# SUBGRAPH DISTRIBUTIONS IN DENSE RANDOM REGULAR GRAPHS 

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#### Abstract

Given connected graph $H$ which is not a star, we show that the number of copies of $H$ in a dense uniformly random regular graph is asymptotically Gaussian, which was not known even for $H$ being a triangle. This addresses a question of McKay from the 2010 International Congress of Mathematicians. In fact, we prove that the behavior of the variance of the number of copies of $H$ depends in a delicate manner on the occurrence and number of cycles of length 3,4,5 as well as paths of length 3 in $H$. More generally, we provide control of the asymptotic distribution of certain statistics of bounded degree which are invariant under vertex permutations, including moments of the spectrum of a random regular graph.

Our techniques are based on combining complex-analytic methods due to McKay and Wormald used to enumerate regular graphs with the notion of graph factors developed by Janson in the context of studying subgraph counts in $\mathbb{G}(n, p)$.


## 1. Introduction

The study of the asymptotic distribution of small subgraph counts in the Erdős-Rényi random graphs $\mathbb{G}(n, p)$ and $\mathbb{G}(n, m)$ has been a topic of central interest in random graph theory. In particular, following a long series of papers, Ruciński [18] established the optimal conditions under which $X_{H}$, the number of unlabelled copies of $H$ in $\mathbb{G}(n, p)$, satisfies a central limit theorem. Furthermore, in general the distribution of small subgraphs in $\mathbb{G}(n, p)$ is known to a substantial degree of precision. We in particular refer the reader to the book $[8]$ and references therein for a more complete account.

With regards to asymptotic distributions, the state of affairs for random $d$-regular graphs is substantially less satisfactory. Let $\mathbb{G}(n, d)$ denote a uniformly random $d$-regular graph. Note that unlike $\mathbb{G}(n, p)$ or $\mathbb{G}(n, m)$, the edges in $\mathbb{G}(n, d)$ exhibit strong and non-obvious correlations and therefore even the question of determining the number of $d$-regular graphs has a rich history drawing on techniques ranging from switchings developed by McKay [12] (and refined by McKay and Wormald [16]), a complex-analytic technique of McKay and Wormald [15], and recent breakthroughs using fixed-point iteration due to Liebenau and Wormald [11]. We refer the reader to the excellent survey of Wormald [20] where the extensive history of this problem and various related enumeration problems are discussed.

McKay [13] in his 2010 ICM survey on graphs with a fixed degree sequence asked for an understanding of the asymptotic distribution of subgraph counts in dense random regular graphs, noting that "there is almost nothing known about the distribution of subgraph counts" for these models; the state of affairs has remained unchanged since. In particular, the only result which applies in this regime is work of McKay [14] which computes the expectation of the number of subgraphs of a fixed size in $\mathbb{G}(n, d)$ (see [5] for an extension to more exotic degree sequences). Our main result establishes a central limit theorem for counting copies of connected graphs $H$ in $\mathbb{G}(n, d)$ for $\min (d, n-d) \geq n / \log n$, and a consequence of our methods demonstrates a joint central limit theorem for so-called "graph factors" in the sense of Janson [6]. We additionally apply our techniques to show an analogous result for moments of the spectrum.

[^0]Though such a result for dense graphs has until now been out of reach, there is a rich literature regarding sparser graphs. A variety of results have been proven based on applications of the moment method and taking sufficiently fast growing moments. When $d$ is constant, the cycle count distribution was shown to asymptotically converge to a Poisson distribution independently by Bollobás [1] and Wormald [21]. This result was extended to strictly balanced graphs near the threshold for existence by Kim, Sudakov, and Vu [9], establishing a Poisson limit theorem in general. For results regarding asymptotic normality, McKay, Wormald, and Wysocka [17] proved asymptotic normality of cycle counts for $d$ tending to infinity sufficiently slowly, in particular proving normality of triangle counts when $d=o\left(n^{1 / 3}\right)$. A later result of Gao and Wormald [3] improved this result for a variety of subgraph structures $H$ by counting isolated copies, including extending the regime for triangle counts to $d=o\left(n^{2 / 7}\right)$. This was further improved by Gao [2] who improved the range of normality for triangle counts to $d=O\left(n^{1 / 2}\right)$. Finally, we note that the study of the asymptotic distribution of the number of spanning structures in random regular graphs has also been of interest (see [2] for further discussion).

Before stating our results let us formally define a random graph with a specified degree sequence.
Definition 1.1. Given nonnegative sequence $\mathbf{d}=\left(d_{1}, \ldots, d_{n}\right)$, let $\mathbb{G}(\mathbf{d})$ be a uniformly random simple graph $G$ with degree sequence $\mathbf{d}$. When $2 \mid d n$ let $\mathbb{G}(n, d)$ be a uniformly random simple graph on $n$ vertices which is $d$-regular, and let $G(n, d)$ be the set of possible outcomes. Given $G \sim \mathbb{G}(n, d)$ we define its density $p=p(G)=e(G) /\binom{v(G)}{2}=d /(n-1)$.

Our results provide a complete understanding of the small subgraph distribution for $d$-regular graphs when $d$ is dense. We first state a corollary of our main result regarding the distribution of subgraph statistics in $\mathbb{G}(n, d)$. Note first that the number of stars with $s \geq 2$ leaves in a $d$-regular graph on $n$ vertices is trivially always $n\binom{d}{s}$, so we exclude this case from consideration. Additionally, given graphs $H$ and $F$ let $N(H, F)$ be the number of unlabelled copies of $F$ in $H$. We write $C_{k}$ for a cycle of $k$ edges and $P_{k}$ for a path of $k$ edges.

Corollary 1.2. Fix a nonempty connected graph $H$ which is not a star and let $X_{H}$ denote the number of unlabelled copies of $H$ in $G \sim \mathbb{G}(n, d)$. If $n / \log n \leq \min (d, n-d), 2 \mid d n$, and $G \sim \mathbb{G}(n, d)$ we have:

- If $H$ contains a $C_{3}$ then

$$
\begin{gathered}
\left(\frac{X_{H}-\mathbb{E} X_{H}}{\sqrt{\operatorname{Var}\left[X_{H}\right]}}\right) \xrightarrow{d .} \mathcal{N}(0,1) \\
\text { with } \operatorname{Var}\left[X_{H}(G)\right]=6 N\left(H, C_{3}\right)^{2} p^{2 e(H)-3}(1-p)^{3} \frac{n^{2 v(H)-3}}{\operatorname{aut}(H)^{2}}+O\left(n^{2 v(H)-3-1 / 6}\right) .
\end{gathered}
$$

- If $H$ contains a $C_{4}$ and no $C_{3}$ then

$$
\left(\frac{X_{H}-\mathbb{E} X_{H}(G)}{\sqrt{\operatorname{Var}\left[X_{H}(G)\right]}}\right) \xrightarrow{d} \mathcal{N}(0,1)
$$

with $\operatorname{Var}\left[X_{H}(G)\right]=8 N\left(H, C_{4}\right)^{2} p^{2 e(H)-4}(1-p)^{4} \frac{n^{2 v(H)-4}}{\operatorname{aut}(H)^{2}}+O\left(n^{2 v(H)-4-1 / 6}\right)$.

- If $H$ does not contain a $C_{3}$ or $C_{4}$ then it contains a $P_{3}$ and

$$
\left(\frac{X_{H}-\mathbb{E}\left[X_{H}(G)\right]}{\sqrt{\operatorname{Var}\left[X_{H}(G)\right]}}\right) \xrightarrow{d .} \mathcal{N}(0,1)
$$

$$
\begin{aligned}
& \text { with } \operatorname{Var}\left[X_{H}(G)\right]=\left(10 p^{2 e(H)-5}(1-p)^{5} N\left(H, C_{5}\right)^{2}+6 p^{2 e(H)-4}(1-p)^{4} N\left(H, P_{3}\right)^{2}\right) \frac{n^{2 v(H)-5}}{\operatorname{aut}(H)^{2}}+ \\
& O\left(n^{2 v(H)-5-1 / 6}\right) \text {. }
\end{aligned}
$$

Remark 1.3. We note that $\mathbb{E} X_{H}(G)=(1+o(1)) n^{v(H)} p^{e(H)} / \operatorname{aut}(\mathrm{H})$ is known due to [14] and in fact our method can be used to compute the expectation to accuracy $o\left(\sqrt{\operatorname{Var}\left[X_{H}\right]}\right)$, but the resulting
expressions are rather involved. Additionally, the above result implies that for $H$ being a triangle we have that the variance of $X_{H}$ is on the order of $p^{3} n^{3}$ for $1 / \log n \leq p \leq 1 / 2$, whereas in $\mathbb{G}(n, p)$ the variance is of order $\max \left(p^{3} n^{3}, p^{5} n^{4}\right)$. Therefore for the range of $p$ under consideration $\operatorname{Var}\left[X_{H}\right]$ is substantially lower than in $\mathbb{G}(n, p)$; this is unlike the results of McKay, Wormald, and Wysocka [17] and Gao and Wormald [3] when $p$ is sufficiently sparse. We also note that for $H$ not containing a triangle, $\operatorname{Var}\left[X_{H}\right]$ is in fact asymptotically smaller than the corresponding variance in $\mathbb{G}(n, m)$.

In general our results are sufficiently powerful to deduce the asymptotic distribution of statistics of fixed degree which is invariant under vertex permutation. To state our main result we will need the notion of graph factors as defined by Janson [6]. Let $x_{e}$ be the indicator random variable for whether an edge $e$ is in included in random graph $G \sim \mathbb{G}(n, d)$ and let $\chi_{e}=\left(x_{e}-p\right) / \sqrt{p(1-p)}$. Note that by symmetry, marginally each $x_{e}$ is distributed as $\operatorname{Ber}(p)$ and thus $\chi_{e}$ has mean 0 and variance 1. However, as $G$ is a random regular graph there are substantial correlations between different edges $\chi_{e}$.
Definition 1.4. Fix a graph $H$ with no isolated vertices and an integer $n \geq|v(H)|$. Then define

$$
\gamma_{H}(\mathbf{x})=\sum_{\substack{E \subset K_{n} \\ E \simeq H}} \prod_{e \in E} \chi_{e} .
$$

Here $\simeq$ denotes graph isomorphism, specifically between $H$ and the graph spanned by the edges $E$. We will frequently adopt the shorthand that $\chi_{S}=\prod_{e \in S} \chi_{e}$. We call $\gamma_{H}(\mathbf{x})$ the graph factor corresponding to the graph $H$. When $H$ is connected and its minimum degree is at least 2, define the normalized graph factor to be

$$
\widetilde{\gamma}_{H}(G)=\left(\gamma_{H}(\mathbf{x})-E_{H}\right) / \sigma_{H}
$$

where $\sigma_{H}=\left(\frac{n^{v(H)}}{\operatorname{aut}(H)}\right)^{1 / 2}$ and $E_{H}=0$ if $H$ is not an even cycle and $E_{H}=\frac{2 n^{v(H) / 2}}{\operatorname{aut}(H)}=\frac{n^{v(H) / 2}}{v(H)}$ if it is.
Remark 1.5. In the original notion [6], all connected graphs $H$ are needed to express symmetric functions of graphs. However, as we will see in Section 4, $d$-regularity means that $\gamma_{H}$ for $H$ with a degree 1 vertex can be expressed as a linear combination of the smaller $\gamma_{H^{\prime}}$ (in terms of $d$ ). Additionally, the $E_{H}$ (approximate expectation) term for even cycles makes an appearance due to regularity. This is another departure from $\mathbb{G}(n, p)$ behavior, as in the independent setting the expectation of every $\gamma_{H}(\mathbf{x})$ is 0 .

We now are in position to state our main result.
Theorem 1.6. Fix a collection of nonisomorphic connected graphs $\mathcal{H}=\left\{H_{i}: 1 \leq i \leq k\right\}$ each of minimum degree at least 2 . Let $n / \log n \leq \min (d, n-d), 2 \mid d n$, and $G \sim \mathbb{G}(n, d)$. Then as $n \rightarrow \infty$ (uniformly in d), we have

$$
\left(\widetilde{\gamma}_{H_{i}}(G)\right)_{1 \leq i \leq k} \xrightarrow{d .} \mathcal{N}(0,1)^{\otimes k} .
$$

Furthermore, $\operatorname{Var}\left[\gamma_{H}(G)\right]=\left(1+O\left(n^{-1 / 6}\right)\right) \sigma_{H}^{2}$ and $\mathbb{E} \gamma_{H}(G)=E_{H}+O\left(n^{-1 / 6} \sigma_{H}\right)$.
Remark 1.7. The convergence in distribution can be made quantitative in terms of Kolmogorov distance by quantifying the convergence in moments and using a suitable version of the Weierstrass approximation theorem. The associated bounds however are quantitatively quite poor.

