\beta^h &= -\frac{\alpha^h}{2} < 0 \\
\lambda^h &= \frac{2\phi}{\alpha^h} > \lambda > 0 \\
\zeta^h &= -\frac{r+2\phi}{r+3\phi} \left(= -\frac{2(r+2\phi)}{2(r+2\phi)+\alpha^h\lambda^h}\right) \in \left(-1,-\frac{2}{3}\right) \\
\mathbb{E}[P^h] &= \mu\frac{(r+\phi)(r+3\phi)}{2(r+\phi)(r+3\phi)+2\phi^2} \left(= \frac{\mu}{2+\frac{\alpha^h\lambda^h(\alpha^h\lambda^h+2\phi)}{2(r+\phi)[2(r+2\phi)+\alpha^h\lambda^h]}}\right) > \mathbb{E}[P] \\
\mathbb{E}[Q^h] &= \mu\frac{(r+\phi)(r+2\phi)}{2(r+\phi)(r+3\phi)+2\phi^2} \left(= -\zeta^h\mathbb{E}[P^h]\right) < \mathbb{E}[Q] \\
G^h &= \frac{\phi\alpha^h}{\phi\alpha^h+\kappa\alpha^h/2} \left(= \frac{\alpha^h\lambda^h}{\phi+\kappa-\beta^h\lambda^h}\right) \\
\Pi^h &= \mu^2\frac{r+2\phi}{r+3\phi}\left[\frac{(r+\phi)(r+3\phi)}{2(r+\phi)(r+3\phi)+2\phi^2}\right]^2 + \mathrm{Var}[\theta]\frac{r+3\phi}{4(r+2\phi)}(\alpha^h)^2\frac{\phi}{\phi+\kappa/2} \\
&\quad \left(= -\zeta^h[\mathbb{E}[(P^h)^2]+\mathrm{Var}[P^h]]\right) \\
CS^h &= \mathrm{Var}[\theta]\alpha^h\left(1-\frac{\alpha^h}{2}\right) + \underbrace{(-\zeta^h\mathbb{E}[P^h])}_{=\mathbb{E}[Q^h]}\left(\mu - \left[\frac{1}{2}-\frac{1}{\zeta^h}\right](-\zeta^h\mathbb{E}[P^h])\right) \\
&\quad +\mathrm{Var}[\theta]\alpha^h\left[\frac{\alpha^h}{4}\left(\frac{3}{2}+\frac{1}{\zeta^h}\right)-\frac{1}{2}\right]G^h < CS
\end{aligned}
$$

S.3.2 Equilibrium Analysis when $\sigma_\xi = 0$

Existence of Linear Markov Equilibria in the Observable Case. The steps taken in the equilibrium analysis performed when $\sigma_\xi > 0$ (Lemma 2, Lemma A.1 and Theorem 1) have direct counterparts when $\sigma_\xi = 0$. Specifically:

1. Lemma 2 (Monopoly Price) does not change.

2. Lemma A.1 (Stationarity and Beliefs). It is easy to see that the long-run stationary variance when $\sigma_\xi = 0$ can be obtained by directly evaluating Γ at that value. This does not alter the stationarity condition $\phi - \beta\lambda > 0$, but it changes (7) to

$$\lambda = \frac{\alpha\sigma_\xi^2(\phi-\beta\lambda)}{\alpha^2\sigma_\theta^2}.$$

3. Proof of Theorem 1.

- Displays (A.3)–(A.6) remain unchanged.
- Lemma A.2. As in the $\sigma_\xi > 0$ case, we cannot have a linear Markov equilibrium with $\alpha = 0$: in this case, the score does not covary with the type, and so $M_t = \mu$; but this implies that $\alpha = 1$ at all times. Thus, from the equation for λ, we obtain $(\alpha + \beta)\lambda = \phi$. Since $\phi > 0$, we have that $\alpha + \beta \neq 0$ and $\lambda \neq 0$. We conclude that

$$\lambda = \frac{\phi}{\alpha + \beta} \neq 0.$$

The cases $\alpha < 0$ and $\alpha \geq 1$ can be ruled out using the same arguments as in the proof (with obvious modifications that employ the new expression for λ and that, by stationarity, $\phi - \beta\lambda > 0 \Leftrightarrow \frac{\alpha\phi}{\alpha+\beta} > 0$.)
- (A.7) is unchanged and (A.8) changes to $\lambda = \Lambda = \frac{\phi}{\alpha+\beta} > 0$.
- We can replace the new expression for λ in (A.9) to obtain the equation

$$(r + \kappa + \phi)(\alpha - 1) - \underbrace{\frac{\phi\alpha B(\phi, \alpha)}{\alpha + B(\phi, \alpha)}}_{=\lambda\alpha\beta} = 0.$$

It can be easily shown that this function is strictly increasing in $[0, 1]$, and hence, that is has a unique root. Rearranging terms, we obtain the quadratic

$$\alpha^2[3\phi^2 + 2r\phi - r\kappa - \kappa^2] + \alpha[r + \kappa + \phi][r + 2\kappa] - [r + \kappa + \phi]^2 = 0,$$

which admits as a root

$$\alpha = \frac{2(r + \kappa + \phi)}{r + 2\kappa + \sqrt{(r + 2\phi)(r + 6\phi)}} \in (0, 1), \;\; \phi > 0.$$

Hence, an analytic expression for α is obtained.
- The function $D(\phi, \alpha)$ that defines δ does not change, so the new intercept in the quantity process is obtained by inserting $\Lambda = \phi/(\alpha + \beta)$ in the same expression. The remaining coefficients of the value function, as well as the transversality and admissibility conditions, are obtained via identical arguments.

Setting $\sigma_\xi = 0$ in the equilibrium objects and outcomes of the hidden case. Recall that the proof of existence of linear Markov equilibria when $\sigma_\xi > 0$ in the hidden case follows from identical steps as those performed in the observable counterpart (the only changes being that $\beta = -\alpha/2$, that $\zeta^h < -1$ is endogenous, and that the equation defines δ is modified;

see the proof of Proposition 7 in section S.2.4 for details).

Given 1–3 above just discussed, it is easy to see that all the equilibrium variables $(\alpha, \beta, \delta, \lambda, G)$ in the hidden-scores model admit an evaluation at $\sigma_\xi = 0$ and can be obtained using the same arguments used in the $\sigma_\xi > 0$ case. In particular, since $\beta^h = -\alpha^h/2$, it follows that $\lambda^h = \phi/(\alpha^h + \beta^h) = 2\phi/\alpha^h$, from where α^h must satisfy

$$(r + \kappa + \phi)(\alpha^h - 1) - \alpha^h \left(\frac{2\phi}{\alpha^h}\right)(\frac{-\alpha^h}{2}) = 0 \Rightarrow \alpha^h = \frac{r + \kappa + \phi}{r + \kappa + 2\phi}.$$

Deriving the rest of the terms in S.3.1 is straightforward from the results in section S.2.4.

Further properties of the equilibrium coefficients and outcomes when $\sigma_\xi = 0$. When $\sigma_\xi = 0$, both in the observable and hidden cases:

- α is strictly decreasing, with $\lim_{\phi \to 0} \alpha(\phi) = 1$ and $\lim_{\phi \to +\infty} \alpha(\phi) \geq 1/2$ (moreover, $\lim_{\phi \to +\infty} \alpha(\phi) \geq 1/\sqrt{3}$ in the observable case);
- $\mathbb{E}[P_t]$ is strictly decreasing in ϕ, taking values in $(\mu/3, \mu/2)$ (moreover, $\mathbb{E}[P_t] \in (3\mu/8, \mu/2)$ in the hidden case)
- $\nu(\alpha, \beta) = +\infty$ follows from $\nu(\alpha, \beta) = \kappa + \frac{\alpha\gamma(\alpha)(\alpha+\beta)}{\sigma_\xi^2}$, $\frac{\alpha\gamma(\alpha)}{\sigma_\xi^2} = \frac{1}{\alpha}\left(\sqrt{\kappa^2 + \alpha^2 \frac{\sigma_\theta^2}{\sigma_\xi^2}} - \kappa\right)$, and from $\alpha \in [1/2, 1]$ and $\alpha + \beta \in [1/4, 1]$ ($\alpha \in [1/\sqrt{3}, 1]$ and $\alpha + \beta \in [-1/2\sqrt{3}, 1]$ in the observable case,). Clearly $\phi^*(\sigma_\xi) \nearrow \phi^*(0) = \infty$ as $\sigma_\xi \searrow 0$. Intuitively, a score that decays infinitely fast essentially the "last purchase" while α^h bounded away from zero uniformly in $\phi > 0$.
- $\lim_{\phi \to +\infty} G(\phi) = 1$.

S.3.3 Continuous Differentiability of the Model at $\sigma_\xi = 0$

We address some continuity and limit properties of the model at $\sigma_\xi^2 = 0$, which allow us to directly evaluate at $\sigma_\xi = 0$ when examining the noiseless limit case for every fixed $\phi > 0$, and to extend results at that point to $\sigma_\xi^2 > 0$ by continuity.

(1) In the observable case, the equilibrium variables $(\alpha, \beta, \delta, \lambda, G)$ are continuously differentiable functions of $(\phi, \sigma_\xi^2) \in (0, \infty) \times [0, \infty)$. We already know from Lemma A.3 in Appendix A in the paper that $\alpha(\phi, \sigma_\xi^2)$ is of class C^1 over the *open* set $(0, \infty)^2$, and so are the rest of the variables, as these are continuously differentiable functions of $(\alpha, \phi, \sigma_\xi^2)$. It remains to show their continuous differentiability at points $(\phi, 0)$, $\phi > 0$.

