## CAYLEY GRAPHS THAT HAVE A QUANTUM ERGODIC EIGENBASIS

ASSAF NAOR, ASHWIN SAH, MEHTAAB SAWHNEY, AND YUFEI ZHAO

ABSTRACT. We investigate which finite Cayley graphs admit a quantum ergodic eigenbasis, proving that this holds for any Cayley graph on a group of size n for which the sum of the dimensions of its irreducible representations is o(n), yet there exist Cayley graphs that do not have any quantum ergodic eigenbasis.

### 1. INTRODUCTION

We will prove here the following theorem; see Theorem 3 for a companion impossibility result.

**Theorem 1.** There exists an absolute constant c > 0 with the following property. Suppose that  $\varepsilon > 0$  and let *G* be a finite group whose irreducible representations have total dimension at most  $c\varepsilon^2 |G|$ , i.e.,

$$\sum_{\sigma \in \widehat{G}} d_{\sigma} \le c\varepsilon^2 |G|.$$
(1)

Then, any Cayley graph on G has an orthonormal eigenbasis  $\mathcal{B}$  consisting of functions  $\phi: G \to \mathbb{C}$  satisfying

$$\forall f: G \to \mathbb{C}, \qquad \mathbb{E}_{\phi \in \mathcal{B}} \left[ \left| \mathbb{E}_{x \in G} \left[ f(x) |\phi(x)|^2 \right] - \mathbb{E}f \right| \right] \le \varepsilon \| f \|_2. \tag{2}$$

In the statement of Theorem 1,  $\hat{G}$  is the set of irreducible unitary representations of a finite group *G* and the dimension of each  $\sigma \in \hat{G}$  is denoted  $d_{\sigma}$ . The representation theory of finite groups that we will use below is rudimentary; see e.g. [15, 24]. A Cayley graph on *G* is a graph whose vertex set is *G* such that there is a symmetric subset  $\mathfrak{S} \subseteq G$  that generates *G* and  $\{g, h\} \subseteq G$  forms an edge if and only if  $gh^{-1} \in \mathfrak{S}$ .

In Theorem 1 and throughout what follows, we will adhere to the convention that a finite set *X* is only equipped with the uniform probability measure; thus, all expectations, scalar products and  $L_p$  norms of functions from *X* to  $\mathbb{C}$  will be with respect to this measure, i.e., for every  $f, g: X \to \mathbb{C}$  and  $1 \le p \le \infty$ ,

$$\mathbb{E}f = \mathbb{E}_{x \in X}[f(x)] = \frac{1}{|X|} \sum_{x \in X} f(x) \quad \text{and} \quad \langle f, g \rangle = \mathbb{E}_{x \in X}\left[\overline{f(x)}g(x)\right] \quad \text{and} \quad \left\|f\right\|_{p} = \left(\mathbb{E}\left[|f|^{p}\right]\right)^{\frac{1}{p}}.$$
 (3)

So, a set of functions  $\phi_1, ..., \phi_{|X|}: X \to \mathbb{C}$  is an orthonormal basis if  $\|\phi_j\|_2 = 1$  and  $\langle \phi_j, \phi_k \rangle = 0$  for every distinct  $j, k \in \{1, ..., |X|\}$ . If X is a graph, then we say that  $\mathcal{B} = \{\phi_1, ..., \phi_{|X|}\}$  is an orthonormal eigenbasis of X if it is an orthonormal basis consisting of eigenfunctions of the adjacency matrix of X.

Theorem 1 is a finitary statement in the spirit of quantum ergodicity on manifolds, e.g. Šnirel'man's classical theorem [26, 8, 27]. Investigations along these lines include notably [4], and we refer also to [2, 5] and the survey [3] for background and motivation. From these works, we extract the following definition.

**Definition 2** (quantum ergodic basis). Given a finite set *X* and  $\varepsilon > 0$ , we say that an orthonormal basis  $\mathcal{B}$  of functions  $\phi: X \to \mathbb{C}$  is  $\varepsilon$ -quantum ergodic if

$$\forall f \colon X \to \mathbb{C}, \qquad \mathbb{E}_{\phi \in \mathcal{B}} \left[ \left| \mathbb{E}_{x \in X} \left[ f(x) |\phi(x)|^2 \right] - \mathbb{E} f \right| \right] \le \varepsilon \| f \|_{\infty}. \tag{4}$$

The only difference between the conclusion (2) of Theorem 1 and the requirement (4) of Definition 2 is that the quantity  $||f||_2$  in the right hand side of (2) is replaced in the right hand side of (4) by the larger quantity  $||f||_{\infty}$ . Therefore, Theorem 1 implies that any Cayley graph of a finite group whose irreducible representations have total dimension at most  $c\varepsilon^2 |G|$  has an  $\varepsilon$ -quantum ergodic eigenbasis. The stronger

Naor was supported by NSF grant DMS-2054875 and a Simons Investigator award. Sah and Sawhney were supported by NSF Graduate Research Fellowship Program DGE-1745302. Sah was supported by the PD Soros Fellowship. Zhao was supported by NSF CAREER Award DMS-2044606, a Sloan Research Fellowship, and the MIT Solomon Buchsbaum Fund.

conclusion (2) of Theorem 1 can be significantly stronger when e.g. in (2) we take *f* to be the indicator of a small nonempty subset *S* of *G*, as in this case  $||f||_{\infty} = 1$  while  $||f||_2 = \sqrt{|S|/|G|}$ .

The reason why we formulated Definition 2 using the  $L_{\infty}$  norm of f rather than its  $L_2$  norm is first and foremost because this is how the subject is treated in the literature, but also because the following impossibility result rules out even the weaker requirement (4).

**Theorem 3.** There are arbitrarily large Cayley graphs that do not admit any c-quantum ergodic orthonormal eigenbasis, where c > 0 is a universal constant.

The groups that we will construct in the proof of Theorem 3 will be a direct product of a cyclic group with an appropriately chosen fixed group (specifically, a group that was constructed in [23]).

**Problem 4.** For a finite group *G* let  $\varepsilon(G)$  be the infimum over  $\varepsilon > 0$  such that every Cayley graph on *G* has an  $\varepsilon$ -quantum ergodic orthonormal eigenbasis. Characterize those sequences  $\{G_n\}_{n=1}^{\infty}$  of groups for which  $\lim_{n\to\infty} \varepsilon(G_n) = 0$ . More ambitiously, how can one compute  $\varepsilon(G)$  up to universal constant factors?