Although Theorem 1.6 is stated in terms of graph factors, a straightforward computation allows one to deduce the asymptotic distribution of any symmetric statistic of the edges of bounded degree; one can think of these as the "building blocks" for all such statistics in $\mathbb{G}(n, d)$.

Our results also imply that the traces of fixed powers of $A_{G}$ (the adjacency matrix of $G$ ), or equivalently the moments of the spectrum, satisfy a joint central limit theorem. Using techniques of Sinai and Soshnikov [19] along with a suitable modifications one could likely extend the result to
prove normal fluctuations for sufficiently nice test functions (e.g. analytic functions with suitably large radius of convergence). However, given that substantially stronger results are likely plausible using Green's function estimates established by He [4] and results connecting such estimates with functional central limit theorems (see [10] and reference therein), we omit such an extension. Additionally, we remark that direct spectral techniques are insufficient to recover Theorem 1.6 since graph factors which do not correspond to cycles are not purely determined by the spectrum.

Corollary 1.8. Given $k \geq 3$, there exists a positive definite matrix $\Sigma_{k} \in \mathbb{R}^{k \times k}$ such that the following holds. For $G \sim \mathbb{G}(n, d)$ with $n / \log n \leq \min (d, n-d)$ and $2 \mid d n, E_{\ell}=\mathbb{E}\left(\operatorname{tr}\left(A_{G}^{\ell}\right)\right)$, and $\sigma_{\ell}^{2}=\operatorname{Var}\left[\operatorname{tr}\left(A_{G}^{\ell}\right)\right]$, we have

$$
\left(\sigma_{\ell}^{-1}\left(\operatorname{tr}\left(A_{G}^{\ell}\right)-E_{\ell}\right)\right)_{3 \leq \ell \leq k} \xrightarrow{d .} \mathcal{N}\left(0, \Sigma_{k}\right) .
$$

Remark 1.9. Note that $\operatorname{tr}\left(A_{G}\right)=0$ and $\operatorname{tr}\left(A_{G}^{2}\right)=d n$ deterministically.
1.1. Proof techniques. Our proof uses techniques from the enumeration of dense graphs with a specific degree sequence given by McKay and Wormald [15] combined with the notion of graph factors introduced by Janson [6]. The crucial technical point is that previous work regarding asymptotic normality relied on computing the raw moments of $X_{H}$ and therefore requires taking a number of moments which grows with $n$. In our approach, one instead notices that any symmetric statistic on $d$-regular graphs can be expressed in terms of simple building blocks, and we can directly prove a joint central limit theorem for this collection. In order to prove the necessary limit theorem, we only require an arbitrarily slowly growing moment of these graph factors and they are particularly well-behaved when using the complex-analytic techniques developed by McKay and Wormald [15]. In particular, the necessary moments of graph factors can be given a natural complex-analytic expression using the multidimensional Cauchy integral formula. Then desired estimates can be computed directly. In fact, certain comparisons to $|G(n, d)|$ simplify the situation, allowing us to avoid repeating a careful saddle point analysis as in the work of McKay and Wormald [15]. The nontrivial expectation contributions $E_{H}$ when $H$ is an even cycle come into play due to counting certain even-power monomials in a polynomial expansion associated to the edges of $H$.

We further believe combining the general method of considering graph factors along with recent work of Liebenau and Wormald [11], which enumerates graphs of degrees of intermediate sparsity, can likely be used to address asymptotic distribution for regular graphs of all sparsities, a direction we plan to pursue in future work.
1.2. Organization. In Section 2 we prove the main estimates regarding the expectation of $\chi_{S}$ for a fixed set of edges $S$ via contour integration techniques. In Section 3, we deduce Theorem 1.6 via the method of moments and a graph-theoretic argument which guarantees that the estimates in Section 2 are of sufficient accuracy. In Section 4 we develop the theory of graph factors in $d$-regular graphs and prove that any symmetric graph statistic of fixed degree, when evaluated on $d$-regular graphs, can be expressed as a polynomial of graph factors that are of the type described in Definition 1.4. Finally in Section 5 we deduce Corollaries 1.2 and 1.8 as straightforward consequences of our main results and the proofs in Section 4.
1.3. Notation. We use standard asymptotic notation throughout, as follows. For functions $f=$ $f(n)$ and $g=g(n)$, we write $f=O(g)$ or $f \lesssim g$ to mean that there is a constant $C$ such that $|f(n)| \leq C|g(n)|$ for sufficiently large $n$. Similarly, we write $f=\Omega(g)$ or $f \gtrsim g$ to mean that there is a constant $c>0$ such that $f(n) \geq c|g(n)|$ for sufficiently large $n$. Finally, we write $f \asymp g$ or $f=\Theta(g)$ to mean that $f \lesssim g$ and $g \lesssim f$, and we write $f=o(g)$ or $g=\omega(f)$ to mean that $f(n) / g(n) \rightarrow 0$ as $n \rightarrow \infty$. We write $O_{H}(1)$ for some unspecified constant that can be chosen as some bounded value depending only on $H$. Additionally we set $k!!=2^{k / 2} \cdot k!$ for even integers $k \geq 0$.

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## 2. Cancellation Estimates Based on Contour Integrals

2.1. Preliminary estimates. We first recall a number of estimates from the work of McKay and Wormald [15].
Lemma 2.1 ([15, Lemma 1]). Let $0 \leq \lambda \leq 1$ and $|x| \leq \pi$. Then we have that

$$
\left|1+\lambda\left(e^{i x}-1\right)\right|=(1-2 \lambda(1-\lambda)(1-\cos x))^{1 / 2} \leq \exp \left(-\frac{1}{2} \lambda(1-\lambda) x^{2}+\frac{1}{24} \lambda(1-\lambda) x^{4}\right) .
$$

Lemma 2.2 ([15, (3.3)]). We have for $x_{j} \in \mathbb{R}$ that

$$
\sum_{1 \leq j<k \leq \ell}\left(x_{j}+x_{k}\right)^{2} \geq(\ell-2) \sum_{1 \leq j \leq \ell} x_{j}^{2}, \quad \sum_{1 \leq j<k \leq \ell}\left(x_{j}+x_{k}\right)^{4} \leq 8(\ell-1) \sum_{1 \leq j \leq \ell} x_{j}^{4} .
$$

We also require the following elementary estimate (a variant of which appears in [15, p. 8]); we provide a proof for the sake of completeness.

Lemma 2.3. We have for $m \geq m_{2.3}$ that

$$
\int_{-\pi / 16}^{\pi / 16} \exp \left(-m x^{2}+m x^{4}\right) d x=\left(1 \pm 2 m^{-1}\right) \sqrt{\pi / m} .
$$

Proof. Notice that for $m$ larger than an absolute constant,

$$
\begin{aligned}
\int_{-\pi / 16}^{\pi / 16} \exp \left(-m x^{2}+m x^{4}\right) d x & =\int_{-m^{-2 / 5}}^{m^{-2 / 5}} \exp \left(-m x^{2}+m x^{4}\right) d x \pm \pi / 8 \cdot \exp \left(-m^{1 / 5}\right) \\
& =\int_{-m^{-2 / 5}}^{m^{-2 / 5}}\left(1 \pm 2 m x^{4}\right) \exp \left(-m x^{2}\right) d x \pm \pi / 8 \cdot \exp \left(-m^{1 / 5}\right) \\
& =\left(1 \pm 2 m^{-1}\right) \sqrt{\pi / m}
\end{aligned}
$$

We will also use an elementary estimate bounding large moments in the following twisted Gaussian integral.

Lemma 2.4. We have

$$
\int_{-\pi / 16}^{\pi / 16}|x|^{k} \exp \left(-m x^{2}+m x^{4}\right) d x \leq \sqrt{2 \pi} k^{k / 2} m^{-(k+1) / 2}
$$

Proof. Note

$$
\begin{aligned}
\int_{-\pi / 16}^{\pi / 16}|x|^{k} \exp \left(-m x^{2}+m x^{4}\right) d x & \leq \int_{-\infty}^{\infty}|x|^{k} \exp \left(-m x^{2} / 2\right) d x=m^{-(k+1) / 2} \int_{-\infty}^{\infty}|x|^{k} \exp \left(-x^{2} / 2\right) d x \\
& =\sqrt{2 \pi} m^{-(k+1) / 2} \mathbb{E}_{Z \sim \mathcal{N}(0,1)}|Z|^{k} \leq k^{k / 2} \sqrt{2 \pi} m^{-(k+1) / 2}
\end{aligned}
$$

We will need another polynomial inequality in the real numbers.
Lemma 2.5. For $x_{1}, \ldots, x_{\ell} \in \mathbb{R}$ we have

$$
k!\sum_{j_{1}<\cdots<j_{k}} x_{j_{1}}^{2} \cdots x_{j_{k}}^{2} \leq\left(\sum_{j} x_{j}^{2}\right)^{k} \leq k!\sum_{\substack{j_{1}<\cdots<j_{k} \\ 5}} x_{j_{1}}^{2} \cdots x_{j_{k}}^{2}+\binom{k}{2}\left(\max _{j} x_{j}^{2}\right)\left(\sum_{j} x_{j}^{2}\right)^{k-1}
$$

Proof. The first inequality is trivial. For the second, consider expanding $\left(\sum_{j} x_{j}^{2}\right)^{k}$ and removing the terms which have no repeated index. For the remaining terms, remove the first term in the sequence that later repeats and bound it by $\max _{j} x_{j}^{2}$. It is easy to check that the resulting map on index sequences has fibers of size at most $\binom{k}{2}$.

Finally, we require the main result of $[15$, Theorem 1] which provides a sharp estimate for $|G(n, d)|$.

Theorem 2.6. There exists $\varepsilon=\varepsilon_{2.6}>0$ such that for $n / \log n \leq \min (d, n-d), 2 \mid d n, \lambda=d /(n-1)$, and $r=\sqrt{\lambda /(1-\lambda)}$, we have

$$
\begin{aligned}
|G(n, d)| & =2^{1 / 2}\left(2 \pi \lambda^{d+1}(1-\lambda)^{n-d} n\right)^{-n / 2} \exp \left(\frac{-1+10 \lambda-10 \lambda^{2}}{12 \lambda(1-\lambda)}+O\left(n^{-\varepsilon}\right)\right) \\
& =\frac{\left(1+r^{2}\right)^{\binom{n}{2}}}{\left(2 \pi r^{d}\right)^{n}}\left(\frac{2 \pi}{\lambda(1-\lambda) n}\right)^{n / 2}\left(2^{1 / 2} \exp \left(\frac{-1+10 \lambda-10 \lambda^{2}}{12 \lambda(1-\lambda)}+O\left(n^{-\varepsilon}\right)\right)\right)
\end{aligned}
$$

2.2. Graph factor estimates. The crucial estimates for the remainder of the proof will be the following inequalities controlling the behavior of the constituent expectations in a graph factor.

Proposition 2.7. There is $C=C_{2.7}>0$ so that for a set of distinct edges $S \subseteq K_{n}$ the following holds. Let $p=d /(n-1), n / \log n \leq \min (d, n-d)$, and $2 \mid d n$. Recall the notation $\chi_{S}$ from Definition 1.4.