To show this, we can equivalently look at the system (A.17)–(A.19) in the paper. In particular, using (A.19), the signaling coefficient $\alpha(\phi,\sigma_\xi^2)\in(1/2,1)$ satisfies $F(\phi,\sigma_\xi^2,\alpha(\phi,\sigma_\xi^2))=0$ for $\phi>0$, where

$$F(\phi,x,\alpha):=\frac{(\alpha-1)(\kappa+r+\phi)(\kappa+\alpha(-\kappa+r+3\phi)+r+\phi)}{\alpha^3(r+2\phi)}+\frac{\alpha(\kappa-\alpha(\kappa+r)+r+\phi)}{\alpha^3+\kappa x(\kappa-\alpha r+r+\phi)/\sigma_\theta^2}.$$

The notation simply emphasizes the system's dependence on the variable x that replaces σ_ξ^2, and that is now allowed to take negative values.

Observe that given any compact set $[\underline{\phi},\overline{\phi}]\subset(0,\infty)$, we can always choose $\epsilon>0$ small enough such that $F(\phi,x,\alpha):[\underline{\phi},\overline{\phi}]\times[-\epsilon,\epsilon]\times[1/2,1]\to\mathbb{R}$ is of class C^1. Moreover, from section S.3.1, $F(\phi,0,\frac{2(r+\kappa+\phi)}{r+2\kappa+\sqrt{(r+2\phi)(r+6\phi)}})=0$ and its is easy to verify that

$$\begin{aligned}\frac{\partial F}{\partial\alpha} &= \frac{\left(2\kappa+r+\sqrt{(r+2\phi)(r+6\phi)}\right)^2}{8(r+2\phi)(\kappa+r+\phi)^2}\\ &\quad\times\left(r^2+2\kappa\sqrt{(r+2\phi)(r+6\phi)}+8r\phi+r\sqrt{(r+2\phi)(r+6\phi)}+12\phi^2\right)>0.\end{aligned}$$

at the same point. By the Implicit Function Theorem, therefore, we conclude that $\alpha(\phi,\sigma_\xi^2)$ can be extended to a continuously differentiable function over the desired set; the same conclusion holds for the rest of the variables by composition of functions of class C^1.

(2) In the hidden case, the equilibrium variables $(\alpha,\beta,\delta,\lambda,G)$ admit a continuously differentiable extension to $\{(\phi,\sigma_\xi^2)|\ \phi>0,\ \sigma_\xi^2=0\}$. The argument is identical to the one in the observable case, but instead using

$$F^h(\phi,\sigma_\xi^2,\alpha^h):=\frac{2\left(\alpha^h-1\right)(\kappa+r+\phi)}{(\alpha^h)^2}+\frac{\alpha^h\left(\kappa-\alpha^h(\kappa+r)+r+\phi\right)}{(\alpha^h)^3+\kappa x\left(\kappa-\alpha^h r+r+\phi\right)/\sigma_\theta^2}=0$$

that $\alpha^h\in[1/2,1]$ always satisfies, that $\alpha^h(\phi,0)=\frac{r+\kappa+\phi}{r+\kappa+2\phi}$ (stated in section S.3.1) and that

$$F_\alpha\left(\phi,0,\frac{r+\kappa+\phi}{r+\kappa+2\phi}\right)=\frac{(\kappa+r+2\phi)^3}{(\kappa+r+\phi)^2}>0.$$

The decision to refer to an 'extension', as opposed to a property of the equilibrium coefficients over the domain under study, is because the hidden model is discontinuous at $\sigma_\xi=0$, in the sense discussed in Appendix C in the paper.

(3) Uniform convergence of equilibrium variables over compact sets as $\sigma_\xi^2\searrow 0$.

Consider any compact set $[\underline{\phi}, \overline{\phi}] \subset (0, \infty)$ and a sequence $(\sigma_n) \searrow 0$. Let $f(\phi, \sigma_n)$ denote any continuously differentiable function of the equilibrium variables. Then,

(i) *Pointwise convergence:* for all $\phi \in [\underline{\phi}, \overline{\phi}]$, $f(\sigma_n, \phi) \to f(0, \phi)$ as $n \to \infty$.

(ii) *Equicontinuity.* $\exists K > 0$ independent of n s.t. $|f(\sigma_n, \phi_1) - f(\sigma_n, \phi_2)| < K|\phi_1 - \phi_2|$.

(Part (i) is by continuity, and (ii) follows from $\frac{\partial f}{\partial \phi}(\phi, \sigma)$ being continuous over the compact set $[\underline{\phi}, \overline{\phi}] \times [0, \max\{\sigma_n : n \in \mathbb{N}\}]$.) We conclude that $(f(\sigma_n, \cdot))_{n \in \mathbb{N}}$ converges uniformly to $f(0, \cdot)$ over $[\underline{\phi}, \overline{\phi}]$, i.e., for all $\epsilon > 0$, there is $N > 0$, s.t., for all $n > N$, $|f(\sigma_n, \phi) - f(0, \phi)| < \epsilon$ for all $\phi \in [\underline{\phi}, \overline{\phi}]$. This uniform continuity can be used in the proofs of Propositions 8, 11 and 12 to strengthen the statements to hold in neighborhoods of $\sigma_\xi = 0$ for compact sets of levels of persistence.

S.4 Discretized Model and Limit Demand Sensitivity

This appendix introduces a sequence of discrete-time counterparts of our continuous-time game that allows us to refine the concept of stationary linear Markov equilibrium by choosing a sensitivity of demand equal to -1. More specifically, we will show that, along such sequence, -1 is the limiting value of the sensitivity of demand arising from the consumer's best-response problem as the period length shrinks to zero.

Fix $\Delta > 0$ and consider a consumer who interacts with a sequence of short-run firms in a stochastic game of period length Δ. Specifically, at each $t \in \mathbb{T} := \{0, \Delta, 2\Delta, 3\Delta, ...\}$ the consumer shops for a product that is supplied by a single firm (*firm* t). The timing of events over $[t, t+\Delta)$ is as in the baseline model: first, firm t posts a price; second, having observed this price, the consumer chooses how much to buy; third, the purchase is recorded with noise, and subsequently incorporated into the score. The same sequence of events then repeats at $[t+\Delta, t+2\Delta)$, but now with the next firm.

The discretized model consists of the dynamics

$$\begin{aligned}\theta_{t+\Delta} &= \theta_t - \kappa\Delta(\theta_t - \mu) + \sqrt{\Delta}\epsilon^{\theta}_{t+\Delta}\\ Y_{t+\Delta} &= Y_t - \phi\Delta Y_t + Q_t\Delta + \sqrt{\Delta}\epsilon^{\xi}_{t+\Delta}\end{aligned}$$

where $\epsilon^{\theta}_t \sim \mathcal{N}(0, \sigma^2_\theta)$ and $\epsilon^{\xi}_t \sim \mathcal{N}(0, \sigma^2_\xi)$ are independent across time, and the sequences $(\epsilon^{\theta}_t)_{t\in\mathbb{T}}$ and $(\epsilon^{\xi}_t)_{t\in\mathbb{T}}$ independent from one another. Finally, the consumer's utility over period $[t, t+\Delta)$ given $(\theta_t, P_t, Q_t) = (\theta, p, q)$ takes the form

$$u^{\Delta}(\theta, p, q) = \left((\theta - p)q - \frac{q^2}{2}\right)\Delta.$$

It is easy to see that if the firms conjecture a strategy for the consumer that is linear in (p, θ, M) with weight $-\zeta \neq 0$ on the current price, then, from their perspective, realized prices, $(P_t)_{t\in\mathbb{T}}$, and realized quantities, $(Q_t)_{t\in\mathbb{T}}$, satisfy $P_t = \mathbb{E}[Q_t|Y_t]/\zeta$, $t \in \mathbb{T}$. Thus, firms set prices and conjecture past quantities according to

$$P_t = \frac{\delta + (\alpha+\beta)M_t}{\zeta} \quad \text{and} \quad Q_t = \delta + \alpha\theta_t + \beta M_t,\ t \in \{0, \Delta, 2\Delta, ...\}, \tag{S.11}$$

respectively, for some coefficients ζ, α, β and δ. We allow this conjectured coefficients to depends on Δ. However, we make two assumptions. First, we restrict the analysis to the case $\zeta > 0$, $\alpha > 0$, $\beta < 0$ and $\alpha + \beta > 0$. Second, we assume that all the coefficients are bounded in a neighborhood of $\Delta = 0$, and also bounded away from zero. Observe that these are minimal properties that a meaningful dynamic extension of the outcome of a static interaction must have. In what follows, we omit the dependence of the coefficients on the period length.

We now proceed in three steps. First, we find an expression for the weight that the consumer's best-response attaches to the current price when firms both set prices and form beliefs using (S.11). Call this weight $-\hat{\zeta}$. Second, we show that at any history at which firm t sets a price different than the one prescribed by (S.11), the consumer optimally responds with the same linear strategy used along the path of (S.11); thus $-\hat{\zeta}$ is effectively the *sensitivity of demand*. Third, we show that $-\hat{\zeta}$ goes to -1 as $\Delta \searrow 0$. Importantly, these steps hold under any linear conjecture by the firms (in particular, for $\zeta \neq \hat{\zeta}$), satisfying our requirements on bounds. Thus, $\hat{\zeta} = 1$ is a limiting property of the consumer's best-response along the sequence of games.