Any Abelian group *G* satisfies  $\varepsilon(G) = 0$ , as seen by considering the eigenbasis  $\mathcal{B}$  of Fourier characters: each  $\phi \in \mathcal{B}$  takes value among the roots of unity, so the left-hand side of (2) vanishes for every  $f : G \to \mathbb{C}$ . Theorem 1 furnishes many more examples of sequences  $\{G_n\}_{n=1}^{\infty}$  of groups with  $\lim_{n\to\infty} \varepsilon(G_n) = 0$ .

If  $\eta > 0$  and *G* is a group with at most  $\eta |G|$  conjugacy classes (e.g. by [19, Theorem 2] this holds with  $\eta = 2^{n-1}/|G|$  if *G* is any subgroup of the permutation group  $S_n$ ), then every Cayley graph on *G* has a  $O(\sqrt[4]{\eta})$ -quantum ergodic orthonormal eigenbasis. Indeed,

$$\sum_{\sigma \in \widehat{G}} d_{\sigma} \leq |\widehat{G}|^{\frac{1}{2}} \left( \sum_{\sigma \in \widehat{G}} d_{\sigma}^{2} \right)^{\frac{1}{2}} \leq \sqrt{\eta} |G|,$$

where the first step uses Cauchy—Schwarz and that  $|\hat{G}|$  equals the number of conjugacy classes of *G*, and the second step uses the above assumption and that  $\sum_{\sigma \in \hat{G}} d_{\sigma}^2 = |G|$ . Hence, (1) holds with  $\varepsilon = \sqrt[4]{\eta}/\sqrt{c}$ .

A special case of the above example is when for some  $D \in \mathbb{N}$  a group *G* is nontrivial and *D*-quasirandom in the sense of Gowers [13], i.e., every nontrivial unitary representation of *G* has dimension at least *D*. This implies that *G* has at most  $2|G|/(D^2 + 1)$  conjugacy classes, and hence every Cayley graph on *G* has a  $O(1/\sqrt{D})$ -quantum ergodic orthonormal eigenbasis. Indeed,

$$|G| = \sum_{\sigma \in \widehat{G}} d_{\sigma}^2 = 1 + \sum_{\sigma \in \widehat{G} \smallsetminus \{\text{triv}\}} d_{\sigma}^2 \ge 1 + D^2 (|\widehat{G}| - 1).$$
(5)

Thus,  $|\hat{G}| \le 1 + (|G| - 1)/D^2 \le 2|G|/(D^2 + 1)$ , where the last step holds as |G| > 1 and therefore the second sum in (5) is nonempty, so in fact  $|G| \ge D^2 + 1$ . By an inspection of the tables on pages 769–770 of [9] and the classification of finite simple groups, if *G* is a non-cyclic simple group, then we can take *D* to be at least a universal constant multiple of  $(\log |G|)/\log \log |G|$ ; for most simple groups a much better lower bound on *D* is available, and many more examples appear in the literature (see e.g. [25, Chapter 1, §1.3]).

At the same time, Theorem 3 demonstrates that some assumption on  $\{G_n\}_{n=1}^{\infty}$  must be imposed to ensure that  $\lim_{n\to\infty} \varepsilon(G_n) = 0$ . Thus, Problem 4 remains an intriguing open question.

## 2. PROOF OF THEOREM 1

The Haar probability measure on a compact topological group  $\Gamma$  will be denoted  $h_{\Gamma}$ . Given  $d \in \mathbb{N}$ , the standard coordinate basis of  $\mathbb{C}^d$  will be denoted  $e_1, \ldots, e_d$  and the unitary group of  $d \times d$  matrices will be denoted  $\mathbb{U}(d)$ . The Hilbert–Schmidt norm of a  $d \times d$  matrix  $A = (a_{ik}) \in M_d(\mathbb{C})$  will be denoted

$$||A||_{\mathrm{HS}} = \left(\sum_{j=1}^{d} \sum_{k=1}^{d} |a_{jk}|^2\right)^{\frac{1}{2}}.$$

Our construction of the basis  $\mathcal{B}$  of Theorem 1 will be randomized; its main probabilistic input is the following lemma whose proof appears in Section 2.1 below.

**Lemma 5.** There exists a universal constant  $0 < \eta < 1$  with the following property. Let *S* be a finite set. For every  $s \in S$  fix an integer  $d_s \in \mathbb{N}$  and a  $d_s \times d_s$  matrix  $A_s \in M_{d_s}(\mathbb{C})$  whose trace satisfies  $\operatorname{Tr}(A_s) = 0$ . Denote

$$\alpha = \left(\frac{\sum_{s \in S} \frac{1}{d_s} \|A_s\|_{\mathrm{HS}}^2}{\sum_{s \in S} d_s}\right)^{\frac{1}{2}} \quad \text{and} \quad T = \bigcup_{s \in S} \left(\{s\} \times \{1, \dots, d_s\}\right) = \{(s, k) : s \in S \land k \in \{1, \dots, d_s\}\}.$$
(6)

Consider the direct product  $\Gamma = \prod_{s \in S} \bigcup (d_s)$  of the unitary groups  $\{\bigcup (d_s)\}_{s \in S}$ . Then, for every  $\beta \ge 2$  we have

$$h_{\Gamma}\left[\left\{U = (U_s)_{s \in S} \in \Gamma : \mathbb{E}_{(s,k) \in T}\left[\left|e_k^* U_s^* A_s U_s e_k\right|\right] \ge \beta \alpha\right\}\right] \le e^{-\eta \beta^2 \sum_{s \in S} d_s}.$$
(7)

Fix a finite group *G* and fix also a symmetric subset  $\mathfrak{S} \subseteq G$  that generates *G*. Let n = |G|. The adjacency matrix  $A(G;\mathfrak{S}) \in M_n(\{0,1\})$  of the Cayley graph that is induced by  $\mathfrak{S}$  on *G* acts on a function  $f: G \to \mathbb{C}$  by  $A(G;\mathfrak{S})f(x) = \sum_{\sigma \in \mathfrak{S}} f(\sigma x)$  for every  $x \in G$ .