- For any $S$ such that $|S| \leq \sqrt{\log n}$ we have

$$
\left|\mathbb{E}_{G \sim \mathbb{G}(n, d)} \chi_{S}\right| \leq C n^{-|S| / 2+1 / 4}
$$

- For $S$ such that $|S| \leq \sqrt{\log n}$, and there is a connected component which is an odd cycle, we have

$$
\left|\mathbb{E}_{G \sim \mathbb{G}(n, d)} \chi_{S}\right| \leq C n^{-1 / 4} n^{-|S| / 2}
$$

- For $S$ such that $|S| \leq \sqrt{\log n}$, the set of edges form a set of vertex disjoint even cycles, and there are $\ell$ disjoint cycles, we have

$$
\left|\mathbb{E}_{G \sim \mathbb{G}(n, d)} \chi_{S}-2^{\ell} n^{-|S| / 2}\right| \leq C n^{-1 / 5} n^{-|S| / 2}
$$

As mentioned the initial reduction in the proof closely mimic that of the proof of [15, Theorem 1]. Proof. Note that by complementing $\mathbb{G}(n, d)$ and replacing $p$ by $1-p$, we have that

$$
\mathbb{E}_{G \sim \mathbb{G}(n, d)} \chi_{S}=\left.(-1)^{\mid S}\right|_{\mathbb{E}_{G \sim \mathbb{G}(n, n-d-1)} \chi_{S}}
$$

Therefore it suffices to treat the case where $p \leq 1 / 2$.
By Cauchy's integral formula and taking the contours for $z_{j}$ to be circles of radius $r=\sqrt{p /(1-p)}$ around the origin we have

$$
\begin{align*}
& \mathbb{E}_{G \sim \mathbb{G}(n, d)} \chi_{S}=\frac{(2 \pi i)^{-n}}{|G(n, d)|} \oint \cdots \oint \frac{\prod_{(j, k) \notin S}\left(1+z_{j} z_{k}\right) \prod_{(j, k) \in S}\left(-p+(1-p) z_{j} z_{k}\right) / \sqrt{p(1-p)}}{\prod_{j \in[n]} z_{j}^{d+1}} d z \\
& =\frac{\left.\left(1+r^{2}\right)^{n} \begin{array}{c}
n \\
2
\end{array}\right)}{\left(2 \pi r^{d}\right)^{n}|G(n, d)|} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\prod_{(j, k) \notin S}\left(1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right) \prod_{(j, k) \in S}(p(1-p))^{1 / 2}\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)}{\exp \left(i d \sum_{j \in[n]} \theta_{j}\right)} d \theta \tag{2.1}
\end{align*}
$$

where $d z=\prod_{j \in[n]} d z_{j}$ and $d \theta=\prod_{j \in[n]} d \theta_{j}$, and the product is over unordered pairs $(j, k)$ which can be thought of as edges of the complete graph $K_{n}$.

Step 1: Localizing $\theta$. As in [15, Theorem 1], which corresponds to the case $S=\emptyset$, the first maneuver is to localize near the origin, and the techniques are similar. Let $t=\pi / 8$ and
fix $\varepsilon$ to be a small numerical constant to be chosen later ( $\varepsilon=10^{-10}$ will suffice). We divide indices based on where they lie on the circle: $S_{1}=\left\{j: \theta_{j} \in[-t, t]\right\}, S_{2}=\left\{j: \theta_{j} \in[t, \pi-t]\right\}$, $S_{3}=\left\{j: \theta_{j} \in[\pi-t, \pi] \cup[-\pi,-\pi+t]\right\}$, and $S_{4}=\left\{j: \theta_{j} \in[-\pi+t,-t]\right\}$. Let $\mathbf{R}$ denote the set of $\theta$ such at least one of $\left|S_{1}\right|\left|S_{3}\right| \geq n^{1+\varepsilon},\left|S_{2}\right|^{2} \geq n^{1+\varepsilon}$, or $\left|S_{4}\right|^{2} \geq n^{1+\varepsilon}$ holds. We have by Lemma 2.1 that

$$
\begin{align*}
& \left|\int_{\mathbf{R}} \frac{\prod_{(j, k) \notin S}\left(1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right) \prod_{(j, k) \in S}(p(1-p))^{1 / 2}\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)}{\exp \left(i d \sum_{j \in[n]} \theta_{j}\right)} d \theta\right| \\
& \leq \int_{\mathbf{R}}(1-2 p(1-p)(1-\cos (2 t)))^{n^{1+\varepsilon} / 3-|S|} d \theta \leq \exp \left(-\Omega\left(n^{1+\varepsilon / 2}\right)\right) \tag{2.2}
\end{align*}
$$

and therefore it will suffice to consider $\theta \notin \mathbf{R}$. Thus $\left|S_{2}\right|,\left|S_{4}\right| \leq n^{1 / 2+\varepsilon / 2}$. Furthermore note that $\theta \notin \mathbf{R}$ implies that $\left|S_{1}\right| \leq n^{\varepsilon}$ or $\left|S_{3}\right| \leq n^{\varepsilon}$. As the integrand is invariant under $\theta \rightarrow \theta+\pi$ (since $2 \mid d n$ ) it suffices to consider when $\left|S_{3}\right| \leq n^{\varepsilon}$ and multiply the resulting integral by a factor of 2 .

Let $\mathbf{R}^{\prime}$ denote the set of $\theta$ such that $\theta \notin \mathbf{R},\left|S_{3}\right| \leq n^{\varepsilon}$, and there is $\theta_{j} \notin\left[-n^{-1 / 2+\varepsilon}, n^{-1 / 2+\varepsilon}\right]$. We have

$$
\begin{align*}
& \left|\int_{\theta \in \mathbf{R}^{\prime}} \frac{\prod_{(j, k) \notin S}\left(1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right) \prod_{(j, k) \in S}(p(1-p))^{1 / 2}\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)}{\exp \left(i d \sum_{j \in[n]} \theta_{j}\right)} d \theta\right| \\
& \leq \int_{\theta \in \mathbf{R}^{\prime}} \prod_{(j, k) \notin S}\left|1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right| d \theta \leq e^{O(|S|)}\left(\frac{2 \pi}{\lambda(1-\lambda) n}\right)^{n / 2} \exp \left(-\Omega\left(n^{\varepsilon}\right)\right), \tag{2.3}
\end{align*}
$$

where we used a slight modification of $[15,(3.4),(3.5)]$ in the second inequality (namely, the analogy to the intermediate upper bound given in [15] is multiplicatively stable with respect to removal of the terms corresponding to $(j, k) \in S)$.

Finally, let $\mathbf{U}$ denote the set of $\theta$ such that $\left|\theta_{j}\right| \leq n^{-1 / 2+\varepsilon}$ for all $j$. Combining (2.1) to (2.3), the above symmetry observation, and Theorem 2.6 yields

$$
\begin{align*}
& \mathbb{E}_{G \sim \mathbb{G}(n, d)} \chi_{S} \pm \exp \left(-\Omega\left(n^{\varepsilon}\right)\right) \\
& =\frac{\left.2\left(1+r^{2}\right)^{n} \begin{array}{c}
n \\
2
\end{array}\right)}{\left(2 \pi r^{d}\right)^{n}|G(n, d)|} \int_{\mathbf{U}} \frac{\prod_{(j, k) \notin S}\left(1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right) \prod_{(j, k) \in S}(p(1-p))^{1 / 2}\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)}{\exp \left(i d \sum_{j \in[n]} \theta_{j}\right)} d \theta . \tag{2.4}
\end{align*}
$$

Step 2: Reducing the $S$ contribution to a polynomial. We next apply a Taylor series transformation in order to reduce to a more symmetric integral where the terms depending on $S$ are polynomial factors within the integrand. First notice that if $\theta_{j}, \theta_{k}$ are sufficiently small then

$$
\left|\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\left(1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right)^{-1}-i\left(\theta_{j}+\theta_{k}\right)\right| \leq\left|\theta_{j}+\theta_{k}\right|^{2} .
$$

Therefore for $\theta \in \mathbf{U}$ we have

$$
\left|\prod_{(j, k) \in S}\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\left(1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right)^{-1}-\prod_{(j, k) \in S} i\left(\theta_{j}+\theta_{k}\right)\right| \leq 2^{|S|}\left(2 n^{-1 / 2+\varepsilon}\right) \prod_{(j, k) \in S}\left|\theta_{j}+\theta_{k}\right| .
$$

Define
$P_{1}(\theta)=\prod_{(j, k) \notin S}\left(1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right) \prod_{(j, k) \in S}\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right), P_{2}(\theta)=\prod_{(j, k)}\left(1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right) \prod_{(j, k) \in S}\left(i\left(\theta_{j}+\theta_{k}\right)\right)$,
and note the above analysis implies

$$
\left|\int_{\mathbf{U}} \frac{P_{1}(\theta)-P_{2}(\theta)}{\exp \left(i d \sum_{j \in[n]} \theta_{j}\right)} d \theta\right| \leq 2^{|S|} \int_{\mathbf{U}} 2 n^{-1 / 2+\varepsilon} \prod_{(j, k) \in S}\left|\theta_{j}+\theta_{k}\right| \prod_{(j, k)}\left|1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right| d \theta
$$

$$
\begin{aligned}
& \leq 8^{|S|} n^{-1 / 2+\varepsilon} \int_{\mathbf{U}}\left|\theta_{1}\right|^{|S|} \prod_{(j, k)}\left|1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right| d \theta \\
& \leq 8^{|S|} n^{-1 / 2+\varepsilon} \int_{\mathbf{U}}\left|\theta_{1}\right|^{|S|} \prod_{(j, k)} \exp \left(-\frac{1}{2} p(1-p)\left(\theta_{j}+\theta_{k}\right)^{2}+\frac{1}{24} p(1-p)\left(\theta_{j}+\theta_{k}\right)^{4}\right) d \theta \\
& \leq 8^{|S|} n^{-1 / 2+\varepsilon} \int_{\mathbf{U}}\left|\theta_{1}\right|^{|S|} \exp \left(\sum_{j}-(n-2) \frac{p(1-p)}{2} \theta_{j}^{2}+(n-1) \frac{p(1-p)}{3} \theta_{j}^{4}\right) d \theta \\
& \leq 8^{|S|} n^{-1 / 2+\varepsilon} \int_{\mathbf{U}}\left|\theta_{1}\right|^{|S|} \exp \left(\sum_{j}-(n-2) \frac{p(1-p)}{2} \theta_{j}^{2}+(n-2) \frac{p(1-p)}{2} \theta_{j}^{4}\right) d \theta \\
& \lesssim 16^{|S|} n^{-1 / 2+\varepsilon} n^{-|S| / 2}|S|^{|S| / 2}(2 \pi /(p(1-p) n))^{n / 2}
\end{aligned}
$$

where we have applied Lemmas 2.1 to 2.4. By (2.4) and Theorem 2.6 it follows that

$$
\begin{equation*}
\mathbb{E}_{G \sim \mathbb{G}(n, d)} \chi_{S}=\frac{2\left(1+r^{2}\right)^{\binom{n}{2}}}{\left(2 \pi r^{d}\right)^{n}|G(n, d)|} \int_{\mathbf{U}} \frac{P_{2}(\theta)(p(1-p))^{|S| / 2}}{\exp \left(i d \sum_{j \in[n]} \theta_{j}\right)} d \theta \pm n^{-(|S|+1) / 2+2 \varepsilon} . \tag{2.5}
\end{equation*}
$$

Step 3: Uniform bound on the integral. We now prove the first bullet point in Proposition 2.7. Note that Lemmas 2.1 and 2.2 gives

$$
\begin{aligned}
& \left|\int_{\mathbf{U}} \frac{P_{2}(\theta)}{\exp \left(i d \sum_{j \in[n]} \theta_{j}\right)} d \theta\right| \leq \int_{\mathbf{U}} \prod_{(j, k) \in S}\left|\theta_{j}+\theta_{k}\right| \prod_{(j, k)}\left|1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right| d \theta \\
& \quad \leq 2^{|S|} \int_{\mathbf{U}}\left|\theta_{1}\right|^{|S|} \prod_{(j, k)} \exp \left(-\frac{1}{2} p(1-p)\left(\theta_{j}+\theta_{k}\right)^{2}+\frac{1}{24} p(1-p)\left(\theta_{j}+\theta_{k}\right)^{4}\right) d \theta \\
& \quad \leq 2^{|S|} \int_{\mathbf{U}}\left|\theta_{1}\right|^{|S|} \exp \left(\sum_{j}-(n-2) \frac{p(1-p)}{2} \theta_{j}^{2}+(n-1) \frac{p(1-p)}{3} \theta_{j}^{4}\right) d \theta \\
& \quad \leq 2^{|S|} \int_{\mathbf{U}}\left|\theta_{1}\right|^{|S|} \exp \left(\sum_{j}-(n-2) \frac{p(1-p)}{2}\left(\theta_{j}^{2}-\theta_{j}^{4}\right)\right) d \theta \\
& \left.\quad \lesssim n^{-|S| / 2}(4|S|)^{|S|}(2 \pi /(p(1-p) n))\right)^{n / 2}
\end{aligned}
$$

which immediately gives the desired initial estimate noting that the final term in enumeration count from Theorem 2.6 is bounded by $n^{1 / 5}$ and since $|S|$ is small.