Step 1. Since from each firm's perspective the score carries past quantities that satisfy (S.11), $M_t := \mathbb{E}[\theta_t|Y_t] = \rho + \lambda Y_t$ for some $\rho \in \mathbb{R}$ and $\lambda > 0$ (potentially depending on Δ). In this case,

$$
\begin{aligned}
& M_{t+\Delta} - M_t = \lambda[Y_{t+\Delta} - Y_t] = \lambda[-\phi\Delta(M_t - \rho)/\lambda + Q_t\Delta + \sqrt{\Delta}\epsilon^{\xi}_{t+\Delta}] \\
\Rightarrow \quad & M_{t+\Delta} = M_t - \phi\Delta(M_t - \rho) + \lambda Q_t\Delta + \lambda\sqrt{\Delta}\epsilon^{\xi}_{t+\Delta}, \ t \in \mathbb{T}.
\end{aligned}
$$

Let V denote the consumer's value function when facing prices as stated in (S.11). Then, the following Bellman equation holds:

$$
\begin{aligned}
V(\theta, M) &= \max_{q\in\mathbb{R}} \left\{ \left[\left(\theta - \frac{\delta + (\alpha+\beta)M}{\zeta} \right) q - q^2/2 \right] \Delta + e^{-r\Delta}\mathbb{E}[V(\theta', M')|(M, \theta)] \right\} \\
s.t. & \\
\theta' &= \theta - \kappa\Delta(\theta - \mu) + \sqrt{\Delta}\epsilon^{\theta} \\
M' &= M - \phi\Delta(M - \rho) + \lambda q\Delta + \lambda\sqrt{\Delta}\epsilon^{\xi}.
\end{aligned}
$$

We look for a quadratic value function, i.e., $V(\theta, M) = v_0 + v_1\theta + v_2 M + v_3 M^2 + v_4\theta^2 + v_5\theta M$, where we omit the dependence of the coefficients on Δ. Letting $X := (\theta, M)$, we have that $V(X') = V(X) + DV(X)(X' - X) + \frac{1}{2}(X' - X)^{\top} D^2 V(X' - X)$, and straightforward algebra shows that the Bellman equation further reduces to

$$\begin{aligned}
V(\theta, M) \quad = \quad & \max_{q\in\mathbb{R}} \left\{ \left[\left(\theta - \frac{\delta + (\alpha+\beta)M}{\zeta} \right) q - q^2/2 \right] \Delta + e^{-r\Delta} V(\theta, M) \right. \\
& + e^{-r\Delta}\Delta(-\kappa[\theta-\mu]V_\theta + [-\phi(M-\rho) + \lambda q]V_M + \frac{1}{2}V_{\theta\theta}[\Delta\kappa^2(\theta-\mu)^2 + \sigma_\theta^2]) \\
& + e^{-r\Delta}\Delta V_{\theta M}[-\kappa(\theta-\mu)(-\phi\Delta(M-\rho) + q\lambda\Delta)] \\
& \left. + e^{-r\Delta}\Delta\frac{1}{2}V_{MM}[\phi^2\Delta(M-\rho)^2 + \lambda^2 q^2\Delta + \lambda^2\sigma_\xi^2 - 2\phi\lambda\Delta(M-\rho)q]) \right\}.
\end{aligned}$$

The first-order condition of this problem reads

$$[1-e^{-r\Delta}\lambda^2\Delta V_{MM}]q = \theta - \underbrace{\frac{\delta + (\alpha+\beta)M}{\zeta}}_{p=} + e^{-r\Delta}(\lambda V_M + \Delta V_{\theta M}[-\kappa(\theta-\mu)\lambda] - V_{MM}\phi\Delta(M-\rho)\lambda),$$

from where the contemporaneous price has a weight equal to

$$-\hat{\zeta} = -\frac{1}{1-e^{-r\Delta}\lambda^2\Delta V_{MM}} = -\frac{1}{1-2e^{-r\Delta}\lambda^2\Delta v_3}$$

in the consumer's linear best-response. As we show in step 3, ζ, which enters as a parameter in the consumer's best-response problem, turns out to affect coefficient v_3. In particular, to show that $\lim_{\Delta\to 0}\hat{\zeta} = 1$, it suffices that $\Delta v_3 \searrow 0$ while the rest of the terms that accompany it remain bounded. We turn to the the sensitivity of demand first.

Step 2. Consider now a history at which firm t posts a price $p \neq [\delta + (\alpha+\beta)M_t]/\zeta$. It is easy to see that at any such history the consumer's problem is of the form

$$\begin{aligned}
\max_{q\in\mathbb{R}} \quad & \left\{ \left[(\theta - p)\, q - q^2/2 \right] \Delta + e^{-r\Delta}\mathbb{E}[V(\theta', M')|(M,\theta)] \right\} \\
s.t. \quad & \theta' = \theta - \kappa\Delta(\theta-\mu) + \sqrt{\Delta}\epsilon^\theta \\
& M' = M - \phi\Delta(M-\rho) + \lambda q\Delta + \lambda\sqrt{\Delta}\epsilon^\xi.
\end{aligned}$$

In fact, since the deviation is not observed by subsequent firms, the consumer's continuation payoff given any fixed continuation strategy is unaffected by the deviation. But this implies that her continuation value—i.e., her best continuation payoff among admissible strategies—must be given by V found by solving the Bellman equation of the previous step. As a result, the consumer's optimal strategy is determined by the same first-order condition. In particular, $-\hat{\zeta}$, the weight that the linear best-response attaches to the current price, is effectively the sensitivity of demand.

Step 3. It is straightforward to verify that v_3 can be found by setting the coefficient on M^2 in the Bellman equation equal to zero. Such equation is

$$4\zeta^2\Delta\lambda^2 v_3^2 + 2\zeta v_3\{\zeta[(1-\Delta\phi)^2 - e^{\Delta r}] - 2\Delta(1-\Delta\phi)\lambda(\alpha+\beta)\} + e^{\Delta r}\Delta(\alpha+\beta)^2 = 0.$$

Letting $\Gamma := \zeta[(1-\Delta\phi)^2 - e^{\Delta r}] - 2\Delta(1-\Delta\phi)\lambda(\alpha+\beta)$, the two solutions are given by

$$v_3^{\pm} = \frac{-\Gamma \pm \sqrt{\Gamma^2 - 4\Delta^2\lambda^2 e^{\Delta r}(\alpha+\beta)^2}}{2\zeta\Delta\lambda^2}.$$

The square root is well defined for small Δ due to $\Delta^2[\zeta(r+2\phi)+2\lambda(\alpha+\beta)]^2$ and $4\Delta^2\lambda^2(\alpha+\beta)^2$ being the terms that dominate for low Δ in Γ^2 and $4\Delta^2\lambda^2 e^{\Delta r}(\alpha+\beta)^2$ respectively.

We now show that $\Delta v_3^{\pm} \searrow 0$ as $\Delta \searrow 0$ (but as we show below, v_3^- is the root associated with the equilibrium examined in the paper). To this end, observe that λ also depends on Δ. A calculation presented at the end of this appendix shows that this value satisfies the equation $F(\Delta,\lambda) = 0$ where

$$F(\Delta,\lambda) := \lambda - \frac{\sigma_\theta^2\alpha(1-\kappa\Delta)[2(\phi-\beta\lambda) - (\phi-\beta\lambda)^2\Delta]}{\sigma_\xi^2[2\kappa-\kappa^2\Delta][(\phi-\beta\lambda)(1-\kappa\Delta)+\kappa] + \sigma_\theta^2\alpha^2[2-\kappa\Delta-(\phi-\beta\lambda)(1-\kappa\Delta)\Delta]}.$$

It is easy to verify that, at $\Delta = 0$, the previous equation reduces to the quadratic function that determines the sensitivity of beliefs in the continuous-time game analyzed (equation (7) in the paper). Let $\lambda_0 = \Lambda(\phi,\alpha,\beta) > 0$, as in the paper. By definition of λ_0, $F(0,\lambda_0) = 0$. Moreover, since $\beta < 0$ and $\lambda_0 > 0$

$$\begin{aligned}\frac{\partial F}{\partial\lambda}(0,\lambda_0) &= \frac{(\sigma_\theta^2\alpha^2 + \sigma_\xi^2\kappa[\phi+\kappa-\beta\lambda_0])^2 + \beta\sigma_\theta^2\alpha[\sigma_\theta^2\alpha^2 + \kappa^2\sigma_\xi^2]}{(\sigma_\theta^2\alpha^2 + \sigma_\xi^2\kappa[\phi+\kappa-\beta\lambda_0])^2} \\ &> \frac{\sigma_\theta^4\alpha^3[\alpha+\beta] + \sigma_\xi^2\sigma_\theta^2\alpha\kappa^2[2\alpha+\beta]}{(\sigma_\theta^2\alpha^2 + \sigma_\xi^2\kappa[\phi+\kappa-\beta\lambda_0])^2} > 0\end{aligned}$$

where the last inequality follows from $\alpha + \beta > 0$. By the Implicit Function Theorem, therefore, the exists $\varepsilon > 0$ and a unique continuously differentiable function $\lambda(\Delta)$ such that $\lambda(0) = \lambda_0$, $F(\Delta,\lambda(\Delta)) = 0$, and $\lambda(\Delta) > 0$, for all $\Delta \in [0,\varepsilon]$.

Since $\lambda(\cdot)$ is bounded in that set, and both $\lambda(\cdot)$ and ζ being bounded away from zero, we conclude that

$$\Delta v_3^{\pm} = \frac{-\Gamma \pm \sqrt{\Gamma^2 - 4\Delta^2\lambda^2(\Delta)e^{\Delta r}(\alpha+\beta)^2}}{2\zeta\lambda^2(\Delta)} \to 0, \text{ as } \Delta \searrow 0,$$

due to the rest of the coefficients being bounded and $\Gamma := \zeta[(1-\Delta\phi)^2 - e^{\Delta r}] - 2\Delta(1-$

$\Delta\phi)\lambda(\alpha+\beta)$ also vanishing in the limit. This proves step 3.

Before showing that $F(\Delta,\lambda)=0$ is the equation for the sensitivity of beliefs that makes the quantity process (S.11) consistent with Bayesian updating, we make two observations.