We will apply Lemma 5 with the index set S

$$S = \bigcup_{\rho \in \widehat{G}} \left( \{\rho\} \times \{1, \dots, d_{\rho}\} \right) = \left\{ (\rho, j) : \rho \in \widehat{G} \land j \in \{1, \dots, d_{\rho}\} \right\}$$

and  $d_s = d_\rho$  for every  $s = (\rho, j) \in S$ . For this *S*, the set *T* in (6) becomes

$$T = \left\{ (\rho, j, k) : \rho \in \widehat{G} \land (j, k) \in \{1, \dots, d_{\rho}\}^2 \right\}.$$

Henceforth,  $\Gamma = \prod_{(\rho,j) \in S} \mathbb{U}(d_{\rho}) \cong \prod_{\rho \in \widehat{G}} \mathbb{U}(d_{\rho})^{d_{\rho}}$  will be the group from Lemma 5.

Suppose that for each  $\rho \in \widehat{G}$  and  $j, k \in \{1, ..., d_{\rho}\}$  we are given  $a_{\rho, j, k} \in \mathbb{C}^{d_{\rho}}$  and  $b_{\rho, j} \in \mathbb{C}^{d_{\rho}}$  such that

$$\forall j, j', k, k' \in \{1, \dots, d_{\rho}\}, \qquad a_{\rho, j, k'}^* a_{\rho, j, k'} = \mathbf{1}_{\{k=k'\}} \qquad \text{and} \qquad b_{\rho, j}^* b_{\rho, j'} = \mathbf{1}_{\{j=j'\}}. \tag{8}$$

This is an orthornormality requirement<sup>1</sup> with respect to the standard (not normalized) scalar product on  $\mathbb{C}^{d_{\rho}}$ . The statement of Schur orthogonality is that whenever (8) holds the following collection of functions from *G* to  $\mathbb{C}$  (indexed by *T*) is orthonormal; as  $|T| = \sum_{\rho \in \widehat{G}} d_{\rho}^2 = n$ , it is an orthonormal basis of *G*:

$$\left\{ (x \in G) \mapsto d_{\rho}^{\frac{1}{2}} a_{\rho,j,k}^* \rho(x)^* b_{\rho,j} \right\}_{(\rho,j,k) \in T}.$$
(9)

These expressions are also natural through the lens of non-Abelian Fourier analysis. It is mechanical to check that (9) consists of eigenfunctions of the adjacency matrix  $A(G; \mathfrak{S})$  if for each  $\rho \in \widehat{G}$  we choose  $b_{\rho,1}, \ldots, b_{\rho,d_{\rho}} \in \mathbb{C}^{d_{\rho}}$  to be eigenvectors of the (Hermitian, as  $\mathfrak{S}$  is symmetric) matrix

$$\widehat{\mathbf{l}_{\mathfrak{S}}}(\rho) = \mathbb{E}_{\sigma \in \mathfrak{S}}[\rho(\sigma)] \in \mathsf{M}_{d_{\rho}}(\mathbb{C}).$$

So, we will henceforth assume that  $\{b_{\rho,j}\}_{j=1}^{d_{\rho}}$  are eigenvectors of  $\widehat{\mathbf{1}_{\mathfrak{S}}}(\rho)$  and satisfy (8) for each  $\rho \in \widehat{G}$ .

We will prove Theorem 1 by choosing the rest of the datum in (9) uniformly at random. Namely, vectors  $\{a_{\rho,j,k}\}_{(\rho,j,k)\in T}$  as above can be parameterized by taking  $U = (U_{\rho,j})_{(\rho,j)\in S} \in \Gamma$  and letting  $a_{\rho,j,k} = U_{\rho,j}e_k$  for every  $(\rho, j, k) \in T$ . Using this notation, the orthonormal eigenbasis of *G* in (9) becomes

$$\mathcal{B}_U = \left\{ (x \in G) \mapsto d_\rho^{\frac{1}{2}} e_k^* U_{\rho,j}^* \rho(x)^* b_{\rho,j} \right\}_{(\rho,j,k) \in T}.$$

We will show that if (1) holds and  $U \in \Gamma$  is distributed according to the Haar probability measure  $h_{\Gamma}$ , then  $\mathcal{B}_U$  satisfies the conclusion of Theorem 1 with probability at least  $1 - e^{-n}$ .

We will see that the following lemma is an instantiation of Lemma 5.

**Lemma 6.** Let  $\eta > 0$  be the universal constant of Lemma 5. For every  $\beta \ge 2$  and  $f: G \to \mathbb{C}$  we have

$$h_{\Gamma}\left[\left\{U \in \Gamma : \mathbb{E}_{\phi \in \mathcal{B}_{U}}\left[\left|\mathbb{E}_{x \in G}\left[f(x)|\phi(x)|^{2}\right] - \mathbb{E}f\right|\right] \ge \beta\left(\frac{1}{n}\sum_{\rho \in \widehat{G}}d_{\rho}\right)^{\frac{1}{2}} \|f\|_{2}\right\}\right] \le e^{-\eta\beta^{2}n}.$$
(10)

<sup>&</sup>lt;sup>1</sup>To be consistent with our normalization convention in (3), for every  $d \in \mathbb{N}$  we will use matrix notation as in (8) when treating the standard scalar product on  $\mathbb{C}^d$ .

Prior to proving Lemma 6, we will explain how it implies Theorem 1.