Step 4: Cancellation from odd degree terms. We next prove that any polynomial factor in terms of the $\theta$ coefficients which is not an even polynomial exhibits additional cancellation. This will immediately imply the second bullet point as there are at most $2^{|S|}$ terms in $\prod_{(j, k) \in S}\left(\theta_{j}+\theta_{k}\right)$ and since there is an odd cycle component, every term has an index of degree 1. In particular it suffices to bound

$$
\int_{\mathbf{U}} \frac{\prod_{j \in[k]} \theta_{j}^{\ell_{j}} \prod_{(j, k)}\left(1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right)}{\exp \left(i d \sum_{j \in[n]} \theta_{j}\right)} d \theta
$$

where $\ell_{k}=1$, and $k \leq 2|S|$. For this, notice that by symmetry

$$
\begin{aligned}
& \left|\int_{\mathbf{U}} \frac{\prod_{j \in[k]} \theta_{j}^{\ell_{j}} \prod_{(j, k)}\left(1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right)}{\exp \left(i d \sum_{j \in[n]} \theta_{j}\right)} d \theta\right| \\
& =\frac{1}{n-k+1}\left|\int_{\mathbf{U}} \frac{\left(\sum_{k \leq j \leq n} \theta_{j}\right) \prod_{j \in[k-1]} \theta_{j}^{\ell_{j}} \prod_{(j, k)}\left(1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right)}{\exp \left(i d \sum_{\substack{j \in[n] \\
8}} \theta_{j}\right)} d \theta\right|
\end{aligned}
$$

$$
\begin{aligned}
& \leq \frac{1}{n-k+1} \int_{\mathbf{U}}\left|\sum_{k \leq j \leq n} \theta_{j}\right| \prod_{j \in[k-1]}\left|\theta_{j}\right|^{\ell_{j}} \prod_{(j, k)}\left|1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right| d \theta \\
& \leq \frac{2}{n} \int_{\mathbf{U}}\left|\sum_{k \leq j \leq n} \theta_{j}\right| \prod_{j \in[k-1]}\left|\theta_{j}\right|^{\ell_{j}} \exp \left(\sum_{j}-(n-2) \frac{p(1-p)}{2}\left(\theta_{j}^{2}-\theta_{j}^{4}\right)\right) d \theta \\
& =\frac{2}{n} \int_{\mathbf{U}} \mathbb{E}_{s \sim \operatorname{Rad}^{\otimes n}}\left|\sum_{k \leq j \leq n} s_{j} \theta_{j}\right| \prod_{j \in[k-1]}\left|\theta_{j}\right|^{\ell_{j}} \exp \left(\sum_{j}-(n-2) \frac{p(1-p)}{2}\left(\theta_{j}^{2}-\theta_{j}^{4}\right)\right) d \theta \\
& \leq \frac{2}{n} \int_{\mathbf{U}}\left(\mathbb{E}_{s \sim \operatorname{Rad}^{\otimes n}}\left(\sum_{k \leq j \leq n} s_{j} \theta_{j}\right)^{2}\right)^{1 / 2} \prod_{j \in[k-1]}\left|\theta_{j}\right|^{\ell_{j}} \exp \left(\sum_{j}-(n-2) \frac{p(1-p)}{2}\left(\theta_{j}^{2}-\theta_{j}^{4}\right)\right) d \theta \\
& \leq \frac{2}{n} \int_{\mathbf{U}}\left(\sum_{k \leq j \leq n} \theta_{j}^{2}\right)^{1 / 2} \prod_{j \in[k-1]}\left|\theta_{j}\right|^{\ell_{j}} \exp \left(\sum_{j}-(n-2) \frac{p(1-p)}{2}\left(\theta_{j}^{2}-\theta_{j}^{4}\right)\right) d \theta \\
& \leq \frac{2}{n^{1-\varepsilon}} \int_{\mathbf{U}} \prod_{j \in[k-1]}\left|\theta_{j}\right|^{\ell_{j}} \exp \left(\sum_{j}-(n-2) \frac{p(1-p)}{2}\left(\theta_{j}^{2}-\theta_{j}^{4}\right)\right) d \theta \\
& \lesssim n^{\left.-\sum_{j \in[k]}^{\ell_{j} / 2-1 / 2+2 \varepsilon}(2 \pi /(p(1-p) n))\right)^{n / 2}}
\end{aligned}
$$

as desired, where in the last line we apply Lemma 2.4 and use that $|S|$ is small.
Step 5: Even cycles. We now handle the third bullet point, proving that the integral is sufficiently close to the desired quantity. Using the technique in the previous step, and noting that given a set $S$ of $\ell$ disjoint even cycles there are $2^{\ell}$ terms in the expansion of $\prod_{(j, k) \in S}\left(\theta_{j}+\theta_{k}\right)$ where every vertex has even degree, we have

$$
\begin{align*}
\mid \mathbb{E}_{G \sim G(n, d)} \chi_{S}- & \left.\frac{2^{\ell+1}\left(1+r^{2}\right)^{\binom{n}{2}}(p(1-p))^{|S| / 2}}{\left(2 \pi r^{d}\right)^{n}|G(n, d)|} \int_{\mathbf{U}} \frac{\prod_{j \in[|S| / 2]} \theta_{j}^{2} \prod_{(j, k)}\left(1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right)}{\exp \left(i d \sum_{j \in[n]} \theta_{j}\right)} d \theta \right\rvert\, \\
& \lesssim n^{-1 / 4-|S| / 2} . \tag{2.6}
\end{align*}
$$

Notice that $\mathbb{E}_{G \sim \mathbb{G}(n, d)} \chi_{\emptyset}=1$ by definition and (2.6) applies with $S$ empty. Subtracting, it therefore suffices to prove

$$
\left|\frac{2^{\ell+1}\left(1+r^{2}\right)^{\binom{n}{2}}}{\left(2 \pi r^{d}\right)^{n}|G(n, d)|} \int_{\mathbf{U}} \frac{\left(\prod_{j \in[|S| / 2]} \theta_{j}^{2}-(p(1-p) n)^{-|S| / 2}\right) \prod_{(j, k)}\left(1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right)}{\exp \left(i d \sum_{j \in[n]} \theta_{j}\right)} d \theta\right| \lesssim n^{-1 / 4-|S| / 2}
$$

From Lemma 2.5 we have

$$
\frac{\left(\sum_{j} x_{j}^{2}\right)^{k}-k^{2}\left(\max _{j} x_{j}^{2}\right)\left(\sum_{j} x_{j}^{2}\right)^{k-1}}{k!} \leq \sum_{j_{1}<\cdots<j_{k}} x_{j_{1}}^{2} \cdots x_{j_{k}}^{2} \leq \frac{\left(\sum_{j} x_{j}^{2}\right)^{k}}{k!}
$$

and using our initial bounds from earlier it follows immediately that

$$
\left|\frac{2^{\ell+1}\left(1+r^{2}\right)^{\binom{n}{2}}\binom{n}{|S| / 2}^{-1}}{\left(2 \pi r^{d}\right)^{n}|G(n, d)|(|S| / 2)!} \int_{\mathbf{U}} \frac{\left((|S| / 2)^{2} n^{-1+2 \varepsilon}\left(\sum_{j} \theta_{j}^{2}\right)^{|S| / 2-1}\right) \prod_{(j, k)}\left(1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right)}{\exp \left(i d \sum_{j \in[n]} \theta_{j}\right)} d \theta\right| \lesssim n^{-1 / 4-|S| / 2} .
$$

Therefore, symmetrizing over all permutations of $[n]$ and trivially boundind some lower order contributions, we see it suffices to bound

$$
\frac{2^{\ell+1}\left(1+r^{2}\right)^{\binom{n}{2}}\binom{n}{(S \mid / 2}^{-1}}{\left(2 \pi r^{d}\right)^{n}|G(n, d)|(|S| / 2)!} \int_{\mathbf{U}} \frac{\left(\left(\sum_{j} \theta_{j}^{2}\right)^{|S| / 2}-(p(1-p))^{-|S| / 2}\right) \prod_{(j, k)}\left(1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right)}{\exp \left(i d \sum_{j \in[n]} \theta_{j}\right)} d \theta
$$

Notice that, once again,

$$
\begin{align*}
& \int_{\mathbf{U}} \frac{\left(\left(\sum_{j} \theta_{j}^{2}\right)^{|S| / 2}-(p(1-p))^{-|S| / 2}\right) \prod_{(j, k)}\left(1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right)}{\exp \left(i d \sum_{j \in[n]} \theta_{j}\right)} d \theta \\
& \leq \int_{\mathbf{U}}\left|\left(\sum_{j} \theta_{j}^{2}\right)^{|S| / 2}-(p(1-p))^{-|S| / 2}\right| \prod_{(j, k)}\left|1+p\left(e^{i\left(\theta_{j}+\theta_{k}\right)}-1\right)\right| d \theta \\
& \leq \int_{\mathbf{U}}\left|\left(\sum_{j} \theta_{j}^{2}\right)^{|S| / 2}-(p(1-p))^{-|S| / 2}\right| \exp \left(\sum_{j}-(n-2) \frac{p(1-p)}{2}\left(\theta_{j}^{2}-\theta_{j}^{4}\right)\right) d \theta \tag{2.7}
\end{align*}
$$

We now proceed via splitting (2.7) based on the size of $\sum_{j} \theta_{j}^{2}$. Defining the region $\mathbf{S}=\left\{\theta: \mid \sum_{j} \theta_{j}^{2}-\right.$ $\left.(p(1-p))^{-1} \mid \geq n^{-1 / 3}\right\}$ we have that

$$
\begin{aligned}
& \int_{\mathbf{U}} \mathbb{1}_{\theta \in \mathbf{S}}\left|\left(\sum_{j} \theta_{j}^{2}\right)^{|S| / 2}-(p(1-p))^{-|S| / 2}\right| \exp \left(\sum_{j}-(n-2) \frac{p(1-p)}{2}\left(\theta_{j}^{2}-\theta_{j}^{4}\right)\right) d \theta \\
& \leq \int_{\mathbf{U}} \mathbb{1}_{\theta \in \mathbf{S}}\left(2 n^{2 \varepsilon}\right)^{|S| / 2} \exp \left(\sum_{j}-(n-2) \frac{p(1-p)}{2}\left(\theta_{j}^{2}-\theta_{j}^{4}\right)\right) d \theta \\
& \leq \int_{\mathbf{U}} \mathbb{1}_{\theta \in \mathbf{S}}\left(2 n^{2 \varepsilon}\right)^{|S| / 2} \exp \left(\sum_{j} \frac{-\left(n-2 n^{2 \varepsilon}\right) p(1-p)}{2} \theta_{j}^{2}\right) d \theta \\
& \leq\left(2 n^{2 \varepsilon}\right)^{|S| / 2}\left(1 /\left(p(1-p)\left(n-2 n^{2 \varepsilon}\right)\right)\right)^{n / 2} \int_{\mathbb{R}^{n}} \mathbb{1}_{\left|\sum_{j \in[n]} x_{j}^{2}-\left(n-2 n^{2 \varepsilon}\right)\right| \geq p(1-p) n^{2 / 3} / 2} \exp \left(-\frac{1}{2} \sum_{j \in[n]} x_{j}^{2}\right) d x \\
& \leq \exp \left(O\left(n^{2 \varepsilon}\right)\right) \cdot(2 \pi /(p(1-p) n))^{n / 2} \mathbb{P}_{Z \sim \mathcal{N}(0,1)^{\otimes n}}\left[\left|\sum_{j} Z_{j}^{2}-n\right| \geq n^{3 / 5}\right] \\
& \leq \exp \left(n^{-1 / 10}\right) \cdot(2 \pi /(p(1-p) n))^{n / 2}
\end{aligned}
$$

which is sufficiently small as desired. For the remaining portion of (2.7) notice that

$$
\begin{aligned}
& \int_{\mathbf{U}} \mathbb{1}_{\theta \notin \mathbf{S}}\left|\left(\sum_{j} \theta_{j}^{2}\right)^{|S| / 2}-(p(1-p))^{|S| / 2}\right| \exp \left(\sum_{j}-(n-2) \frac{p(1-p)}{2}\left(\theta_{j}^{2}-\theta_{j}^{4}\right)\right) d \theta \\
& \leq \int_{\mathbf{U}} \mathbb{1}_{\theta \notin \mathbf{S}}(4 p(1-p))^{|S| / 2}\left(n^{-1 / 3}\right) \left\lvert\, \exp \left(\sum_{j}-(n-2) \frac{p(1-p)}{2}\left(\theta_{j}^{2}-\theta_{j}^{4}\right)\right) d \theta\right. \\
& \leq \int_{\mathbf{U}}(4 p(1-p))^{|S| / 2}\left(n^{-1 / 3}\right) \left\lvert\, \exp \left(\sum_{j}-(n-2) \frac{p(1-p)}{2}\left(\theta_{j}^{2}-\theta_{j}^{4}\right)\right) d \theta\right. \\
& \lesssim n^{-2 / 7}(2 \pi /(p(1-p) n))^{n / 2}
\end{aligned}
$$

where we have used Lemma 2.3 in the final step. The desired result follows immediately.