1. It is easy to see that when $\zeta=1$, then, as $\Delta\searrow 0$,

$$v_3^{\pm}\to\frac{2\lambda_0(\alpha+\beta)+(r+2\phi)\pm\sqrt{[2\lambda_0(\alpha+\beta)+(r+2\phi)]^2-4\lambda_0^2(\alpha+\beta)^2}}{4\lambda_0^2}$$

the right-hand side being the two positive roots for the equation that v_3 must satisfy in the continuous-time program.[10] However, an equilibrium condition of the continuous-time model is $2\lambda v_3=\alpha+2\beta$ (last equation in (A.3)). As a result, either

$$2\lambda v_3^{+}=\alpha+2\beta \ \text{ or } \ 2\lambda v_3^{-}=\alpha+2\beta$$

must hold. However, the previous conditions reduce to

$$r+2\phi\pm\sqrt{(r+2\phi)^2+4\lambda(\alpha+\beta)(r+2\phi)}=2\beta\lambda.$$

Since $\beta<0$ in the equilibrium found, only v_3^- converges to the value of v_3 in the equilibrium studied.

2. In equilibrium, $\hat{\zeta}=\zeta$. Using v_3^-, straightforward algebra shows that this condition becomes

$$\begin{aligned} e^{r\Delta}(\zeta-1) &= -\Gamma(\zeta)-\sqrt{\Gamma^2(\zeta)-4\Delta^2\lambda^2(\Delta)e^{r\Delta}(\alpha+\beta)^2} \\ &= \frac{4\Delta^2\lambda^2(\Delta)e^{r\Delta}(\alpha+\beta)^2}{-\Gamma(\zeta)+\sqrt{\Gamma^2(\zeta)-4\Delta^2\lambda^2(\Delta)e^{r\Delta}(\alpha+\beta)^2}}. \end{aligned}$$

where the dependence of Γ on ζ is being made explicit. For sufficiently small Δ, however, $(1-\Delta\phi)^2-e^{\Delta r}<0$ and so $-\Gamma(\zeta)>0$ for all $\zeta\geq 1$. The linearity of both $2(\zeta-1)$ and $\Gamma(\zeta)$ in ζ then yields the existence of ζ^* such that the previous equality holds. In particular, the convergence to of ζ to 1 along a sequence of equilibria must be from above.

Equation for λ. We conclude with the derivation of the equation that λ must satisfy for small Δ. For notational simplicity, we set $\mu=\rho=\delta=0$, as the means and intercepts do

[10] This equation can be obtained as follows: first, use (A.3) to solve for (α,β,δ) as a function of (v_2,v_3,v_5); second, insert the first-order condition (display preceding (A.3)) into (A.4); and, finally, equate the coefficient on M to zero in the resulting equation.

not affect the sensitivity of beliefs.

Define the matrices

$$X := \begin{bmatrix} \theta \\ Y \end{bmatrix};\ A_\Delta := \begin{bmatrix} 1-\kappa\Delta & 0 \\ \alpha\Delta & 1-(\phi-\beta\lambda)\Delta \end{bmatrix};\ B := \begin{bmatrix} \sigma_\theta & 0 \\ 0 & \sigma_\xi \end{bmatrix}\ \vec{\epsilon} := \begin{bmatrix} \epsilon^\theta \\ \epsilon^\xi, \end{bmatrix}$$

and notice that

$$X_{(j+1)\Delta} = A_\Delta X_{j\Delta} + \sqrt{\Delta} B\vec{\epsilon}_{(j+1)\Delta},\ j \in \mathbb{N}.$$

The solution to this difference equation is given by

$$X_{(j+1)\Delta} = A_\Delta^{j+1} X_0 + \sqrt{\Delta} A_\Delta^{j+1} \sum_{i=0}^{j} A_\Delta^{-(j+1-i)} B\vec{\epsilon}_{(j+1-i)\Delta}.$$

To obtain a stationary Gaussian process, therefore, we impose first that X_0 is Gaussian and independent of $(\vec{\epsilon}_{j\Delta})_{j\in\mathbb{N}}$. Moreover, stationary requires that $\vec{\mu} := \mathbb{E}[X_0] = 0$, so as to obtain $\mathbb{E}[X_{j\Delta}] = 0$ for all $j \in \mathbb{N}$. In addition, omitting the dependence on Δ, let Γ denote the candidate covariance matrix of $(X_{j\Delta})_{j\in\mathbb{N}}$. It follows that

$$\Gamma = A_\Delta^{j+1}\Gamma(A_\Delta^{j+1})^\top + \Delta A_\Delta^{j+1}\left[\sum_{i=0}^{j} A_\Delta^{-(j+1-i)} B^2 (A_\Delta^{-(j+1-i)})^\top\right](A_\Delta^{j+1})^\top,\ \forall j \in \mathbb{N}.$$

Moreover, taking consecutive differences leads to

$$0 = A_\Delta^j \Big\{ A_\Delta \Gamma A_\Delta^\top - \Gamma + \underbrace{\Delta A_\Delta \left[\sum_{i=0}^{j} A_\Delta^{-(j+1-i)} B^2 (A_\Delta^{-(j+1-i)})^\top\right] A_\Delta^\top - \Delta\left[\sum_{i=0}^{j-1} A_\Delta^{-(j-i)} B^2 (A_\Delta^{j-i})^\top\right]}_{=\Delta B^2} \Big\} (A_\Delta^j)^\top,$$

and thus, Γ is defined by the equation

$$A_\Delta \Gamma A_\Delta^\top - \Gamma + \Delta B^2 = 0.$$

Straightforward algebra leads to the following equations for the unknowns $\Gamma_{11} = \mathrm{Var}[\theta_{j\Delta}]$,

$\Gamma_{12} = \Gamma_{21} = \mathrm{Cov}[\theta_{j\Delta}, Y_{j\Delta}]$, and $\Gamma_{22} = \mathrm{Var}[Y_{j\Delta}]$, $j \in \mathbb{N}$:

$$\begin{aligned}
\Gamma_{11}(1-\kappa\Delta)^2 - \Gamma_{11} + \Delta\sigma_\theta^2 &= 0 \\
\Gamma_{11}\alpha\Delta(1-\kappa\Delta) + \Gamma_{12}(1-(\phi-\beta\lambda)\Delta)(1-\kappa\Delta) - \Gamma_{12} &= 0 \\
\Gamma_{11}(\alpha\Delta)^2 + 2\Gamma_{12}(1-(\phi-\beta\lambda)\Delta)\alpha\Delta + \Gamma_{22}(1-(\phi-\beta\lambda)\Delta)^2 - \Gamma_{22} + \Delta\sigma_\xi^2 &= 0.
\end{aligned}$$

This system has as a solution

$$\begin{aligned}
\Gamma_{11} &= \frac{\sigma_\theta^2}{2\kappa - \kappa^2\Delta} \\
\Gamma_{12} &= \frac{\alpha\sigma_\theta^2(1-\kappa\Delta)}{[2\kappa-\kappa^2\Delta][\phi-\beta\lambda+\kappa-(\phi-\beta\lambda)\kappa\Delta]} \\
\Gamma_{22} &= \frac{1}{2(\phi-\beta\lambda)-(\phi-\beta\lambda)^2\Delta}\left[\sigma_\xi^2 + \frac{\sigma_\theta^2\alpha^2\Delta}{2\kappa-\kappa^2\Delta} + \frac{2\alpha[1-(\phi-\beta\lambda)\Delta]\sigma_\theta^2\alpha(1-\kappa\Delta)}{[2\kappa-\kappa^2\Delta][\phi-\beta\lambda+\kappa-(\phi-\beta\lambda)\kappa\Delta]}\right].
\end{aligned}$$

(In particular, observe that we recover the expression for Γ in continuous time by letting $\Delta \to 0$ and replacing λ by λ_0.) To conclude, because

$$\lambda = \frac{\mathrm{Cov}[\theta_{j\Delta}, Y_{j\Delta}]}{\mathrm{Var}[\mathrm{Y_{j\Delta}}]} = \frac{\Gamma_{12}(\Delta,\lambda)}{\Gamma_{22}(\Delta,\lambda)},$$

straightforward algebra yields

$$\lambda = \frac{\sigma_\theta^2\alpha(1-\kappa\Delta)[2(\phi-\beta\lambda)-(\phi-\beta\lambda)^2\Delta]}{\sigma_\xi^2[2\kappa-\kappa^2\Delta][(\phi-\beta\lambda)(1-\kappa\Delta)+\kappa] + \sigma_\theta^2\alpha^2[2-\kappa\Delta-(\phi-\beta\lambda)(1-\kappa\Delta)\Delta]}.$$

This concludes the proof. □

S.5 Convexity Parameter in the Consumer's Cost Function

S.5.1 Main Results

Consider a more general flow utility of the form

$$(\theta_t - P_t)Q_t - \psi \frac{Q_t^2}{2}, \ \psi > 0.$$

We show that the main qualitative results of the paper still hold in this case.

Observable scores. With a flow utility as above, the static Nash equilibrium is

$$Q_t = \frac{\theta_t - P_t}{\psi} \ \text{ and } \ P_t = \frac{M_t}{2}.$$

Thus, the ex ante static outcome is $\mathbb{E}[Q_t^{static}] = \mu/2\psi = \mathbb{E}[P_t^{static}]/\psi$.

We aim to characterize a stationary linear Markov equilibrium supported by a sensitivity of demand of $1/\psi$. In this line, standard monopoly pricing with linear demand (i.e., the analog of Lemma 2 in the paper to this setting) shows that the dynamic outcome (Q_t, P_t) must satisfy

$$P_t = \psi \mathbb{E}[Q_t | Y_t].$$

Thus, $P_t = \psi[(\alpha+\beta)M_t + \delta\mu]$ if $Q_t = \alpha\theta_t + \beta M_t + \delta\mu$ along the path of play.