Deduction of Theorem 1 from Lemma 6. It is a classical fact (see e.g. [11, Lemma 2.4]) that there exist  $f_1, \ldots, f_{5^{2n}} : G \to \mathbb{C}$  with  $||f_1||_2 = \ldots = ||f_{5^{2n}}||_2 = 1$  such that every  $f : G \to \mathbb{C}$  with  $||f||_2 = 1$  belongs to the convex hull of  $\{2f_1, \ldots, 2f_{5^{2n}}\}$  (better bounds on such polytopal approximation of balls can be found in [7, 6, 22], but they only affect the constant *c* in Theorem 1). Since for every fixed  $U \in \Gamma$  the mapping

$$(f: G \to \mathbb{C}) \mapsto \mathbb{E}_{\phi \in \mathcal{B}_U} \Big[ \big| \mathbb{E}_{x \in G} \big[ f(x) |\phi(x)|^2 \big] - \mathbb{E}f \big| \Big]$$

is convex (in the variable f), it follows that

$$\sup_{\substack{f:G \to \mathbb{C} \\ \|f\|_2 = 1}} \mathbb{E}_{\phi \in \mathcal{B}_U} \Big[ \Big| \mathbb{E}_{x \in G} \big[ f(x) |\phi(x)|^2 \big] - \mathbb{E}f \Big| \Big] \leq 2 \max_{\ell \in \{1, \dots, 5^{2n}\}} \mathbb{E}_{\phi \in \mathcal{B}_U} \Big[ \Big| \mathbb{E}_{x \in G} \big[ f_\ell(x) |\phi(x)|^2 \big] - \mathbb{E}f_\ell \Big| \Big]$$

Consequently, if  $\eta$  is the universal constant in (10), then

$$\begin{split} &h_{\Gamma} \left[ \left\{ U \in \Gamma : \forall f : G \to \mathbb{C}, \quad \mathbb{E}_{\phi \in \mathcal{B}_{U}} \left[ \left| \mathbb{E}_{x \in G} [f(x) | \phi(x)|^{2} \right] - \mathbb{E}f \right| \right] \leq \frac{5}{\sqrt{\eta}} \left( \frac{1}{n} \sum_{\rho \in \widehat{G}} d_{\rho} \right)^{\frac{1}{2}} \|f\|_{2} \right\} \right] \\ &\geq 1 - \sum_{\ell=1}^{5^{2n}} h_{\Gamma} \left[ \left\{ U \in \Gamma : \forall \ell \in \{1, \dots, 5^{2n}\}, \quad \mathbb{E}_{\phi \in \mathcal{B}_{U}} \left[ \left| \mathbb{E}_{x \in G} [f_{\ell}(x) | \phi(x)|^{2} \right] - \mathbb{E}f_{\ell} \right| \right] \geq \frac{5}{2\sqrt{\eta}} \left( \frac{1}{n} \sum_{\rho \in \widehat{G}} d_{\rho} \right)^{\frac{1}{2}} \right\} \right] \\ &\geq 1 - 5^{2n} \cdot e^{-5n} \geq 1 - e^{-n} > 0. \end{split}$$

Hence, there is  $U \in \Gamma$  such that if (1) holds with  $c = \sqrt{\eta}/5$ , then the orthonormal eigenbasis  $\mathcal{B}_U$  satisfies

$$\forall f \colon G \to \mathbb{C}, \qquad \mathbb{E}_{\phi \in \mathcal{B}_U} \Big[ \left| \mathbb{E}_{x \in G} [f(x) | \phi(x) |^2 \right] - \mathbb{E}f \Big| \Big] \leq \frac{5}{\sqrt{\eta}} \left( \frac{1}{n} \sum_{\rho \in \widehat{G}} d_\rho \right)^{\frac{1}{2}} \|f\|_2 \leq \frac{5c}{\sqrt{\eta}} \varepsilon \|f\|_2 = \varepsilon \|f\|_2. \qquad \Box$$

We will next prove Lemma 6 assuming Lemma 5, after which we will pass (in Section 2.1) to the proof of Lemma 5, thus completing the proof of Theorem 1.

*Deduction of Lemma 6 from Lemma 5.* As  $||f - \mathbb{E}f||_2 \le ||f||_2 \le 1$ , it suffices to prove (10) under the additional assumptions  $\mathbb{E}f = 0$  and  $||f||_2 = 1$ . Observe that for every  $(\rho, j, k) \in T$  and  $U \in \Gamma$  we have

$$\mathbb{E}_{x\in G}\left[f(x)\left|d_{\rho}^{\frac{1}{2}}e_{k}^{*}U_{\rho,j}^{*}\rho(x)^{*}b_{\rho,j}\right|^{2}\right] = e_{k}^{*}U_{\rho,j}^{*}A_{\rho,j}^{f}U_{\rho,j}e_{k},$$

where we introduce the notation

$$A_{\rho,j}^f = d_\rho \mathbb{E}_{x \in G} \big[ f(x) \rho(x)^* b_{\rho,j} b_{\rho,j}^* \rho(x) \big] \in M_{d_\rho}(\mathbb{C}).$$

For every  $(\rho, j) \in S$ ,

$$\operatorname{Tr}[A_{\rho,j}^{f}] = d_{\rho} \mathbb{E}\Big[f(x) \operatorname{Tr}\big[\rho(x)^{*} b_{\rho,j} b_{\rho,j}^{*} \rho(x)\big]\Big] = d_{\rho} \big(\mathbb{E}f\big) \operatorname{Tr}\big[b_{\rho,j} b_{\rho,j}^{*}\big] = 0,$$

where we used the cyclicity of the trace and that  $\rho(x)$  is unitary for every  $x \in G$ . Also,

$$\|A_{\rho,j}^{f}\|_{\mathrm{HS}}^{2} = \mathrm{Tr}[(A_{\rho,j}^{f})^{*}A_{\rho,j}^{f}] = d_{\rho}^{2}\mathbb{E}_{(x,y)\in G\times G}[\overline{f(x)}f(y)\mathrm{Tr}[\rho(x)^{*}b_{\rho,j}b_{\rho,j}^{*}\rho(x)\rho(y)^{*}b_{\rho,j}b_{\rho,j}^{*}\rho(y)]].$$
(11)

Using the cyclicity of the trace once more, for every  $x, y \in G$  we have

$$\operatorname{Tr}[\rho(x)^* b_{\rho,j} b_{\rho,j}^* \rho(x) \rho(y)^* b_{\rho,j} b_{\rho,j}^* \rho(y)] = |b_{\rho,j}^* \rho(x) \rho(y)^* b_{\rho,j}|^2$$