## 3. Deduction of Theorem 1.6

In order to prove Theorem 1.6 we proceed via the method of moments. We will require the following standard result regarding converting control on moments to distributional control.
Lemma 3.1. Fix a vector $\mu \in \mathbb{R}^{d}$ and a positive definite matrix in $\Sigma \in \mathbb{R}^{d \times d}$. Given a sequence of random vectors $X_{n} \in \mathbb{R}^{d}$, suppose that for any sequence of nonnegative integers $\left(\ell_{i}\right)_{1 \leq i \leq d}$ that

$$
\mathbb{E}\left[\prod_{i=1}^{d}\left(\left(X_{n}\right)_{i}\right)^{\ell_{i}}\right] \rightarrow \mathbb{E}_{G \sim \mathcal{N}(\mu, \Sigma)}\left[\prod_{i=1}^{d}\left(G_{i}\right)^{\ell_{i}}\right]
$$

as $n \rightarrow \infty$. Then it follows that

$$
X_{n} \xrightarrow{d .} \mathcal{N}(\mu, \Sigma) .
$$

We also require the following graph-theoretic input which will be used when applying the method of moments. For a multigraph $G$, let $E_{\text {sing }}(G)$ be the set of edges of multiplicity 1 .
Lemma 3.2. Let $\mathcal{H}=\left(H_{i}\right)_{1 \leq i \leq k}$ be a sequence of connected graphs each of minimum degree at least 2 (not necessarily distinct). Consider overlaying the $H_{i}$ in order to obtain a graph $G$. Let $E_{\text {sing }}=E_{\text {sing }}(G)$, which is precisely those edges of $G$ which are contained in a single graph in $\mathcal{H}$. Then we have

$$
v(G)-\frac{1}{2}\left|E_{\text {sing }}\right| \leq \frac{1}{2} \sum_{i=1}^{k} v\left(H_{i}\right)
$$

with equality if and only if each component of $G$ is a cycle which is isolated or a graph which has been laid upon itself with multiplicity 2.

Proof. It trivially suffices to prove the claim for each connected component of $G$ individually. Notice that

$$
\begin{aligned}
\frac{1}{2} \sum_{i=1}^{k} v\left(H_{i}\right)-v(G)+\frac{\left|E_{\text {sing }}\right|}{2} & =\sum_{v \in V(G)}\left(\left(\frac{1}{2} \sum_{i=1}^{k} \mathbb{1}_{v \in H_{i}}\right)-1\right)+\frac{\left|E_{\text {sing }}\right|}{2} \\
& \geq-\frac{1}{2} \sum_{v \in V(G)} \mathbb{1}\left[\left|\left\{i \in[k]: v \in H_{i}\right\}\right|=1\right]+\frac{\left|E_{\text {sing }}\right|}{2} \\
& \geq-\frac{1}{2} \sum_{v \in V(G)} \mathbb{1}\left[\left|\left\{i \in[k]: v \in H_{i}\right\}\right|=1\right]+\frac{2 \mathbb{1}\left[\left|\left\{i \in[k]: v \in H_{i}\right\}\right|=1\right]}{4} \\
& \geq 0 .
\end{aligned}
$$

In the second to last line, the inequality is justified as follows: consider distributing a mass of $1 / 2$ on each edge in $E_{\text {sing }}$ into $1 / 4$ on both its vertices. Note that every vertex that appears in exactly one $H_{i}$ must be contributed by at least 2 such edges, since the minimum degree is at least 2 and such edges clearly must be singletons.

For equality to occur notice that every vertex must have multiplicity 1 or 2 (i.e. appears in 1 or 2 of the $H_{i}$ ), each singleton edge must occur between two vertices of multiplicity 1 , and no multiplicity 1 vertex has degree larger than 2. Notice that as we assumed $G$ is connected, we must have that either all vertices have multiplicity 1 or all have multiplicity 2 . In the former case, it is a single graph, and the equality case is immediately seen to be a cycle using that $G$ is connected and every vertex has degree exactly 2 .

We now focus on the complementary case that every vertex has multiplicity 2 and hence there are no singleton edges. Next assume for the sake of contradiction that there are at least 3 components used to build $G$. Call two components connected if they share a vertex $v$. Notice that the corresponding graph on components is connected (as otherwise $G$ is not connected) and since there are at least 3 components there must be one with at least 2 connections. Let this component be $H_{1}$ and let $H_{2}, \ldots, H_{\ell}$ be the components to which $H_{1}$ is connected for $\ell \geq 3$. Since no vertex appears in 3 components in our equality case, the vertices of $H_{1}$ can be partitioned into disjoint classes $V_{2}, \ldots, V_{\ell}$ based on whether they are in $H_{2}, \ldots, H_{\ell}$, respectively (this is a partition since each vertex in $G$ has multiplicity 2). But since $H_{1}$ is connected, there exists an edge of $H_{1}$ that connects some $V_{i}$ and $V_{j}$ for $i \neq j$. This demonstrates that $\left|E_{\text {sing }}\right|>0$ since $V_{2}, \ldots, V_{\ell}$ form a partition, a contradiction!

Thus there are only 2 graphs $H_{i}$ which build $G$. Since there are no singleton edges, this must be a direct overlay of isomorphic graphs. Finally, the claimed cases are easily seen to indeed give equality.

We now prove Theorem 1.6. Given the results proven so far this is essentially a routine computation with the method of moments.

Proof of Theorem 1.6. Fix a collection of connected graphs $\mathcal{H}=\left\{H_{i}: 1 \leq i \leq k\right\}$ of minimum degree at least 2. In order to apply the method of moments consider fixed values $\ell_{1}, \ldots, \ell_{k}$ and write

$$
\begin{equation*}
\mathbb{E}_{G \sim \mathbb{G}(n, d)}\left[\prod_{i=1}^{k} \gamma_{H_{i}}(G)^{\ell_{i}}\right]=\left(\prod_{i=1}^{k} \sigma_{H_{i}}^{-\ell_{i}}\right) \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell_{i} \\ H_{i, j} \simeq H_{i}}} \mathbb{E}_{G \sim \mathbb{G}(n, d)}\left[\prod_{i=1}^{k} \prod_{j=1}^{\ell_{i}} \chi_{H_{i, j}}\right] . \tag{3.1}
\end{equation*}
$$

Here the $H_{i, j}$ are embedded into $K_{n}$, and we are summing over possible simultaneous choices of such unlabeled copies. Recall the definition of $E_{H}, \sigma_{H}$ from Definition 1.4, and note this means $\prod_{i=1}^{k} \sigma_{H_{i}}(G)^{\ell_{i}}=\Theta\left(n^{\sum_{i=1}^{k} \ell_{i} v\left(H_{i}\right) / 2}\right)=\Theta\left(n^{\sum_{i, j} v\left(H_{i, j}\right) / 2}\right)$.

We consider the terms based on the isomorphism class of $G=\bigcup_{1 \leq i \leq k} \bigcup_{1 \leq j \leq \ell_{i}} H_{i, j}$, treating $G$ as a multigraph. Let $E_{\text {sing }}=E_{\operatorname{sing}}(G)$, the set of singleton edges in the isomorphism class. Notice that the contribution of terms based on $G$ is bounded by $O\left(n^{v(G)} n^{-\left|E_{\text {sing }}\right| / 2+1 / 3}\right)$ using the first bullet of Proposition 2.7 (and using that if an edge is repeated multiple times, the corresponding $\chi_{e}^{t}$ term can be reduced to a linear combination of $1, \chi_{e}$ with coefficients depending only on $t$ and $p)$. Thus if $v(G)-\left|E_{\text {sing }}\right| / 2<\sum_{i, j} v\left(H_{i, j}\right) / 2$ the terms contribute negligibly (namely, $O\left(n^{-1 / 6}\right)$ ) to the quantity (3.1), since this implies $v(G)-\left|E_{\text {sing }}\right| / 2 \leq-1 / 2+\sum_{i, j} v\left(H_{i, j}\right) / 2$.

But recall that by Lemma 3.2 we have $v(G)-\left|E_{\text {sing }}\right| / 2 \leq \sum_{i, j} v\left(H_{i, j}\right) / 2$, and equality occurs only certain specialized cases where graphs are perfectly overlaid with isomorphic copies, or are lone cycles. The earlier analysis shows we may restrict attention to such equality cases, so now we more closely characterize which such terms contribute. Without loss of generality, let us assume that $H_{1}, \ldots, H_{m}$ are cycles, if any, while $H_{m+1}, \ldots, H_{k}$ are not cycles. Notice that if any $\ell_{t}$ for $t \in[m+1, k]$ is odd then it is impossible to pair up all copies $H_{t}$. This violates the equality condition, so is not possible. Thus if $\ell_{t}$ for some $t \in[m+1, k]$ is odd, then the total contribution to (3.1) is $O\left(n^{-1 / 6}\right)$.

Now consider $H_{t}$ with $1 \leq t \leq m$. If an odd cycle is not overlaid with another, then the second bullet of Proposition 2.7 again shows the total contribution of such terms to (3.1) is $O\left(n^{-1 / 6}\right)$.

Finally, we have a situation where all cycles except even cycles must be paired among themselves and the even cycles are either disjoint or paired with another even cycle and overlaid. Without loss of generality let $H_{1}, \ldots, H_{m^{\prime}}$ be the even cycles, if any.

The number of choices for pairing up the graphs other than even cycles is $\prod_{i=m^{\prime}+1}^{k} \ell_{i}!!$. The number of choices for pairing up $s_{i} \leq \ell_{i} / 2$ even cycles for $i \in\left[m^{\prime}\right]$ is $\prod_{i=1}^{m^{\prime}}\binom{\ell_{i}}{2 s_{i}}\left(2 s_{i}\right)$ !!. In such a pairing, let $\mathcal{U}_{i} \subseteq\left[\ell_{i}\right]$ be the list of unpaired indices for $i \in\left[m^{\prime}\right]$. We find that $\prod_{i=1}^{k} \prod_{j=1}^{\ell_{i}} \chi_{H_{i, j}}$ is a product of various terms of the form $\chi_{e}$ for $e \in H_{i, j}$ where $i \in\left[m^{\prime}\right]$ and $j \in \mathcal{U}_{i}$, as well as terms of the form $\chi_{e}^{2}$ in certain connected components. There are $\sum_{i=1}^{m^{\prime}}\left(\ell_{i}-2 s_{i}\right) v\left(H_{i}\right)$ vertices of the former type and $\sum_{i=1}^{m^{\prime}} s_{i} v\left(H_{i}\right)+\sum_{i=m^{\prime}+1}^{k}\left(\ell_{i} / 2\right) v\left(H_{i}\right)$ of the latter type. Note that $\chi_{e}^{2}=1-(2 p-1) \chi_{e} / \sqrt{p(1-p)}$, and expanding out the repeated terms in such a way yields one term of the form $\prod_{i=1}^{m^{\prime}} \prod_{j \in \mathcal{U}_{i}} \chi_{H_{i, j}}$ and others with additional terms of the form $(2 p-1) \chi_{e} / \sqrt{p(1-p)}$ tacked on. The contribution of such other terms totals at most, by the first bullet of Proposition 2.7,

$$
O\left(n^{\sum_{i=1}^{m^{\prime}}\left(\ell_{i}-s_{i}\right) v\left(H_{i}\right)+\sum_{i=m^{\prime}+1}^{k}\left(\ell_{i} / 2\right) v\left(H_{i}\right)} \cdot n^{-\sum_{i=1}^{m^{\prime}} \sum_{j \in \mathcal{U}_{i}} e\left(H_{i, j}\right) / 2-1 / 2+1 / 3}\right) .
$$

The exponent is bounded by $v(G) / 2-1 / 3+\sum_{i=1}^{m^{\prime}} \sum_{j \in \mathcal{U}_{i}}\left(v\left(H_{i, j}\right)-e\left(H_{i, j}\right) / 2\right)=\sum_{i=1}^{k} \ell_{i} v\left(H_{i}\right) / 2-1 / 6$ since $H_{i, j}$ for $i \in\left[m^{\prime}\right]$ is a cycle, so in (3.1) this amounts to a total contribution of $O\left(n^{-1 / 6}\right)$.