We present next the generalized version of Theorem 1 in the paper–to highlight the changes relative to that result, we highlight ψ in what follows.

Proposition 5 (Observable scores)**.** *There exists a unique stationary linear Markov equilibrium in the observable-scores model. In this equilibrium, $\alpha \in (0, 1/\psi)$, is the unique solution to the equation*

$$a = \frac{1}{\psi} + \frac{\Lambda(\phi, a, B(\phi, a)) a B(\phi, a)}{r + \kappa + \phi}, \ a \in [0, 1/\psi], \tag{S.12}$$

where Λ is as in the $\psi = 1$ case ((A.8)). In contrast, β and δ become

$$\beta = B(\phi, \alpha) := -\frac{\alpha^2(r + 2\phi)}{2(r+2\phi)\alpha - (r+\kappa+\phi)(\alpha - 1/\psi)} \in (-\alpha/2, 0) \tag{S.13}$$

$$\delta = D(\phi, \alpha) := \frac{\kappa(\alpha - 1/\psi) + [\alpha + 2B(\phi,\alpha)][\phi - (\alpha + B(\phi,\alpha))\Lambda(\phi,\alpha,B(\phi,\alpha))]}{2(r+\phi) + (\alpha + B(\phi,\alpha))\Lambda(\phi,\alpha,B(\phi,\alpha))},$$

and the ratchet effect is at play: the average price and quantity satisfy

$$\mathbb{E}[Q_t] = \frac{\mathbb{E}[P_t]}{\psi} = \frac{\mu}{\psi}\frac{r+\phi}{2(r+\phi)+(\alpha+\beta)\lambda} < \frac{\mu}{2\psi} = \mathbb{E}[Q_t^{static}].$$

Proof: Refer to the proofs subsection.

In other words, the equilibrium analysis and ratchet effect are qualitatively unchanged, as the convexity parameter ψ essentially scales the static benchmark. Of course, introducing $\psi \neq 1$ introduces a wedge between the price and the quantity, as the static demand ceases to have unit elasticity: as it is clear from the proposition, however, this wedge does not drive the economics of the paper.

More critically, this convexity parameter does not affect the conclusions drawn from the comparison of the observable- and hidden-score cases, as those are driven exclusively by the *relative* sensitivity of demand across cases. The next result shows that, in the hidden case, the equilibrium characterization is virtually identical to the $\psi = 1$ case, and that demand is less price sensitive than its observable counterpart. In addition, the same ranking of signaling coefficients ensues.

Proposition 6 (Hidden scores)**.** *There exists a unique stationary linear Markov equilibrium in the hidden-scores model. In this equilibrium, $\alpha^h \in (0, 1/\psi)$, is the unique solution to the equation*

$$a = \frac{1}{\psi} + \frac{\Lambda(\phi, a, -a/2)a[-a/2]}{r+\kappa+\phi},\ a \in [0, 1/\psi], \tag{S.14}$$

with Λ as in the paper. The sensitivity of demand is given by

$$\zeta^h = -\frac{2(r+2\phi)}{2(r+2\phi)+\alpha^h\Lambda(\phi, a^h, -a^h/2)}\frac{1}{\psi}.$$

Moreover, $\alpha(\phi) > \alpha^h(\phi)$ for all $\phi > 0$.

Proof: Refer to the proofs subsection.

In other words, the sensitivity of the demand in the hidden case scales down the observable counterpart, $1/\psi$, by the same function as in the $\psi = 1$ case (of course the magnitude is endogenous due to a^h depending on ψ as well).

S.5.2 Proofs

Proof of Proposition 1. The proof for existence proceeds in the exact same way as Theorem 1 in the paper, now with a price process of the form $P_t = \psi[(\alpha+\beta)M_t + \delta\mu]$. In particular, the systems (A.3)–(A.6) are modified as follows. First, the new first-order condition in the HJB equation reads

$$q = \frac{1}{\psi}\{\theta - \psi[\delta\mu + (\alpha+\beta)M] + \lambda[v_2 + 2v_3 M + v_5\theta]\}$$

which leads to the following conditions on the equilibrium coefficients:

$$\delta\mu = -\delta\mu + \frac{\lambda v_2}{\psi}, \ \alpha = \frac{1}{\psi} + \frac{\lambda v_5}{\psi}, \ \text{ and } \beta = \frac{2\lambda v_3}{\psi} - (\alpha+\beta). \tag{S.15}$$

Second, by the Envelope Theorem,

$$\begin{aligned}(r+\phi)[v_2 + 2v_3 M + v_5\theta] &= -\psi(\alpha+\beta)[\delta\mu + \alpha\theta + \beta M] - \kappa(\theta-\mu)v_5 \\ &\quad +[\lambda(\delta\mu + \alpha\theta + \beta M) - \phi(M - \mu + \lambda\bar{Y})]2v_3,\end{aligned} \tag{S.16}$$

which yields the following system of equations

$$\begin{cases}(r+\phi)v_2 = -\psi(\alpha+\beta)\delta\mu + \kappa\mu v_5 + [\lambda\delta\mu + \phi(\mu - \lambda\bar{Y})]2v_3 \\ (r+2\phi)2v_3 = -\psi(\alpha+\beta)\beta + 2v_3\lambda\beta \\ (r+\kappa+\phi)v_5 = -\psi(\alpha+\beta)\alpha + 2v_3\lambda\alpha.\end{cases} \tag{S.17}$$

Using that v_2, v_3 and v_5 can be written as a function of α, β and $\delta\mu$, this system becomes

$$\begin{cases}(r+\phi)\frac{2\delta\mu\psi}{\lambda} = -\psi(\alpha+\beta)\delta\mu + \psi\kappa\mu\frac{\alpha-1/\psi}{\lambda} + \psi[\lambda\delta\mu + \phi(\mu-\lambda\bar{Y})]\frac{\alpha+2\beta}{\lambda} \\ (r+2\phi)\frac{\alpha+2\beta}{\lambda}\psi = \psi\underbrace{[-(\alpha+\beta)\beta + \beta(\alpha+2\beta)]}_{=(\beta)^2} \\ (r+\kappa+\phi)\frac{\alpha-1/\psi}{\lambda}\psi = \psi\underbrace{[-(\alpha+\beta)\alpha + \alpha(\alpha+2\beta)]}_{=\alpha\beta}.\end{cases} \tag{S.18}$$

and simplifying ψ on both sides:

$$\begin{cases} (r+\phi)\frac{2\delta\mu}{\lambda} = -(\alpha+\beta)\delta\mu + \kappa\mu\frac{\alpha-1/\psi}{\lambda} + [\lambda\delta\mu + \phi(\mu-\lambda\bar{Y})]\frac{\alpha+2\beta}{\lambda} \\ (r+2\phi)\frac{\alpha+2\beta}{\lambda} = \underbrace{[-(\alpha+\beta)\beta + \beta(\alpha+2\beta)]}_{=(\beta)^2} \\ (r+\kappa+\phi)\frac{\alpha-1/\psi}{\lambda} = \underbrace{[-(\alpha+\beta)\alpha + \alpha(\alpha+2\beta)]}_{=\alpha\beta}. \end{cases} \tag{S.19}$$

From from the last two equations we obtain the expression for $B(\phi,\alpha)$ in the proposition, which is then inserted into the last equation to obtain the one for α. On the other hand, the expression for Λ that defines λ is unchanged, as this one is derived using a quantity process of the form $Q_t = \alpha\theta_t + \beta_t M_t + \delta\mu$. It is then straightforward to verify that the same argument used to prove the existence and uniqueness of a solution to α-equation over $[0,1]$ goes through for (S.12) when the domain becomes $[0,1/\psi]$ and $B(\phi,\alpha)$ is as in (**??**) (Lemma A.3 in Appendix A in the paper). Likewise, both the remaining coefficients of the value function and the transversality conditions follow from identical arguments to those in the $\psi=1$ case.

To obtain the expression for the average quantity, we proceed as in the proof of (ii) in Proposition 2. Specifically, using that the last two equations in (**??**) imply that $(\alpha+2\beta)(\alpha+\beta)\lambda = (r+2\phi)(\alpha+2\beta) + (r+\kappa+\phi)(\alpha-1/\psi) + (\alpha+\beta)^2\lambda$, we obtain

$$\begin{aligned} \delta &= \frac{\kappa(\alpha-1/\psi) + [\alpha+2\beta][\phi-(\alpha+\beta)\lambda]}{2(r+\phi)+(\alpha+\beta)\lambda} \\ &= \frac{\kappa(\alpha-1/\psi) + (\alpha+2\beta)\phi - (r+2\phi)(\alpha+2\beta) - (r+\kappa+\phi)(\alpha-1) - (\alpha+\beta)^2\lambda}{2(r+\phi)+(\alpha+\beta)\lambda} \\ &= \frac{-(r+\phi)[2(\alpha+\beta)-1/\psi] - (\alpha+\beta)^2\lambda}{2(r+\phi)+(\alpha+\beta)\lambda}. \end{aligned}$$

From here, it is easy to conclude that

$$\mathbb{E}[Q_t] = \mu[\alpha+\beta+\delta] = \frac{1}{\psi}\frac{\mu(r+\phi)}{2(r+\phi)+(\alpha+\beta)\lambda} < \frac{\mu}{2\psi} = \mathbb{E}[Q_t^{static}].$$

due to $\lambda(\alpha+\beta) > 0$. This concludes the proof. □

Proof of Proposition 2. The proof for existence proceeds in the exact same way as Proposition 9 in the paper. In particular, since the sensitivity of demand is endogenous, then, given a demand function the $Q(p) = \delta\mu + \alpha^h\theta + \zeta^h p$, the relationship

$$P_t = -\mathbb{E}[Q_t|Y_t]/\zeta^h$$

continues to hold on path. This implies, as in the $\psi = 1$ case, that (i) $\beta^h = \alpha^h/2$ and that (ii) we can use the price process as a state variable (the functional form of which is not directly affected, as it depends only the previous displayed expression and on $Q_t = \delta\mu + \alpha^h\theta_t + \beta^h M_t$ via the learning process). The factor ψ, however, will affect ζ^h via the first-order condition in the consumer's problem.