In combination with (11), this gives that

$$\begin{split} \|A_{\rho,j}^{f}\|_{\mathrm{HS}}^{2} &= d_{\rho}^{2} \mathbb{E}_{(x,y)\in G\times G} \left[ \left( \overline{f(x)b_{\rho,j}^{*}\rho(x)\rho(y)^{*}b_{\rho,j}} \right) \left( f(y)b_{\rho,j}^{*}\rho(x)\rho(y)^{*}b_{\rho,j} \right) \right] \\ &\leq d_{\rho}^{2} \mathbb{E}_{(x,y)\in G\times G} \left[ |f(x)|^{2} \left| b_{\rho,j}^{*}\rho(x)\rho(y)^{*}b_{\rho,j} \right|^{2} \right] = d_{\rho} \mathbb{E}_{x\in G} \left[ |f(x)|^{2} d_{\rho} \mathbb{E}_{y\in G} \left[ \left| b_{\rho,j}^{*}\rho(x)\rho(y)^{*}b_{\rho,j} \right|^{2} \right] \right], \end{split}$$

where the penultimate step uses Cauchy—Schwarz. By Schur orthogonality, for every  $x \in G$  we have

$$d_{\rho}\mathbb{E}_{y\in G}\Big[\left|b_{\rho,j}^{*}\rho(x)\rho(y)^{*}b_{\rho,j}\right|^{2}\Big] = \big((\rho(x)^{*}b_{\rho,j})^{*}\rho(x)^{*}b_{\rho,j}\big)\big(b_{\rho,j}^{*}b_{\rho,j}\big) = (b_{\rho,j}^{*}b_{\rho,j})^{2} = 1.$$

Therefore,  $\|A_s^f\|_{\text{HS}}^2 \leq d_{\rho} \|f\|_2 \leq d_{\rho}$  for every  $s \in S$ . The desired estimate (10) now follows from (7) because

$$\sum_{s \in S} d_s = \sum_{\rho \in \widehat{G}} d_\rho^2 = n \quad \text{and} \quad \sum_{s \in S} \frac{1}{d_s} \|A_s^f\|_{\mathrm{HS}}^2 \le |S| = \sum_{\rho \in \widehat{G}} d_\rho. \quad \Box$$

2.1. **Concentration.** Given  $d \in \mathbb{N}$ , let  $g_d$  be the standard Riemannian metric on  $\mathbb{U}(d)$ , namely the geodesic distance that is induced by taking the Hilbert–Schmidt metric on all of the tangent spaces.

The following theorem is a concatenation of known results that we formulate for ease of later reference. Its quick justification below uses fundamental properties of logarithmic Soboloev inequalities [14] on metric probability spaces; good expositions of what we need can be found in the monographs [18, 20].

**Theorem 7** (concentration of measure on Pythagorean products of rescaled unitary groups). *Let S be a finite set and*  $\{d_s\}_{s \in S} \subseteq \mathbb{N}$ . *Denote*  $\Gamma = \bigcup (d_1) \times \ldots \times \bigcup (d_m)$ . *Suppose that* K > 0 *and that*  $f : \Omega \to \mathbb{R}$  *satisfies* 

$$\forall U = (U_s)_{s \in S}, V = (V_s)_{s \in S} \in \Gamma, \qquad |f(U) - f(V)| \le K \left(\sum_{s \in S} d_s g_{d_s} (U_s, V_s)^2\right)^{\frac{1}{2}}.$$
 (12)

In other words, (12) is the requirement that f is K-Lipschitz with respect on the Pythagorean product of the metric spaces  $\{(U(d_s), \sqrt{d_s g_{d_1}})\}_{s \in S}$ . Then, for every  $\varepsilon > 0$  we have

$$h_{\Gamma}\left[f \ge \varepsilon + \int_{\Gamma} f \,\mathrm{d}h_{\Gamma}\right] \le \exp\left(-\frac{\varepsilon^2}{3\pi^2 K^2}\right). \tag{13}$$

*Proof.* By the paragraph after Theorem 15 in [21], for every  $d \in \mathbb{N}$  the logarithmic Sobolev constant of the metric probability space  $(\mathbb{U}(d), g_d, h_{\mathbb{U}(d)})$  is at most  $3\pi^2/(2d)$ . As the logarithmic Sobolev constant scales quadratically with rescaling of the metric, it follows that the metric probability space  $(\mathbb{U}(d), \sqrt{d}g_d, h_{\mathbb{U}(d)})$  has logarithmic Sobolev constant at most  $3\pi^2/2$ . By the tensorization property of the logarithmic Sobolev constant constant under Pythagorean products (see [18, Corollary 5.7]), if we define

$$\forall U = (U_s)_{s \in S}, V = (V_s)_{s \in S} \in \Omega, \qquad \rho(U, V) = \left(\sum_{s \in S} d_s g_{d_s} (U_s, V_s)^2\right)^{\frac{1}{2}},$$

then the logarithmic Sobolev constant of the metric probability space  $(\Gamma, \rho, h_{\Gamma})$  is at most  $3\pi^2/2$ . The desired conclusion (13) follows by the classical Herbst argument [10, 1, 17] (see [18, Theorem 5.3]).

It is worthwhile to formulate separately the following quick corollary of Theorem 7.

**Corollary 8.** Continuing with the notation of Theorem 7, suppose that  $\{K_s\}_{s\in S} \subseteq (0,\infty)$  and that for each  $s \in S$  we are given a function  $f_s : \mathbb{U}(d_s) \to \mathbb{R}$  that is  $K_s$ -Lipschitz with respect to the geodesic metric  $g_{d_s}$ , i.e.,  $|f_s(U) - f_s(V)| \leq K_s g_{d_s}(U, V)$  for every  $U, V \in \mathbb{U}(d_s)$ . Then, for every  $\varepsilon > 0$  we have

$$h_{\Gamma}\left[\left\{U = (U_{s})_{s \in S} \in \Gamma : \mathbb{E}_{s \in S}\left[f_{s}(U_{s})\right] \ge \mathbb{E}_{s \in S}\left[\int_{\mathbb{U}(d_{s})} f_{s} \,\mathrm{d}h_{\mathbb{U}(d_{s})}\right] + \varepsilon\right\}\right] \le \exp\left(-\frac{\varepsilon^{2}|S|^{2}}{3\pi^{2}\sum_{s \in S}\frac{1}{d_{s}}K_{s}^{2}}\right)$$

*Proof.* Define  $f : \Gamma \to \mathbb{R}$  by setting  $f(U) = \mathbb{E}_{s \in S} [f_s(U_s)]$  for  $U = (U_s)_{s \in S} \in \Gamma$ . If  $U, V \in \Gamma$ , then

$$|f(U) - f(V)| \leq \mathbb{E}_{s \in S} \Big[ |f_s(U_s) - f_s(V_s)| \Big] \leq \frac{1}{|S|} \sum_{s \in S} K_s g_s(U_s, V_s) \leq \frac{1}{|S|} \Big( \sum_{s \in S} \frac{1}{d_s} K_s^2 \Big)^{\frac{1}{2}} \Big( \sum_{s \in S} d_s g_{d_s}(U_s, V_s)^2 \Big)^{\frac{1}{2}},$$

where the final step is Cauchy–Schwarz. Now apply Theorem 7.