Finally, what remains is
$\mathbb{E}_{G \sim \mathbb{G}(n, d)}\left[\prod_{i=1}^{k} \gamma_{H_{i}}(G)^{\ell_{i}}\right]=\left(\prod_{i=1}^{k} \sigma_{H_{i}}^{-\ell_{i}} \prod_{i=m^{\prime}+1}^{k} \ell_{i}!!\right) \sum_{s_{i} \leq \ell_{i} / 2} \sum_{H_{i, j}}^{*} \mathbb{E} \prod_{i=1}^{m^{\prime}}\left(\binom{\ell_{i}}{2 s_{i}}\left(2 s_{i}\right)!!\prod_{j=2 s_{i}+1}^{\ell_{i}} \chi_{H_{i, j}}\right)+O\left(n^{-1 / 6}\right)$,
where $\sum_{s_{i} \leq \ell_{i} / 2}$ denotes a simultaneous choice of such nonnegative integers $s_{i}$ for $i \in\left[m^{\prime}\right]$ and where $\sum^{*}$ denotes a sum over choices of $H_{i, j}$ such that they are all vertex-disjoint other than pairs $H_{i, 2 j-1}=H_{i, 2 j}$ for $1 \leq j \leq s_{i} / 2$ when $1 \leq i \leq m^{\prime}$ as well as for $1 \leq j \leq \ell_{i} / 2$ when $m^{\prime}+1 \leq i \leq k$. This equation basically means that we can validly pair up the necessary graphs and then replace the $\chi_{e}^{2}$ terms by 1. Furthermore, it is not hard to see based on the considerations so far that we can remove the vertex-disjointness condition between different $H_{i, j}$ without changing the error rate, and thus we can write

$$
\begin{aligned}
\mathbb{E}_{G \sim \mathbb{G}(n, d)}\left[\prod_{i=1}^{k} \gamma_{H_{i}}(G)^{\ell_{i}}\right]= & \left(\prod_{i=1}^{k} \sigma_{H_{i}}^{-\ell_{i}} \prod_{i=m^{\prime}+1}^{k} \ell_{i}!!\left(\binom{n}{v\left(H_{i}\right)} \frac{v\left(H_{i}\right)!}{\operatorname{aut}\left(H_{i}\right)}\right)^{\ell_{i} / 2}\right) \times \\
& \sum_{s_{i} \leq \ell_{i} / 2}\left(\prod_{i=1}^{m^{\prime}}\binom{\ell_{i}}{2 s_{i}}\left(2 s_{i}\right)!!\left(\binom{n}{v\left(H_{i}\right)} \frac{v\left(H_{i}\right)!}{\operatorname{aut}\left(H_{i}\right)}\right)^{s_{i}}\left(\mathbb{E} \gamma_{H_{i}}\right)^{\ell_{i}-2 s_{i}}\right)+O\left(n^{-1 / 6}\right), \\
= & \prod_{i=m^{\prime}+1}^{k} \ell_{i}!!\sum_{s_{i} \leq \ell_{i} / 2} \prod_{i=1}^{m^{\prime}}\binom{\ell_{i}}{2 s_{i}}\left(2 s_{i}\right)!!\left(\sigma_{H_{i}}^{-1} \mathbb{E} \gamma_{H_{i}}(G)\right)^{\ell_{i}-2 s_{i}}+O\left(n^{-1 / 6}\right),
\end{aligned}
$$

using the formula for $\sigma_{H}$ in the second step. Finally, the third bullet of Proposition 2.7 shows that $\mathbb{E}_{\gamma_{H_{i}}}(G)=\left(1+O\left(n^{-1 / 5}\right)\right) 2 n^{-v\left(H_{i}\right) / 2} \cdot\binom{n}{v\left(H_{i}\right)} \frac{v\left(H_{i}\right)!}{\operatorname{aut}\left(H_{i}\right)}=\left(1+O\left(n^{-1 / 5}\right)\right) E_{H}$ for $i \in\left[m^{\prime}\right]$. We also know that $E_{H}=(2 / v(H))^{1 / 2} \sigma_{H}$ hence we find

$$
\mathbb{E}_{G \sim \mathbb{G}(n, d)}\left[\prod_{i=1}^{k} \gamma_{H_{i}}(G)^{\ell_{i}}\right]=\prod_{i=m^{\prime}+1}^{k} \ell_{i}!!\sum_{s_{i} \leq \ell_{i} / 2} \prod_{i=1}^{m^{\prime}}\binom{\ell_{i}}{2 s_{i}}\left(2 s_{i}\right)!!\left(E_{H} / \sigma_{H}\right)^{\ell_{i}-2 s_{i}}+O\left(n^{-1 / 6}\right),
$$

which can be seen to match the moments of $\mathcal{N}\left(E_{H} / \sigma_{H}, 1\right)^{\otimes m^{\prime}} \otimes \mathcal{N}(0,1)^{\otimes\left(k-m^{\prime}\right)}$. Using Lemma 3.1 and shifting appropriately, this implies the desired

$$
\left(\widetilde{\gamma}_{H_{i}}(G)\right)_{1 \leq i \leq k} \xrightarrow{d .} \mathcal{N}(0,1)^{\otimes k} .
$$

Finally, we briefly note that the moment computations above where $\ell_{i} \in\{1,2\}$ and all $\ell_{j}=0$ for $j \neq i$ show that the means and variances are as claimed.

## 4. Computations with Graph Factors

We now prove that any fixed degree polynomial in the indicator functions $x_{e} \in\{0,1\}$ which is symmetric under vertex permutation can be rewritten (so that it agrees on the set of $d$-regular graphs) as a function of connected graph factors of the form in Definition 1.4. The reduction specifically to connected graph factors appears essentially in the work of Janson [7, p. 347].

Lemma 4.1. Given a disconnected graph $H$ (with no isolated vertices) with connected components $H_{1}, \ldots, H_{k}, \gamma_{H}(\mathbf{x})-\prod_{i=1}^{k} \gamma_{H_{i}}(G)$ can be expressed (as a function on graphs) as a sum of $\gamma_{H^{\prime}}$ with $v\left(H^{\prime}\right)<v(H)$ (though $H^{\prime}$ may be itself disconnected). Furthermore, the coefficients of the sum are bounded by $O\left(1 /(p(1-p))^{O_{H}(1)}\right)$.

This can clearly be inductively applied to show that the connected graph factors generate all graph factors using polynomial expressions. The crucial lemma for our work is that given a connected
graph $H$ with a vertex of degree 1 , the graph factor $\gamma_{H}(\mathbf{x})$ can be simplified further (since our input graphs are regular).
Lemma 4.2. Given a graph $H$ (with no isolated vertices) with a vertex of degree 1 then $\gamma_{H}(\mathbf{x})$ can be expressed, as a function on d-regular graphs, as a sum of $\gamma_{H^{\prime}}(\mathbf{x})$ with $v\left(H^{\prime}\right)<v(H)$. Furthermore the coefficients of the sum are bounded by $O\left(1 /(p(1-p))^{O_{H}(1)}\right)$.

Proof. Let $v$ be a vertex in $H$ of degree 1 and $(u, v)$ be the unique edge in $H$ connected to $v$. Notice that, considering this as a sum over possible choices of $v$, we have $\sum_{v \neq u} \chi_{(v, u)}=0$ by $d$-regularity. Therefore it follows that

$$
\begin{aligned}
\gamma_{H}(\mathbf{x}) & =\sum_{\substack{E \subset K_{n} \\
E \simeq H}} \prod_{e \in E} \chi_{e}=\sum_{\substack{E \subseteq K_{n} \\
E \simeq H}} \chi_{(u, v)} \prod_{\substack{E \subseteq E \backslash\{(u, v)\} \\
E \simeq H}} \chi_{e} \\
& =\sum_{u \in V(E) \backslash\{v\}}\left(-\sum_{(u, v)}\right) \prod_{e \in E \backslash\{(u, v)\}} \chi_{e}
\end{aligned}
$$

and the desired result follows immediately using that $\chi_{e}^{2}=1-(2 p-1) \chi_{e} / \sqrt{p(1-p)}$.
Note that iterating Lemmas 4.1 and 4.2 shows we can write any graph factor on $d$-regular graphs as a function (in terms of $d$ ) of ones that are connected and with minimum degree at least 2 .
Lemma 4.3. Suppose that $n / \log n \leq \min (d, n-d)$. Given a graph $H$ (with no isolated vertices) we have $\operatorname{Var}_{G \sim \mathbb{G}(n, d)}\left(\gamma_{H}(G)\right)=O\left(n^{v(H)}\right)$. Furthermore if the $H$ has a degree 1 vertex, we have $\operatorname{Var}_{G \sim \mathbb{G}(n, d)}\left(\gamma_{H}(G)\right) \leq n^{v(H)-2 / 3}$.
Proof. We induct on $v(H)$. For $H$ being a connected graph with minimum degree at least 2 , the desired result follows immediately from the moments calculation given in the proof of Theorem 1.6. In the remaining cases, for $G$ a $d$-regular graph we find that if $H$ has connected components $H_{1}, \ldots, H_{k}$, then $\gamma_{H}(G)-\prod_{i=1}^{k} \gamma_{H_{i}}(G)$ can be written as a sum of graph factors each involving coefficients bounded by $1 /(p(1-p))^{O_{H}(1)}$ and with at most $v(H)-1$ vertices by Lemma 4.1. If there is a vertex of degree 1 , we can apply Lemma 4.2 and then the same argument shows that we obtain a similar type of sum after expanding. Thus the total variance, by induction, is $(p(1-p))^{-O_{H}(1)} \cdot O\left(n^{v(H)-1}\right) \leq n^{v(H)-2 / 3}$. If all the $H_{i}$ are minimum degree at least 2 , then we see that the "lower" portion contributes $O\left(n^{v(H)-2 / 3}\right)$ similar to before, hence the problem reduces to understanding the variance of $\prod_{i=1}^{k} \gamma_{H_{i}}(G)$. We have

$$
\operatorname{Var}\left(\prod_{i=1}^{k} \gamma_{H_{i}}(G)\right) \leq \mathbb{E} \prod_{i=1}^{k} \gamma_{H_{i}}(G)^{2} \leq \prod_{i=1}^{k}\left(\mathbb{E} \gamma_{H_{i}}(G)^{2 k}\right)^{1 / k} .
$$

Again, the moment-based proof of Theorem 1.6 gives a bound of $O\left(n^{v\left(H_{1}\right)+\cdots+v\left(H_{k}\right)}\right)$.