Specifically, the new first-order condition in the HJB equation reads

$$q = \frac{1}{\psi}\left[\theta - P - \frac{\alpha^h\lambda^h}{2\zeta^h}(v_2 + 2v_3 P + v_5\theta)\right].$$

As a result, we obtain the matching-coefficients conditions

$$\delta^h\mu = -\frac{1}{\psi}\frac{\alpha^h\lambda^h}{2\zeta^h}v_2, \; \alpha^h = \frac{1}{\psi}\left[1 - \frac{\alpha^h\lambda^h}{2\zeta^h}\right]v_5 \;\text{ and }\; \zeta^h = \frac{1}{\psi}\left[-1 - \frac{\alpha^h\lambda^h}{\zeta^h}v_3\right]. \tag{S.20}$$

Moreover, by the Envelope Theorem,

$$(r+\phi)[v_2 + 2v_3 P + v_5\theta] = q\left[-1 - v_3\frac{\alpha^h\lambda^h}{\zeta^h}\right] - 2v_3\phi\left[P + \frac{\delta^h\mu + \alpha^h\rho^h\mu}{2\zeta^h}\right] - \kappa v_5(\theta - \mu),$$

which leads to the system

$$\begin{aligned}
(r+\phi)v_2 &= \delta^h\mu\left[-1 - v_3\frac{\alpha^h\lambda^h}{\zeta^h}\right] - 2v_3\phi\frac{\delta^h\mu + \alpha^h\rho^h\mu}{2\zeta^h} + \kappa\mu v_5 \\
2(r+\phi)v_3 &= \zeta^h\left[-1 - v_3\frac{\alpha^h\lambda^h}{\zeta^h}\right] - 2v_3\phi \\
(r+\phi)v_5 &= \alpha^h\left[-1 - v_3\frac{\alpha^h\lambda^h}{\zeta^h}\right] - \kappa v_5.
\end{aligned}$$

Using that v_2, v_3 and v_5 can be written as a function of $\delta^h\mu, \alpha^h$ and ζ^h, respectively, and dividing by $\zeta^h\psi$ in each equation, we obtain the following system

$$\begin{cases}
-(r+\phi)\frac{2\delta^h\mu}{\alpha^h\lambda^h} = \delta^h\mu + 2\phi\frac{\zeta^h + 1/\psi}{\lambda^h\alpha^h}\frac{\delta^h\mu + \alpha^h\rho^h\mu}{2\zeta^h} + \kappa\mu\frac{2(1/\psi - \alpha^h)}{\alpha^h\lambda^h} \\
-2(r+2\phi)\frac{\zeta^h + 1/\psi}{\alpha^h\lambda^h} = \zeta^h \\
(r+\phi+\kappa)\frac{2(1/\psi - \alpha^h)}{\alpha^h\lambda^h} = \alpha^h.
\end{cases} \tag{S.21}$$

From the last equation in (S.4) it follows that α^h must satisfy $A^h(\phi, \alpha^h) = 0$, where

$$A^h(\phi, a) := (r + \kappa + \phi)(a - 1/\psi) - a\Lambda(\phi, a, -a/2)\left[-\frac{a}{2}\right],$$

thus proving (S.14). Also, from the second equation in (S.4),

$$\zeta^h = -\frac{2(r + 2\phi)}{2(r + 2\phi) + \alpha^h \lambda^h} \frac{1}{\psi}$$

as stated in the Proposition. One can then prove with identical arguments that α^h is characterized as the unique solution to this (S.14) over $[0, 1/\psi]$, and that all the remaining steps in the proof of Proposition 9 (analog of Lemma A.3, derivation of the remaining coefficients, and transversality conditions) can be derived with the same ideas.

Finally, we can parallel the proof of (ii) in Proposition 10 to show the point-wise ranking of the signaling coefficients. Specifically, we can write α and α^h as solutions to

$$\begin{aligned} -\frac{2}{\psi}(r + \kappa + \phi) + \alpha\,(2r + \kappa + \phi) + \alpha h(B(\phi, \alpha); \alpha) &= 0 \\ -\frac{2}{\psi}(r + \phi + \kappa) + \alpha(2r + \phi + \kappa) + \alpha h(-\alpha/2; \alpha) &= 0 \end{aligned}$$

respectively, where $B(\phi, \alpha) \in (-\alpha/2, 0)$ and

$$y \mapsto h(y; \alpha) := \left[\left(\frac{\sigma_\theta^2 \alpha[\alpha + y]}{\kappa \sigma_\xi^2} + \phi + \kappa\right)^2 - \frac{4\sigma_\theta^2 \alpha y \phi}{\kappa \sigma_\xi^2}\right]^{1/2} - \frac{\sigma_\theta^2 \alpha[\alpha + y]}{\kappa \sigma_\xi^2}.$$

As shown in the paper, $y \mapsto h(y; \alpha)$ is strictly decreasing over $\mathbb{R}_-$ for each $\alpha > 0$. But since $-\alpha/2 < B(\phi, \alpha) < 0$, and the left-hand sides of the two equalities are increasing functions of α (Lemma A.3 and its hidden-case analog for $\psi \neq 1$), it follows that $\alpha(\phi) > \alpha^h(\phi)$. This concludes the proof. □

S.6 Sum of Two Scores

Overview. Let us start with a slightly more general specification, the justification of which will be given after Lemma 1 below. There are two scores that obey

$$\begin{aligned} dY_{1t} &= (Q_t - \phi_1 Y_{1t})dt + \sigma_1 dZ_t^1 \\ dY_{2t} &= (Q_t - \phi_1 Y_{2t})dt + \rho\sigma_2 dZ_t^1 + \sigma_2\sqrt{1-\rho^2}dZ_t^2, \end{aligned}$$

where $\phi_2 > \phi_1$, and where $\rho \in [0,1]$ measures the degree of correlation of the shocks.

As in the baseline model, firm t chooses its price based on the observation of $Y_t := Y_{1t}+Y_{2t}$ only, whereas the consumer observes the history of $(W_t)_{t\geq 0}$.[11]

The first thing to notice is that $(W_t)_{t\geq 0}$ is not Markov, which implies that the consumer's problem ceases to be recursive in $(\theta_t, W_t)_{t\geq 0}$. In fact, it is easy to verify that

$$dW_t = (2Q_t - \phi_2 W_t + (\phi_2 - \phi_1)Y_{1t})dt + (\sigma_1 + \rho\sigma_2)dZ_t^1 + \sigma_2\sqrt{1-\rho^2}dZ_t^2, \tag{S.22}$$

so knowledge of Y_{1t} is needed to assess the evolution of W_t. In particular, this implies that the consumer will attempt to filter the components of Y_1 and Y_2 from the observations of $(W_t)_{t\geq 0}$ to forecast the evolution of prices.

To make the problem recursive, therefore, we can use as states the consumer's beliefs about Y_1 and Y_2.[12] This can be done by filtering Y_1 from the observations of W, and then using that $W_t - Y_{1t} = Y_{2t}$ also holds when (Y_1, Y_2) is replaced with the corresponding posterior mean. Moreover, in order to have a stationary model, we require those beliefs to be stationary.

Let $\hat{Y}_{it} := \mathbb{E}[Y_{it}|\mathcal{F}_t^W]$, where $(\mathcal{F}_t^W)_{t\geq 0}$ is the filtration generated by $(W_t)_{t\geq 0}$, and observe that $(Q_t)_{t\geq 0}$ is ultimately a function of the paths of $(\theta_t, W_t)_{t\geq 0}$. The following lemma is a direct application of the filtering equations for conditionally Gaussian systems (Chapter 12 in Liptser and Shiryaev, 1977):

[11] The case in which the consumer observes the histories of $(Y_{1t}, Y_{2t})_{t\geq 0}$ can be analyzed with the same approach displayed here, as will become clear soon.

[12] The fact that the consumer can filter first (Y_1, Y_2) from W, and then optimize using the dynamics of the posterior mean, is a consequence of the *separation principle* of Wonham (1960); for an application to a particular class of linear-quadratic control problems, see chapter 16 in Liptser and Shiryaev (1977)—the idea in that chapter easily extends to our setting.

Lemma 7. *Suppose that beliefs are stationary. Then,*

$$\begin{aligned} d\hat{Y}_{1t} &= (Q_t - \phi_1 \hat{Y}_{1t})dt + \frac{\sigma_1(\sigma_1+\rho\sigma_2)+\gamma^*(\phi_2-\phi_1)}{\sqrt{\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2}}dZ_t \\ d\hat{Y}_{2t} &= (Q_t - \phi_2 \hat{Y}_{2t})dt + \frac{\sigma_2(\sigma_2+\rho\sigma_1)-\gamma^*(\phi_2-\phi_1)}{\sqrt{\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2}}dZ_t \end{aligned} \tag{S.23}$$

where

$$\gamma^* = \frac{\sqrt{b^2+(\sigma_1\sigma_2)^2(1-\rho^2)(\phi_2-\phi_1)^2}-b}{(\phi_2-\phi_1)^2}$$

is the steady state variance and $b : -\sigma_1^2\phi_2 + \sigma_2^2\phi_1 + \rho\sigma_1\sigma_2(\phi_1+\phi_2)$. *Moreover,*

$$dZ_t := \frac{dW_t - [2Q_t - \phi_2 W_t + (\phi_2-\phi_1)\hat{Y}_{1t}]dt}{\sqrt{\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2}}$$

is a Brownian motion w.r.t. $(\mathcal{F}^W)_{t\geq 0}$.