The following lemma connects the above general discussion to Lemma 5.

**Lemma 9.** Suppose that  $\varphi_1, \ldots, \varphi_d : \mathbb{C} \to \mathbb{C}$  are 1-Lipschitz and  $A \in M_n(\mathbb{C})$ . Define  $f : U(d) \to \mathbb{C}$  by setting

$$\forall U \in \mathbb{U}(d), \qquad f(U) = \sum_{k=1}^{d} \varphi_k(e_k^* U^* A U e_k)$$

Then, the Lipschitz constant of f with respect to the geodesic distance  $g_d$  is at most  $2||A||_{HS}$ , i.e.,

$$\forall U, V \in \mathbb{U}(d), \qquad |f(U) - f(V)| \leq 2 \|A\|_{\mathrm{HS}} g(U, V).$$

*Proof.* Fix  $U, V \in U(d)$ . By the definition of  $g = g_d(U, V)$ , there is a smooth curve (unit-speed geodesic)  $\gamma : [0,g] \to U(d)$  that satisfies  $\gamma(0) = U$ ,  $\gamma(g) = V$ , and such that  $\|\gamma'(t)\|_{\text{HS}} = 1$  for every  $t \in [0,g]$ . Then,

$$\begin{split} |f(U) - f(V)| &\leq \sum_{k=1}^{d} \left| \varphi_k(e_k^* U^* A U e_k) - \varphi_k(e_k^* V^* A V e_k) \right| \leq \sum_{k=1}^{d} \left| e_k^* U^* A U e_k - e_k^* V^* A V e_k \right| \\ &= \sum_{k=1}^{d} \left| \int_0^g \frac{\mathrm{d}}{\mathrm{d}t} \left( e_k^* \gamma(t)^* A \gamma(t) e_k \right) \mathrm{d}t \right| \leq \int_0^g \left( \sum_{k=1}^{d} \left| \frac{\mathrm{d}}{\mathrm{d}t} \left( e_k^* \gamma(t)^* A \gamma(t) e_k \right) \right| \right) \mathrm{d}t. \end{split}$$

It therefore suffices to prove the following point-wise estimate:

$$\forall t \in [0,g], \qquad \sum_{k=1}^{d} \left| \frac{\mathrm{d}}{\mathrm{d}t} \left( e_k^* \gamma(t)^* A \gamma(t) e_k \right) \right| \le 2 \|A\|_{\mathrm{HS}}. \tag{14}$$

This indeed holds because by Cauchy–Schwarz for every  $t \in [0, g]$  and  $k \in \{1, ..., d\}$ ,

$$\begin{aligned} \left| \frac{\mathrm{d}}{\mathrm{d}t} (e_k^* \gamma(t)^* A \gamma(t) e_k) \right| &= e_k^* \gamma'(t)^* A \gamma(t) e_k + e_k^* \gamma(t)^* A \gamma'(t) e_k \\ &\leq \left( e_k^* \gamma'(t)^* \gamma'(t) e_k \right)^{\frac{1}{2}} \left( e_k^* \gamma(t)^* A^* A \gamma(t) e_k \right)^{\frac{1}{2}} + \left( e_k^* \gamma(t)^* A A^* \gamma(t) e_k \right)^{\frac{1}{2}} \left( e_k^* \gamma'(t)^* \gamma'(t) e_k \right)^{\frac{1}{2}}. \end{aligned}$$

By summing this over  $k \in \{1, ..., d\}$  and using Cauchy–Schwarz, we conclude the proof of (14) as follows.

$$\begin{split} \sum_{k=1}^{d} \left| \frac{\mathrm{d}}{\mathrm{d}t} \left( e_{k}^{*} \gamma(t)^{*} A \gamma(t) e_{k} \right) \right| &\leq \left( \sum_{k=1}^{d} e_{k}^{*} \gamma'(t)^{*} \gamma'(t) e_{k} \right)^{\frac{1}{2}} \left( \left( \sum_{k=1}^{d} e_{k}^{*} \gamma(t)^{*} A^{*} A \gamma(t) e_{k} \right)^{\frac{1}{2}} + \left( \sum_{k=1}^{d} e_{k}^{*} \gamma(t)^{*} A A^{*} \gamma(t) e_{k} \right)^{\frac{1}{2}} \right) \\ &= \left( \mathrm{Tr}[\gamma'(t)^{*} \gamma'(t)] \right)^{\frac{1}{2}} \left( \left( \mathrm{Tr}[\gamma(t)^{*} A^{*} A \gamma(t)] \right)^{\frac{1}{2}} + \left( \mathrm{Tr}[\gamma(t)^{*} A A^{*} \gamma(t)] \right)^{\frac{1}{2}} \right) \\ &= \left( \mathrm{Tr}[\gamma'(t)^{*} \gamma'(t)] \right)^{\frac{1}{2}} \left( \left( \mathrm{Tr}[A^{*} A] \right)^{\frac{1}{2}} + \left( \mathrm{Tr}[A A^{*}] \right)^{\frac{1}{2}} \right) = 2 \|A\|_{\mathrm{HS}}. \end{split}$$

We can now prove Lemma 5, thus completing the proof of Theorem 1.

*Proof of Lemma 5.* For every  $d \in \mathbb{N}$  and  $k \in \{1, ..., d\}$  we have

$$\forall A \in \mathsf{M}_{d}(\mathbb{C}), \qquad \int_{\mathbb{U}(d)} \left| e_{k}^{*} U^{*} A U e_{k} \right|^{2} \mathrm{d}h_{\mathbb{U}(d)}(U) = \frac{\|A\|_{\mathrm{HS}}^{2} + |\mathrm{Tr}(A)|^{2}}{d(d+1)}. \tag{15}$$

One checks (15) by noting that if U is distributed according to the Haar measure on U(d), then  $Ue_k$  is distributed according to the normalized surface area measure on  $\{z \in \mathbb{C}^d : |z_1|^2 + \ldots + |z_d|^2 = 1\}$ , expanding the squares and substituting the resulting standard spherical integrals that are computed in e.g. [12].