## 5. Deduction of Subgraph Count and Trace Count Normality

We now consider a subgraph count $X_{H}(G)$ for $G \sim \mathbb{G}(n, d)$ and prove the desired normality as in Corollary 1.2. This is essentially an immediate consequence of Theorem 1.6 and expanding into the appropriate graph factors. The precise nature of the contributing terms however depends in an intricate manner on the precise structure of $H$.
Proof of Corollary 1.2. Let $H$ be a connected graph at least 2 vertices which is not a star. For $G \sim \mathbb{G}(n, d)$ write

$$
W=X_{H}(G)=\sum_{\substack{H^{\prime} \subseteq K_{n} \\ H^{\prime} \simeq H \\ 14}} \prod_{e \in E\left(H^{\prime}\right)} x_{e}
$$

Letting $\chi_{e}=\left(x_{e}-p\right) / \sqrt{p(1-p)}$ as usual, we find that

$$
\begin{equation*}
W=\sum_{\substack{H^{\prime} \subseteq K_{n} \\ H^{\prime} \simeq H}} \prod_{e \in E\left(H^{\prime}\right)}\left(p+\sqrt{p(1-p)} \chi_{e}\right)=\sum_{S \subseteq H} p^{e(H)-e(S)}(\sqrt{p(1-p)})^{e(S)} c_{S, H} d_{S, H}\binom{n-v(S)}{v(H)-v(S)} \gamma_{S}(\mathbf{x}), \tag{5.1}
\end{equation*}
$$

where $c_{S, H}=(v(H)-v(S))$ !aut $(S) / \operatorname{aut}(H), d_{S, H}=N(H, S)$ (the number of times $S$ appears as a subgraph of $H$ ), and the sum is over subgraphs $S$ (lacking isolated vertices) of $H$ up to isomorphism. For the empty graph, we have $c_{\emptyset, H}=v(H)$ !/aut $H$ and $d_{\emptyset, H}=1$.

Notice that the graph factors $\gamma_{S}$ with $e(S) \leq 2$ (the empty graph, an edge, a star with two edges, and two disjoint edges) are deterministic since $G$ is a $d$-regular graph. If $H$ contains a $C_{3}$ notice that all other graph factors $\gamma_{S}$ in the expansion have $v(S) \geq 4$ hence the corresponding terms have variance bounded by $O\left(n^{2(v(H)-v(S))} \cdot n^{v(S)}\right)=O\left(n^{2 v(H)-4}\right)$ by Lemma 4.3 while the $\gamma_{C_{3}}$ term has variance

$$
\left(\frac{6(v(H)-3)!}{\operatorname{aut}(H)} N\left(H, C_{3}\right) p^{e(H)-3 / 2}(1-p)^{3 / 2}\binom{n-3}{v(H)-3}\right)^{2} \operatorname{Var}\left[\gamma_{C_{3}}\right] .
$$

Since the variance determination in Theorem 1.6 allows us to compute $\operatorname{Var}\left[\gamma_{C_{3}}\right]=\left(1+O\left(n^{-1 / 6}\right)\right) n^{3} / 6$, we easily obtain the first bullet point of Corollary 1.2: we can write $W=X+Y$ where $X$ is the term coming from $\gamma_{C_{3}}$ and $Y$ is the rest. We have that $\operatorname{Var}[Y]=O\left(n^{-1} \operatorname{Var}[X]\right)$. Thus since $X$ satisfies a central limit theorem, so does $X+Y$. Furthermore, the variance can be written

$$
\operatorname{Var}[X+Y]=\operatorname{Var}[X]+\operatorname{Var}[Y]+2 \mathbb{E}(X-\mathbb{E} X)(Y-\mathbb{E} Y)
$$

and $|\mathbb{E}(X-\mathbb{E} X)(Y-\mathbb{E} Y)| \leq \sqrt{\operatorname{Var}[X] \operatorname{Var}[Y]}$ by Cauchy-Schwarz, which gives an appropriate bound for the change in the variance going from $X$ to $X+Y$.

Next suppose that $H$ contains a $C_{4}$ but no $C_{3}$. Then all potential contributing graph factors which are not deterministic are on at least 4 vertices. Notice that any graph factor $\gamma_{S}$ with $v(S) \geq 5$ has corresponding variance at most $O\left(n^{2 v(H)-5}\right)$ by Lemma 4.3. Furthermore for $v(S)=4$, notice that if some vertex has degree 1 then by Lemma 4.3 we obtain corresponding variance of order $O\left(n^{2 v(H)-4-2 / 3}\right)$. All remaining $S$ must have 4 vertices, minimum degree at least 2 , and contain no $C_{3}$, so $S=C_{4}$. The variance of the $\gamma_{C_{4}}$ term is

$$
\left(\frac{8(v(H)-4)!}{\operatorname{aut}(H)} N\left(H, C_{4}\right) p^{e(H)-2}(1-p)^{2}\binom{n-4}{v(H)-4}\right)^{2} \operatorname{Var}\left[\gamma_{C_{4}}\right] .
$$

The second bullet point of Corollary 1.2 follows similar to before.
The last case is when $H$ contains neither $C_{3}$ nor $C_{4}$. Since $H$ is not a star, $H$ contains a $P_{3}$, i.e. a path on 3 vertices. First note that graph factors $\gamma_{S}$ with $v(S) \geq 6$ have corresponding variance $O\left(n^{2 v(H)-6}\right)$ and graph factors $\gamma_{S}$ with $v(S)=5$ and some vertex of degree 1 have corresponding variance $O\left(n^{2 v(H)-5-2 / 3}\right)$ by Lemma 4.3. Furthermore, since $H$ has no $C_{3}$ and no $C_{4}$, we see that the only possible $S$ with $v(S) \leq 4$ for which $\gamma_{S}$ is not deterministic is $S=P_{3}$. Also, the possible $S$ with $v(S)=5$ are those with minimum degree at least 2 and no $C_{3}$ and no $C_{4}$, which is easily seen to force $S=C_{5}$. The variance of the $\gamma_{C_{5}}$ term is

$$
\begin{aligned}
\left(\frac{10(v(H)-5)!}{\operatorname{aut}(H)}\right. & \left.N\left(H, C_{5}\right) p^{e(H)-5 / 2}(1-p)^{5 / 2}\binom{n-5}{v(H)-5}\right)^{2} \operatorname{Var}\left[\gamma_{C_{5}}\right] \\
& =\left(1+O\left(n^{-1 / 6}\right)\right) \frac{10 N\left(H, C_{5}\right)^{2}}{\operatorname{aut}(H)^{2}} p^{2 e(H)-5}(1-p)^{5} n^{2 v(H)-5}
\end{aligned}
$$

by Theorem 1.6. The $\gamma_{P_{3}}$ is more delicate, as we must use the observation in Lemma 4.2 that we can reduce its complexity using $\sum_{v \neq u} \chi_{(v, u)}=0$ for all fixed $u$. We obtain

$$
\begin{aligned}
\gamma_{P_{3}}(\mathbf{x}) & =\frac{1}{2} \sum_{u, v, w}\left(\chi_{(u, v)} \chi_{(v, w)} \sum_{u^{\prime} \neq u, v, w} \chi_{\left(w, u^{\prime}\right)}\right)=\frac{1}{2} \sum_{u, v, w} \chi_{(u, v)} \chi_{(v, w)}\left(-\chi_{(w, u)}-\chi_{(w, v)}\right) \\
& =-\frac{1}{2} \sum_{u, v, w}\left(\chi_{(u, v)} \chi_{(v, w)} \chi_{(w, u)}+\chi_{(u, v)} \chi_{(v, w)}^{2}\right),
\end{aligned}
$$

where the sum is over tuples of distinct $u, v, w \in[n]$. Using $\chi_{e}^{2}=1-(2 p-1) \chi_{e} / \sqrt{p(1-p)}$ and that $\gamma_{K_{2}}, \gamma_{P_{2}}$ are deterministic, we find that $\gamma_{P_{3}}+3 \gamma_{C_{3}}$ is deterministic. Therefore the variance of the $\gamma_{P_{3}}$ term is

$$
\begin{aligned}
\left(\frac{2(v(H)-4)!}{\operatorname{aut}(H)}\right. & \left.N\left(H, P_{3}\right) p^{e(H)-2}(1-p)^{2}\binom{n-4}{v(H)-4}\right)^{2} \cdot 9 \operatorname{Var}\left[\gamma_{C_{3}}\right] \\
& =\left(1+O\left(n^{-1 / 6}\right)\right) \frac{6 N\left(H, P_{3}\right)^{2}}{\operatorname{aut}(H)^{2}} p^{2 e(H)-4}(1-p)^{4} n^{2 v(H)-5} .
\end{aligned}
$$

Finally, writing $X_{1}$ for the $\gamma_{C_{5}}$ term and $X_{2}$ for the $\gamma_{P_{3}}$ term, using the moment computations in the proof of Theorem 1.6 (applied to $C_{3}$ and $\left.C_{5}\right)$ we easily find that $\operatorname{Cov}\left(X_{1}, X_{2}\right)=$ $O\left(n^{2 v(H)-5-1 / 6}\right)$. (Or, we can directly see this from the joint central limit theorem satisfied by $\gamma_{C_{3}}, \gamma_{C_{5}}$ in Theorem 1.6.) The third bullet of Corollary 1.2 follows similar to before.

Finally, we prove Corollary 1.8. Again, this is mostly rearranging terms in order to apply Theorem 1.6. Our analysis is more complicated than typical trace expansion arguments as one cannot trivially rule out walks where an edge appears with multiplicity 1 in various expectation computations. To perform the necessary analysis, we will need the following modified version of Lemma 3.2 which allows for some of the $H_{i}$ to be a doubled edge (but we may otherwise restrict to cycles). As a consequence, the equality case is more complicated.

Lemma 5.1. Let $\mathcal{H}=\left(H_{i}\right)_{1 \leq i \leq k}$ be a sequence of cycles or multigraphs consisting of a lone doubled edge. Consider overlaying the $H_{i}$ in order to obtain a multigraph $G$. Let $E_{\text {sing }}=E_{\text {sing }}(G)$, the set of edges of $G$ of multiplicity 1. Suppose that every connected component of $G$ had at least one participating cycle of $\mathcal{H}$. Then we have

$$
v(G)-\frac{1}{2}\left|E_{\text {sing }}\right| \leq \frac{1}{2} \sum_{i=1}^{k} v\left(H_{i}\right)=\frac{e(G)}{2}
$$

with equality only if every connected component of $G$ is obtained by first taking a cycle of $\mathcal{H}$ or perfectly overlaying two cycles of $\mathcal{H}$, and second attaching pendant trees of doubled edges (i.e., removing the initial cycle portion leaves a forest of doubled edges). Here $e(G)$ is computed with multiplicity.

Proof. Without loss of generality we may assume $G$ is connected, as this clearly preserves the inequality as well as equality cases. Also, the equality $\sum_{i=1}^{k} v\left(H_{i}\right)=e(G)$ is trivial since cycles and doubled edges have the same edge and vertex counts. Now let $H_{1}, \ldots, H_{k^{\prime}}$ be the cycles and the rest the doubled edges. Let $G^{\prime}$ be the overlay of $H_{1}, \ldots, H_{k^{\prime}}$ and define $E_{\text {sing }}^{\prime}$ as the edges contained once in $G^{\prime}$, i.e., in a single $H_{i}$. By Lemma 3.2, we have

$$
v\left(G^{\prime}\right)-\frac{1}{2}\left|E_{\text {sing }}^{\prime}\right| \leq \frac{1}{2} \sum_{i=1}^{k^{\prime}} v\left(H_{i}\right)
$$

and equality can only occur if the cycles are isolated or overlaid with multiplicity 2 .

Now consider adding in the doubled edges in a specified order, starting at $G_{k^{\prime}}=G^{\prime}$ and ending at $G_{k}=G$. We choose the order as follows: at time $k^{\prime} \leq i \leq k-1$, once we have $G_{i}$, since we know $G$ is connected there must be a doubled edge to add which shares a vertex with $G_{i}$; add one of those edges. Define $E_{\text {sing }}^{(i)}$ in the obvious way. We see that

$$
\left(v\left(G_{i+1}\right)-v\left(G_{i}\right)\right)-\frac{1}{2}\left(\left|E_{\text {sing }}^{(i+1)}\right|-\left|E_{\text {sing }}^{(i)}\right|\right) \leq 1=\frac{1}{2} v\left(H_{i+1}\right)
$$

since either we add 0 vertices and at worst reduce the number of singleton edges by 1 , or we add 1 vertex and thus leave the number of singleton edges unchanged (here we are using that a cycle participated in the creation of this connected component, otherwise it could be possible to add 2 vertices at the beginning). Equality occurs here only if we add 1 new pendant vertex.