Proof. Refer to Theorem 12.7 in Lipster and Shiryaev (1977).

Observe that since $W_t = \hat{Y}_{1t} + \hat{Y}_{2t}$ holds at all times, the process $(\theta, \hat{Y}_1, \hat{Y}_2)$ carries all the relevant information for the consumer's decision making (i.e., the problem is recursive).

There are two important observations that follow from the lemma. First, if $\rho = 1$—i.e., there is only one source of noise—we have that $\gamma^* = 0$. In order words, if the consumer only observes the sum of the scores, then, starting from any (non-trivial) initial degree of uncertainty, learning is always non-stationary when $\rho = 1$.

To have a stationary model, therefore, we require $\rho < 1$. However, when $\rho < 1$, the model just described is observationally equivalent to one in which the consumer observes a vector (Y_1, Y_2) of scores subject to only one source of noise and volatilities as in Lemma 1. Thus, in the next section we work with a generic tuple

$$\begin{aligned} dY_{1t} &= (Q_t - \phi_1 Y_{1t})dt + \sigma_1 dZ_t \\ dY_{2t} &= (Q_t - \phi_2 Y_{2t})dt + \sigma_2 dZ_t, \end{aligned}$$

assuming that the consumer observes the histories of each component, whereas firm t only observe the contemporaneous value of the sum, $t \geq 0$.

Equilibrium analysis. In this context, the natural object to characterize is an equilibrium quantity process

$$Q_t = \delta\mu + \alpha\theta + \beta_1 Y_{1t} + \beta_2 Y_{2t}.$$

The existence of a linear Markov equilibrium can be reduced to the existence of a negative solution for a single equation that characterizes β_1.

To derive the main equation, we take the sequence of steps followed in the determination of a linear Markov equilibrium for a single score:

1. Steady state distribution (θ, Y^1, Y^2). Let

$$X_t = \begin{bmatrix} \theta_t \\ Y_{1t} \\ Y_{2t} \end{bmatrix},\ A_0 = \begin{bmatrix} \kappa\mu \\ \delta\mu \\ \delta\mu \end{bmatrix},\ A_1 = \begin{bmatrix} \kappa & 0 & 0 \\ -\alpha & \phi_1 - \beta_1 & -\beta_2 \\ -\alpha & -\beta_1 & \phi_2 - \beta_2 \end{bmatrix},\ B = \begin{bmatrix} \sigma_\theta & 0 \\ 0 & \sigma_1 \\ 0 & \sigma_2 \end{bmatrix},\ Z_t^X = \begin{bmatrix} Z_t^\theta \\ Z_t \end{bmatrix}. \tag{S.24}$$

When $\beta_i < 0$, $i = 1, 2$ (as we would expect in this model of the ratchet effect), the matrix A_1 is invertible, and we can write

$$dX_t = A_1(A_1^{-1}A_0 - X_t)dt + BdZ_t^X$$

which is a three-dimensional Ornstein-Uhlenbeck process. Replicating the proof of Proposition A.1 in the paper, its stationary distribution is Gaussian with mean

$$\vec{\mu} := \begin{bmatrix} \mu \\ \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix} := A_1^{-1}A_0 \tag{S.25}$$

and covariance matrix

$$\Gamma := \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{12} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{13} & \Gamma_{23} & \Gamma_{33} \end{bmatrix} \text{ satisfying } BB^T = A_1\Gamma + \Gamma A_1^T \tag{S.26}$$

2. Consumer's best-response problem. Equipped with $\vec{\mu}$ and Γ, setting up the consumer's problem requires determining the price process. With unit demand sensitivity (which also holds here), the price process must satisfy

$$P_t = \mathbb{E}[Q_t|W_t] = \mathbb{E}[\delta\mu + \alpha\theta + \beta_1 Y_{1t} + \beta_2 Y_{2t}|W_t].$$

However, the firms know that $W_t = Y_{1t} + Y_{2t} \sim \mathcal{N}(\bar{W}, \text{Var}[W_t])$ where

$$\begin{aligned} \bar{W} &= \bar{Y}_1 + \bar{Y}_2 \\ \text{Var}[W_t] &= \Gamma_{22} + \Gamma_{33} + 2\Gamma_{23} \end{aligned} \tag{S.27}$$

Define the sensitivities

$$\begin{aligned} \lambda_\theta &:= \frac{\text{Cov}[\theta_t, W_t]}{\text{Var}[W_t]} = \frac{\Gamma_{12} + \Gamma_{13}}{\Gamma_{22} + \Gamma_{33} + 2\Gamma_{23}} \\ \lambda_1 &:= \frac{\text{Cov}[Y_{1t}, W_t]}{\text{Var}[W_t]} = \frac{\Gamma_{11} + \Gamma_{23}}{\Gamma_{22} + \Gamma_{33} + 2\Gamma_{23}} \\ \lambda_2 &:= \frac{\text{Cov}[Y_{2t}, W_t]}{\text{Var}[W_t]} = \frac{\Gamma_{22} + \Gamma_{23}}{\Gamma_{22} + \Gamma_{33} + 2\Gamma_{23}}. \end{aligned} \tag{S.28}$$

Thus, from firm t's perspective,

$$\begin{aligned} \mathbb{E}[\theta_t | W_t] &= \mu + \lambda_\theta (W_t - \bar{W}) \\ \mathbb{E}[Y_{1t} | W_t] &= \bar{Y}_1 + \lambda_1 (W_t - \bar{W}) \\ \mathbb{E}[Y_{2t} | W_t] &= \bar{Y}_1 + \lambda_2 (W_t - \bar{W}), \end{aligned} \tag{S.29}$$

and so the price process takes the form

$$\begin{aligned} P_t &= \delta\mu + \alpha[\mu - \lambda_\theta(\bar{Y}_1 + \bar{Y}_2)] + \beta_1[\bar{Y}_1 - \lambda_1(\bar{Y}_1 + \bar{Y}_2)] + \beta_2[\bar{Y}_2 - \lambda_1(\bar{Y}_1 + \bar{Y}_2)] \\ &\quad + (\alpha\lambda_\theta + \beta_1\lambda_1 + \beta_2\lambda_2)[Y_{1t} + Y_{2t}] \\ &= \pi_0(\vec{\phi}, \alpha, \beta_1, \beta_2) + \pi_1(\vec{\phi}, \alpha, \beta_1, \beta_2)[Y_{1t} + Y_{2t}] \end{aligned} \tag{S.30}$$

The HJB equation reads

$$\begin{aligned} rV = \sup_{a \in \mathbb{R}} \quad & (\theta - p)q - q^2/2 + V_{Y_1}(q - \phi_1 y_1) + V_{Y_2}(q - \phi_2 y_2) + V_\theta \kappa(\mu - \theta) \\ & + \frac{1}{2}\left[\sigma_\theta^2 V_{\theta\theta} + \sigma_1^2 V_{Y_1 Y_1} + \sigma_2^2 V_{Y_2 Y_2} + 2\sigma_1\sigma_2 V_{Y_1 Y_2}\right] \end{aligned} \tag{S.31}$$

and we look for a solution $V = v_0 + v_1 Y_1 + v_2 Y_2 + v_3\theta + v_4 Y_1^2 + v_5 Y_2^2 + v_6\theta^2 + v_7 Y_1 Y_2 + v_8 Y_1 \theta + v_9 Y_2 \theta$.

3. Equilibrium conditions. The FOC reads

$$q = \theta - \pi_0 - \pi_1(y_1 + y_2) + v_1 + v_2 + 2v_4 y_1 + 2v_5 y_2 + v_7 y_2 + v_7 y_1 + v_8\theta + v_9\theta$$

which yields the relationships

$$\begin{aligned}\alpha &= 1 + v_8 + v_9\\ \beta_1 &= 2v_4 + v_7 - \pi_1\\ \beta_2 &= 2v_5 + v_7 - \pi_1\\ \delta\mu &= v_1 + v_2 - \pi_0.\end{aligned}$$

In particular, we have that $\beta_2 = \beta_1 + 2(v_5 - v_4)$, so we obtain

$$\beta_1 = 2v_4 + v_7 - \pi_1(\vec{\phi}, 1 + v_8 + v_9, \beta_1, \beta_1 + 2(v_5 - v_4)) = \beta_1(v_4, v_5, v_7, v_8, v_9).$$

By the envelope theorem, moreover,

$$\begin{aligned}(\phi_1 + r)[v_1 + 2v_4 y_1 + v_7 y_2 + v_8\theta] &= [2v_4 + v_7 - \pi_1][\delta\mu + \alpha\theta + \beta_1 y_1 + \beta_2 y_2]\\ &\quad -2v_4\phi_1 y_1 - v_7\phi_2 y_2 + v_8\kappa(\mu - \theta)\\ (\phi_2 + r)[v_2 + 2v_5 y_2 + v_7 y_1 + v_9\theta] &= [2v_5 + v_7 - \pi_1][\delta\mu + \alpha\theta + \beta_1 y_1 + \beta_2 y_2]\\ &\quad -2v_5\phi_2 y_2 - v_7\phi_1 y_1 + v_9\kappa(\mu - \theta),\\ (\kappa + r)[v_3 + 2v_6\theta + v_8 y_1 + v_9 y_2] &= [v_8 + v_9 + 1][\delta\mu + \alpha\theta + \beta_1 y_1 + \beta_2 y_2]\\ &\quad +2v_6\kappa[\mu - \theta] - v_8\phi_1 y_1 - v_9\phi_2 y_2\end{aligned}$$

so letting $\vec{v} = (v_4, v_5, v_7, v_8, v_9)$, we obtain the following system of equations for $\vec{v}$:

$$\begin{aligned}2(2\phi_1 + r)v_4 &= \underbrace{[2v_4 + v_7 - \pi_1(\vec{v})]}_{=\beta_1(\vec{v})}\beta_1(\vec{v})\\ 2(2\phi_2 + r)v_5 &= \underbrace{[2v_5 + v_7 - \pi_1(\vec{v})]}_{=\beta_2(\vec{v})}\beta_2(\vec{v})\\ (\phi_1 + \phi_2 + r)v_7 &= [2v_4 + v_7 - \pi_1(\vec{v})]\beta_2(\vec{v})\\ (\phi_1 + r + \kappa)v_8 &= [2v_4 + v_7 - \pi_1(\vec{v})]\underbrace{[1 + v_8 + v_9]}_{\alpha}\\ (\phi_2 + r + \kappa)v_9 &= [2v_5 + v_7 - \pi_1(\vec{v})]\underbrace{[1 + v_8 + v_9]}_{\alpha}\end{aligned}$$

(It is clear from here that we cannot have $\beta_1 = \beta_2$: the FOC would imply that $v_4 = v_5$, and so the first two equations yield $\phi_1 = \phi_2$, a contradiction.)