Returning to the setting and notation of Lemma 5, for every  $s \in S$  and  $U \in U(d_s)$  define

$$f_s(U) = \sum_{k=1}^{d_s} \left| e_k^* U^* A_s U e_k \right|.$$

By Lemma 9, the assumption of Corollary 8 holds with  $K_s = 2 \|A_s\|_{HS}$ . By Cauchy–Schwarz and (15),

$$\int_{\mathbb{U}(d_s)} f_s \, \mathrm{d}h_{\mathbb{U}(d_s)} = \sum_{k=1}^{d_s} \int_{\mathbb{U}(d_s)} \left| e_k^* U^* A_s U e_k \right| \mathrm{d}h_{\mathbb{U}(d_s)}(U) \leq \sum_{k=1}^{d_s} \left( \int_{\mathbb{U}(d_s)} \left| e_k^* U^* A_s U e_k \right|^2 \mathrm{d}h_{\mathbb{U}(d_s)}(U) \right)^{\frac{1}{2}} \leq \|A_s\|_{\mathrm{HS}}.$$

Using Cauchy–Schwarz and recalling the definition of  $\alpha$  in (6), we therefore have

$$\mathbb{E}_{s\in S}\left[\int_{\mathbb{U}(d_s)} f_s \,\mathrm{d}h_{\mathbb{U}(d_s)}\right] \leq \mathbb{E}_{s\in S}\left[\|A_s\|_{\mathrm{HS}}\right] \leq \frac{1}{|S|} \left(\sum_{s\in S} d_s\right)^{\frac{1}{2}} \left(\sum_{s\in S} \frac{1}{d_s} \|A_s\|_{\mathrm{HS}}^2\right)^{\frac{1}{2}} = \frac{\sum_{s\in S} d_s}{|S|} \alpha.$$

Corollary 8 therefore implies the following estimate for every  $\beta \ge 2$ :

$$h_{\Gamma}\left[\left\{U = (U_s)_{s \in S} \in \Gamma : \mathbb{E}_{s \in S}\left[f_s(U_s)\right] \ge \frac{\sum_{s \in S} d_s}{|S|} \beta \alpha\right\}\right] \le \exp\left(-\frac{(\beta - 1)^2}{3\pi^2} \sum_{s \in S} d_s\right) \le \exp\left(-\frac{\beta^2}{12\pi^2} \sum_{s \in S} d_s\right).$$

This coincides with the desired estimate (7) with  $\eta = 1/(12\pi^2)$ .

# 3. PROOF OF THEOREM 3

For the statement of the following proposition, observe that if *H* is a finite group and  $\mathfrak{S}$  a symmetric generating subset of *H*, then  $\mathfrak{S} \times \{-1, 1\}$  generates  $H \times (\mathbb{Z}/m\mathbb{Z})$  for any odd integer  $m \in 1 + 2\mathbb{N}$ . Indeed, if  $(h, k) \in H \times (\mathbb{Z}/m\mathbb{Z})$ , then take  $a \in \mathbb{N}$  and  $\sigma_1, \ldots, \sigma_a \in \mathfrak{S}$  such that  $h = \sigma_1 \cdots \sigma_a$ . Since *m* is odd, there exists  $b \in \mathbb{N}$  such that  $a + 2b \equiv k \mod m$ . We then have  $(h, k) = (\sigma_1, 1) \cdots (\sigma_a, 1)(\sigma_1, 1)^b(\sigma_1^{-1}, 1)^b$ .

**Proposition 10** (from quantum ergodicity to existence of delocalized eigenfunctions). Let *H* be a finite group and fix a symmetric generating subset  $\mathfrak{S}$  of *H*. There is  $\ell = \ell(H, \mathfrak{S}) \in \mathbb{N}$  with the following property. Let p > 3 be a prime that does not divide  $\ell$ . Consider the direct product  $G = H \times (\mathbb{Z}/p\mathbb{Z})$ . Suppose that the Cayley graph that is induced on *G* by the generating set  $\mathfrak{S} \times \{-1, 1\}$  has an  $\varepsilon$ -quantum ergodic eigenbasis for some  $\varepsilon > 0$ . Then, for every nonzero eigenvalue  $\lambda$  of the Cayley graph that is induced on *H* by  $\mathfrak{S}$  there exists an eigenfunction  $\psi : H \to \mathbb{C}$  whose eigenvalue is  $\lambda$  and  $0 < \|\psi\|_{\infty} \le \sqrt{2(1+2|H|^3\varepsilon)} \|\psi\|_2$ .

Prior to proving Proposition 10, we will explain how it implies Theorem 3. This deduction uses (a very small part of) the following theorem from [23]:

**Theorem 11.** There exists a universal constant  $\kappa > 0$  with the following property. For arbitrarily large  $n \in \mathbb{N}$  there exists a group H with |H| = n and a symmetric generating subset  $\mathfrak{S}$  of H such that the adjacency matrix  $A(H;\mathfrak{S})$  has a nonzero eigenvalue  $\lambda$  with the property that  $\|\psi\|_{\infty}/\|\psi\|_{2} \ge \kappa \sqrt{\log n}/\log\log n$  for every nonzero eigenfunction  $\psi$  of  $A(H;\mathfrak{S})$  whose eigenvalue is  $\lambda$ .

The statement of Theorem 1.2 in [23] coincides with Theorem 11, except that it does not include the assertion that the eigenvalue is nonzero, but this is stated in the proof of [23, Theorem 1.2].

*Deduction of Theorem 3 from Proposition 10.* If Theorem 3 does not hold, then by Proposition 10 for any nonzero eigenvalue  $\lambda$  of any finite Cayley graph there is an eigenfunction  $\psi$  of that Cayley graph whose eigenvalue is  $\lambda$  and  $\|\psi\|_{\infty} \leq \sqrt{2} \|\psi\|_2$ . This contradicts Theorem 11.