Adding these inequalities over all $i$, we obtain the desired inequality. Furthermore, equality can only occur if we start with isolated or doubled cycles, and then we only add pendant trees of doubled edges. But since the final multigraph $G$ is connected, this means we must have started with at most one component as we cannot connect components with a doubled edge while simultaneously increasing the vertex count by 1 . The result follows.

Finally, we demonstrate Corollary 1.8.
Proof of Corollary 1.8. Note by taking complements it suffices to prove the result for $p \leq 1 / 2$ (up to a slight shift in eigenvalues which we will see to be inconsequential). Furthermore notice that deterministically we have that the all 1 vector is an eigenvector with eigenvalue $d$. Therefore we have that $M:=A_{G}-p J+p I$ (where $J$ is the all 1 matrix) has eigenvalues $\lambda_{i}+p$ for $2 \leq i \leq n$ and one eigenvalue of 0 .

In order to prove Corollary 1.8 it suffices to prove that if $E_{\ell}^{*}=\mathbb{E} \operatorname{tr}\left(M^{\ell}\right), \sigma_{\ell}^{* 2}=\operatorname{Var}\left[\operatorname{tr}\left(M^{\ell}\right)\right]$ then

$$
\left(\sigma_{\ell}^{*-1 / 2}\left(\operatorname{tr}\left(M^{\ell}\right)-E_{\ell}^{*}\right)\right)_{3 \leq \ell \leq k} \xrightarrow{\text { d. }} \mathcal{N}\left(0, \Sigma_{k}\right)
$$

and $\sigma_{\ell}^{*}=\Theta\left((p(1-p) n)^{\ell / 2}\right)$ for fixed $\ell \geq 3$. To see that this implies the desired result note that each term of $\sum_{i=2}^{n}\left(\lambda_{i}+p\right)^{\ell}-\sum_{i=2}^{n} \lambda_{i}^{\ell}$ can be represented as a degree at most $\ell-1$ polynomial in $\lambda_{i}+p$ with coefficients bounded by $O_{k}\left(p^{O_{k}(1)}\right)$. These terms are lower order due to the order of the variance (and use that the first two moments of the eigenvalues are deterministic).

Given $\ell \geq 3$ note that

$$
\operatorname{tr}\left(\left(A_{G}-p J+p I\right)^{\ell} /(p(1-p))^{\ell / 2}\right)=\sum_{u_{1}, \ldots, u_{\ell} \in[n]} \prod_{i=1}^{\ell} \chi_{\left(u_{i}, u_{i+1}\right)}
$$

where we define $\chi_{(u, u)}=0$ and take indices modulo $\ell$. The sum is over closed walks of length $\ell$.
Consider the closed walk $u_{1}, \ldots, u_{\ell}$. As $\chi_{(u, u)}=0$, we have that the walk has no self-loops corresponding to $u_{t+1}=u_{t}$. The edges traced out thus form a multigraph when superimposed. Let $\mathcal{W}_{\ell}$ be the collection of possible isomorphism types of multigraphs and for $(u, v) \in G$ and $G \in \mathcal{W}_{\ell}$ let $G(u, v)$ be the multiplicity of $(u, v)$ in $G$. We see

$$
\operatorname{tr}\left(\left(A_{G}-p J+p I\right)^{\ell} /(p(1-p))^{\ell / 2}\right)=\sum_{G \in \mathcal{W}_{\ell}} c_{G}\left(\sum_{\substack{V\left(G^{\prime}\right) \subseteq V\left(K_{n}\right) \\ G^{\prime} \simeq G}} \prod_{(u, v) \in G} \chi_{(u, v)}^{G(u, v)}\right),
$$

were $c_{G}$ is the number of choices of vertices in $G$ and closed walks of length $\ell$ starting at that vertex and traversing each edge $(u, v) \in G$ in either direction exactly $G(u, v)$ times. If $G$ is a simple graph, the term on the inside is just $\gamma_{G}(\mathbf{x})$. We therefore abusively define

$$
\gamma_{G}(\mathbf{x})=\sum_{\substack{V\left(G^{\prime}\right) \subseteq V\left(K_{n}\right) \\ G^{\prime} \simeq G \\ 17}} \prod_{(u, v) \in G} \chi_{(u, v)}^{G(u, v)}
$$

for multigraphs $G$ without isolated vertices. However, we will later use $\chi_{e}^{2}=1-(2 p-1) \chi_{e} / \sqrt{p(1-p)}$ and similar relations for higher powers to reduce to a linear combination of graph factors $\gamma_{F}$.

Furthermore, every multigraph in $\mathcal{W}_{\ell}$ can be decomposed (with multiplicity preserved) into a collection of cycles and doubled edges: move along the walk until the first vertex repetition, then remove a portion corresponding to a doubled edge or cycle, and keep doing this. We can further further turn the doubled edges into a multitree by iteratively removing cycles of doubled edges and turning them into two cycles. Given $G \in \mathcal{W}_{\ell}$, let $\mathcal{H}_{G}$ be the sequence of multigraphs thus generated. Let $\mathcal{T}_{\ell}$ be the collection of $G \in \mathcal{W}_{\ell}$ that are composed only of doubled edges, which therefore compose a tree as $G$ is connected.

First consider $G \in \mathcal{W}_{\ell} \backslash \mathcal{T}_{\ell}$, so that $\mathcal{H}_{G}$ contains at least one cycle. Let $\mathcal{W}_{G}^{\prime}$ be all possible isomorphism classes $G_{1} \cup G_{2}$ for the multigraph union of two copies $G_{1}, G_{2} \simeq G$. We see

$$
\operatorname{Var}\left[\gamma_{G}\right] \leq \mathbb{E} \gamma_{G}^{2} \lesssim \sum_{G^{\prime} \in \mathcal{W}_{G}^{\prime}} n^{v\left(G^{\prime}\right)} \cdot n^{-\left|E_{\text {sing }}\left(G^{\prime}\right)\right| / 2+1 / 3}
$$

by expansion and Proposition 2.7. Now consider the collection of cycles and doubled edges which make up $G^{\prime}$. Since they are overlaid in a way that form two copies of $G$, our condition on $\mathcal{H}_{G}$ implies that every connected component of $G^{\prime}$ has at least one cycle participating in its creation, Lemma 5.1 applies. For cases where equality does not hold, we have $v\left(G^{\prime}\right)-\left|E_{\text {sing }}\left(G^{\prime}\right)\right| / 2<\ell$ since $e\left(G^{\prime}\right)=2 \ell$. This implies $v\left(G^{\prime}\right)-\left|E_{\text {sing }}\left(G^{\prime}\right)\right| / 2+1 / 3 \leq \ell-1 / 6$. For cases where equality does hold, by Lemma 5.1 every connected component of $G^{\prime}$ must consist of a cycle or doubled cycle (which come from our specified list of cycles that create $G_{1}, G_{2}$ ) and then pendant trees of doubled edges. Note that $G_{1}, G_{2}$ are each connected, so we either have that these are disjoint and of this form or they are connected and form such a graph. In the former case $G$ is clearly either a cycle or doubled cycle with pendant trees of doubled edges. In the latter case we easily deduce that $G$ is a single cycle with pendant trees of doubled edges. Let $\mathcal{C}_{\ell}$ be the set of isomorphism classes of these more special forms, so that $\operatorname{Var}\left[\gamma_{G}\right]=O\left(n^{\ell-1 / 6}\right)$ for $G \in \mathcal{W}_{\ell} \backslash\left(\mathcal{T}_{\ell} \cup \mathcal{C}_{\ell}\right)$.

Next we study $G \in \mathcal{T}_{\ell}$, in which case $\ell$ must be even (thus $\ell \geq 4$ ) and $G$ has $\ell / 2$ doubled edges and at most $\ell / 2+1$ vertices (being a multitree). We write

$$
\gamma_{G}(\mathbf{x})=\sum_{F \subseteq G} c_{F, G}(p)\left(\sum_{\substack{F^{\prime} \subseteq K_{n} \\ F^{\prime} \simeq F}} \prod_{(u, v) \in F^{\prime}} \chi_{(u, v)}\right)
$$

where the sum is over graphs $F$ up to isomorphism obtained by either including 1 or 0 edges for each edge in $G$ (without multiplicity). Here $c_{F, G}(p)$ are appropriately computed constants. This is shown by expanding via $\chi_{e}^{2}=1-(2 p-1) \chi_{e} / \sqrt{p(1-p)}$ and similar for higher powers, and collecting the patterns that can result. Note also that $F$ may have isolated vertices, and that it is a forest since $G$ is a multitree. Regardless of these isolated vertices, let us abusively denote the inside term as $\gamma_{F}(\mathbf{x})$ (this agrees with the usual definition). If $e(F) \leq 2$ then the term corresponding to $F$ is deterministic since we are considering $d$-regular graphs. Hence we may restrict to just terms with $e(F) \geq 3$. We have for such $F$ that

$$
\operatorname{Var}\left[\gamma_{F}\right] \leq \mathbb{E} \gamma_{F}^{2} \lesssim \max _{v \geq 0} n^{2(\ell / 2+1)-v} \cdot n^{-(2 e(F)-\max (v-1,0)) / 2+1 / 3}=\max _{v \geq 0} n^{\ell+2-e(F)+\max (-v-1,-2 v) / 2+1 / 3}
$$

by Proposition 2.7: two copies of $F$ with $v$ overlapping vertices can share at most max $(v-1,0)$ edges. This clearly yields $\operatorname{Var}\left[\gamma_{F}\right]=O\left(n^{\ell-2 / 3}\right)$, and thus we find $\operatorname{Var}\left[\gamma_{G}\right]=O\left(n^{\ell-1 / 6}\right)$ for $G \in \mathcal{T}_{\ell}$.

Finally, consider $G \in \mathcal{C}_{\ell}$. In $\gamma_{G}(\mathbf{x})$, the highest degree of any term $\chi_{e}$ is 2 . Using $\chi_{e}^{2}=1-(2 p-$ 1) $\chi_{e} / \sqrt{p(1-p)}$ and expanding out, it is easy to see similar to above that the sum of the terms involving any $-(2 p-1) \chi_{e} / \sqrt{p(1-p)}$ in the expansion, call this $\gamma_{G}^{\prime}$, has total variance bounded by $O\left(n^{\ell-1 / 6}\right)$. Finally, if $G$ is a doubled cycle with pendant trees of double edges then the remaining term is deterministic, while if $G$ is a single with with such pendant trees then the remaining term
is a cycle of say length $r$ with $(\ell-r) / 2$ isolated vertices. Finally, recall that the variance of $\gamma_{C_{r}}$ is $O\left(n^{r}\right)$ by Theorem 1.6.

Overall, combining all this information and noting that a $\gamma_{C_{\ell}}$ term only comes from a walk that repeats no vertices, we see

$$
\operatorname{tr}\left(\left(A_{G}-p J+p I\right)^{\ell} /(p(1-p))^{\ell / 2}\right)=2 \ell \gamma_{C_{\ell}}(\mathbf{x})+\sum_{\substack{3 \leq r<\ell \\ r \equiv \ell(\bmod 2)}} \alpha_{\ell, r} n^{(\ell-r) / 2} \gamma_{C_{r}}(\mathbf{x})+X_{\ell}
$$

for some random variable $X_{\ell}$ satisfying $\operatorname{Var}\left[X_{\ell}\right]=O\left(n^{\ell-1 / 6}\right)$ and for appropriate combinatorially definable rational numbers $\alpha_{\ell, r}$.

Equivalently,

$$
\operatorname{tr}\left(\left(A_{G}-p J+p I\right)^{\ell} /(p(1-p) n)^{\ell / 2}\right)=2 \ell\left(n^{-\ell / 2} \gamma_{C_{\ell}}\right)+\sum_{\substack{3 \leq r<\ell \\ r \equiv \ell(\bmod 2)}} \alpha_{\ell, r}\left(n^{-r / 2} \gamma_{C_{r}}\right)+n^{-\ell / 2} X_{\ell}
$$

and now the result clearly follows from Theorem 1.6 as the error terms $X_{\ell}$ are negligible (using a similar argument as in the proof of Corollary 1.2). Since this representation in terms of the cycle graph factors is triangular, we furthermore see that the resulting $\Sigma_{k}$ that arises is indeed positive definite.

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