Since v_4 and v_5 are the coefficients on Y_1^2 and Y_2^2, and both of these states are being controlled, they are strictly positive: In fact, this is implied by the first two equations:

$$v_4 = \frac{\beta_1^2}{2(2\phi_1 + r)} \quad \text{and} \quad v_5 = \frac{\beta_2^2}{2(2\phi_2 + r)}.$$

Thus, $v_4 = v_4(\beta_1)$ and

$$\begin{aligned}
& 2(2\phi_2 + r)v_5 = \beta_2^2 = [\beta_1 + 2(v_5 - v_4)]^2 = [\beta_1 - 2v_4(\beta_1)]^2 + 4[\beta_1 - 2v_4(\beta_1)]v_5 + 4v_5^2 \\
\Rightarrow \quad & v_5 = \frac{-2[2\beta_1 - 4v_4(\beta_1) - 2\phi_2 - r] \pm \sqrt{4[2\beta_1 - 4v_4(\beta_1) - 2\phi_2 - r]^2 - 16[\beta_1 - 2v_4(\beta_1)]^2}}{8} \\
\Rightarrow \quad & v_5 = \frac{-2[2\beta_1 - 4v_4(\beta_1) - 2\phi_2 - r] \pm \sqrt{-16[\beta_1 - 2v_4(\beta_1)][2\phi_2 + r] + [2\phi_2 + r]^2]}}{8}
\end{aligned}$$

We are seeking for $\beta_1 < 0$, so both roots are positive. (As in our model, however, we presume, that it is the smaller one that will work, in the sense that it will deliver a negative β_1.) Given a choice of root, we have $v_4(\beta_1)$ and $v_5(\beta_1)$.

On the other hand, we can subtract the 4th equation from the first in the system v_4–v_9 above to obtain

$$v_7 = \frac{2(2\phi_1 + r)v_4(\beta_1) - 2\beta_1[v_5(\beta_1) - v_4(\beta_1)]}{\phi_1 + \phi_2 + r} = v_7(\beta_1)$$

Finally, from the last two equations and $\beta_2 = \beta_1 + 2(v_5 - v_4)$,

$$v_9 = \frac{\phi_1 + r + \kappa}{\phi_2 + r + \kappa}\frac{\beta_2}{\beta_1}v_8 = \underbrace{\frac{\phi_1 + r + \kappa}{\phi_2 + r + \kappa}\left[1 + \frac{2[v_5(\beta_1) - v_4(\beta_1)]}{\beta_1}\right]}_{\rho(\vec{\phi},\beta_1):=} v_8 = \rho(\vec{\phi}, \beta_1)v_8.$$

Plugging this into the second to last equation we get

$$\begin{aligned}
v_8 &= \frac{\beta_1}{\phi_1 + r + \kappa - \beta_1[1 + \rho(\vec{\phi}, \beta_1)]} \\
\Rightarrow v_8 + v_9 &= \frac{\beta_1[1 + \rho(\vec{\phi}, \beta_1)]}{\phi_1 + r + \kappa - \beta_1[1 + \rho(\vec{\phi}, \beta_1)]}.
\end{aligned}$$

All this together, we obtain a single equation for β_1 given by

$$\beta_1 = \underbrace{\frac{2\beta_1^2}{2(2\phi_1+r)}}_{=2v_4} + \underbrace{\frac{2(2\phi_1+r)v_4(\beta_1) - 2\beta_1[v_5(\beta_1) - v_4(\beta_1)]}{\phi_1+\phi_2+r}}_{=v_7}$$
$$\underbrace{- \pi_1\left(\vec{\phi}, 1 + \frac{\beta_1[1+\rho(\vec{\phi},\beta_1)]}{\phi_1+r+\kappa-\beta_1[1+\rho(\vec{\phi},\beta_1)]}, \beta_1, \beta_1 + 2[v_5(\beta_1) - v_4(\beta_1)]\right)}_{\pi(\vec{\phi},\alpha,\beta_1,\beta_2)=} \tag{S.32}$$

where

$$v_4 = \frac{\beta_1^2}{2(2\phi_1+r)}$$
$$v_5 = \frac{-2[2\beta_1 - 4v_4(\beta_1) - 2\phi_2 - r] \pm \sqrt{-16[\beta_1 - 2v_4(\beta_1)][2\phi_2+r] + [2\phi_2+r]^2]}}{8}$$

Ratcheting properties.

- $\underline{\beta_2 < \beta_1 < 0}$: If instead $0 > \beta_2 > \beta_1$, we would have $\beta_2^2 < \beta_1^2$. However, from the first two envelope conditions

  $$\frac{2(2\phi_1+r)}{\beta_1^2}v_4 = \frac{2(2\phi_2+r)}{\beta_2^2}v_5.$$

  Since $\phi_2 > \phi_1$, it must be that $v_5 < v_4$. But $\beta_2 = \beta_1 + 2(v_5 - v_4)$ then would imply that $\beta_2 < \beta_1$, a contradiction.

- $\underline{\alpha < 1}$. Follows from $\beta_2 < \beta_1 < 0 \Rightarrow v_5 < v_4 \Rightarrow \rho > 0 \Rightarrow v_8 < 0 \Rightarrow v_8 + v_9 < 0$.

Finally, the expressions for Γ coefficients as well as the equation for β_1 are in a Mathematica code, and can be send upon request.

References

Abeille, M., C. Calauzènes, N. E. Karoui, T. Nedelec, and V. Perchet (2018): "Explicit shading strategies for repeated truthful auctions," *arXiv preprint arXiv:1805.00256*.

Amin, K., A. Rostamizadeh, and U. Syed (2014): "Repeated contextual auctions with strategic buyers," in *Advances in Neural Information Processing Systems*, pp. 622–630.

Beyhaghi, H., N. Golrezaei, R. Paes Leme, M. Pal, and B. Siva (2018): "Improved approximations for free-order prophets and second-price auctions," *arXiv preprint arXiv:1807.03435*.

Celis, L., G. Lewis, M. Mobius, and H. Nazerzadeh (2011): "Buy-it-now or Take-a-chance: A simple sequential screening mechanism," in *Proceedings of the 20th international conference on World wide web*, pp. 147–156. ACM.

Decarolis, F., M. Goldmanis, and A. Penta (2017): "Marketing agencies and collusive bidding in online ad auctions," Discussion Paper 23962, National Bureau of Economic Research.

Edelman, B., and M. Ostrovsky (2007): "Strategic bidder behavior in sponsored search auctions," *Decision support systems*, 43(1), 192–198.

Goldfarb, A., and C. Tucker (2011): "Search engine advertising: Channel substitution when pricing ads to context," *Management Science*, 57(3), 458–470.

Golrezaei, N., A. Javanmard, and V. Mirrokni (2018): "Dynamic Incentive-aware Learning: Robust Pricing in Contextual Auctions," Discussion paper, MIT Sloan.

Golrezaei, N., M. Lin, V. Mirrokni, and H. Nazerzadeh (2017): "Boosted second price auctions for heterogeneous bidders," *SSRN Electronic Journal (2017)*, pp. 1–41.

Hummel, P. (2018): "Reserve prices in repeated auctions," *International Journal of Game Theory*, 47(1), 273–299.

Kanoria, Y., and H. Nazerzadeh (2017): "Dynamic reserve prices for repeated auctions: Learning from bids," Discussion paper, University of Southern California.

Lahaie, S., A. Munoz Medina, B. Sivan, and S. Vassilvitskii (2018): "Testing Incentive Compatibility in Display Ad Auctions," in *Proceedings of the 2018 World Wide Web*

Conference on World Wide Web, pp. 1419–1428. International World Wide Web Conferences Steering Committee.

Liptser, R., and A. Shiryaev (1977): *Statistics of Random Processes, I*. Springer-Verlag, New York.

Milgrom, P. (2004): *Putting Auction Theory to Work*. Cambridge University Press.

Mirrokni, V., R. Paes Leme, P. Tang, and S. Zuo (2016): "Dynamic Auctions with Bank Accounts," in *Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence*, pp. 387–393.

Ostrovsky, M., and M. Schwarz (2016): "Reserve Prices in Internet Advertising Auctions: A Field Experiment," Discussion paper, Stanford University.

Paes Leme, R., M. Pal, and S. Vassilvitskii (2016): "A Field Guide to Personalized Reserve Prices," in *Proceedings of the 25th International World Wide Web Conference*, pp. 1093–1102. International World Wide Web Conferences Steering Committee.

Yuan, S., J. Wang, B. Chen, P. Mason, and S. Seljan (2014): "An empirical study of reserve price optimisation in real-time bidding," in *Proceedings of the 20th ACM SIGKDD international conference on Knowledge discovery and data mining*, pp. 1897–1906. ACM.