Our proof of Proposition 10 uses the following basic lemma about algebraic numbers; the rudimentary facts from Galois theory and cyclotomic fields that appear in its proof can be found in e.g. [16].

**Lemma 12.** Let  $\mathbb{K}$  be a finite degree number field. There exists  $\ell = \ell(\mathbb{K}) \in \mathbb{N}$  such that if p > 3 is a prime that does not divide  $\ell$ , then  $\cos(2\pi j/p)/\cos(2\pi k/p) \notin \mathbb{K}$  for all distinct  $j, k \in \{0, 1, ..., (p-1)/2\}$ .

*Proof.* Denote  $\mathbb{Q}^{\text{cyc}} = \mathbb{Q}(\{\exp(2\pi i/k)\}_{k=1}^{\infty})$ . Let  $\mathbb{K}' = \mathbb{K} \cap \mathbb{Q}^{\text{cyc}} \subseteq \mathbb{K}$ . By the primitive element theorem, there exists  $\alpha \in \mathbb{K}'$  such that  $\mathbb{K}' = \mathbb{Q}(\alpha)$ . Since  $\alpha \in \mathbb{Q}^{\text{cyc}}$ , there exists  $\ell \in \mathbb{N}$  such that  $\alpha \in \mathbb{Q}(\exp(2\pi i/\ell))$ . Therefore,  $\mathbb{K} \cap \mathbb{Q}^{\text{cyc}} \subseteq \mathbb{Q}(\exp(2\pi i/\ell))$ . Observe that  $\mathbb{Q}(\exp(2\pi i/\ell)) \cap \mathbb{Q}(\exp(2\pi i/p)) = \mathbb{Q}$  for any prime *p* that does not divide  $\ell$  (as the field generated by  $\mathbb{Q}(\exp(2\pi i/\ell))$  and  $\mathbb{Q}(\exp(2\pi i/p))$  is  $\mathbb{Q}(\exp(2\pi i/(\ell p)))$ , and its degree is  $\varphi(\ell p) = \varphi(\ell)\varphi(p)$ , where  $\varphi(\cdot)$  is Euler's totient function, while the degrees of  $\mathbb{Q}(\exp(2\pi i/\ell))$  and  $\mathbb{Q}(\exp(2\pi i/p))$  are, respectively,  $\varphi(\ell)$  and  $\varphi(p)$ ). Therefore

$$\mathbb{K} \cap \mathbb{Q}\left(e^{\frac{2\pi i}{p}}\right) = \left(\mathbb{K} \cap \mathbb{Q}^{\operatorname{cyc}}\right) \cap \mathbb{Q}\left(e^{\frac{2\pi i}{p}}\right) \subseteq \mathbb{Q}\left(e^{\frac{2\pi i}{\ell}}\right) \cap \mathbb{Q}\left(e^{\frac{2\pi i}{p}}\right) = \mathbb{Q}.$$
(16)

Denoting  $\zeta = \exp(2\pi i/p)$ , it follows from (16) that if  $\cos(2\pi j/p)/\cos(2\pi k/p) = (\zeta^j + \zeta^{-j})/(\zeta^k + \zeta^{-k}) \in \mathbb{K}$ for some distinct  $j, k \in \{0, 1, \dots, (p-1)/2\}$ , then actually in  $(\zeta^j + \zeta^{-j})/(\zeta^k + \zeta^{-k}) \in \mathbb{Q}$ . This cannot happen

for the following reason. Suppose that there are  $a, b \in \mathbb{Z} \setminus \{0\}$  for which  $a(\zeta^j + \zeta^{-j}) - b(\zeta^k + \zeta^{-k}) = 0$ . Given  $r \in (\mathbb{Z}/p\mathbb{Z}) \setminus \{0\}$ , we can apply the automorphism of  $\mathbb{Q}(\zeta)$  which maps  $\zeta$  to  $\zeta^r$ . Since p > 3, we can choose r so that  $jr, kr \neq (p-1)/2 \pmod{m}$ . We therefore deduce that  $a(\zeta^u + \zeta^{-u}) - b(\zeta^v + \zeta^{-v}) = 0$  for some distinct integers  $0 \le u, v < (p-1)/2$ . Without loss of generality, u < v. Then  $a(\zeta^{u+v} + \zeta^{v-u}) - b(\zeta^{2v} + 1) = 0$ . We have thus found a nonzero polynomial with integer coefficients of degree 2v < m-1 that vanishes at  $\zeta$ , contradicting the fact that the minimal polynomial of  $\zeta$  is  $P(t) = t^{p-1} + \dots + t + 1$ .

We can now prove Proposition 10, thus completing the proof of Theorem 3.

*Proof of Proposition 10.* Denote the distinct nonzero eigenvalues of the adjacency matrix  $A(H; \mathfrak{S})$  by  $\lambda_1, \ldots, \lambda_s \in \mathbb{R} \setminus \{0\}$ , and for each  $j \in \{1, \ldots, s\}$  let  $\Lambda_j \subseteq \mathbb{C}^H$  be the eigenspace of  $A(H; \mathfrak{S})$  that corresponds to the eigenvalue  $\lambda_j$ . Also, let  $\Lambda_0 \subseteq \mathbb{C}^H$  be the kernel of  $A(H; \mathfrak{S})$ . Define

$$M = \max\left\{\inf_{\psi \in \Lambda_1 \sim \{0\}} \frac{\|\psi\|_{\infty}}{\|\psi\|_2}, \dots, \inf_{\psi \in \Lambda_s \sim \{0\}} \frac{\|\psi\|_{\infty}}{\|\psi\|_2}\right\}$$

The desired conclusion of Proposition 10 is the same as requiring that  $M \le \sqrt{2(1+2|H|^3 \varepsilon)}$ . If  $M \le \sqrt{2}$ , then there is nothing to prove, so suppose from now on that  $M > \sqrt{2}$ .

Let  $\ell$  be as in Lemma 12 applied to the field  $\mathbb{K} = \mathbb{Q}(\lambda_1, ..., \lambda_s)$ . Fix a prime p > 3 that does not divide  $\ell$ and let  $G = H \times (\mathbb{Z}/p\mathbb{Z})$  be as in the statement of Proposition 10. For  $k \in \mathbb{Z}$  denote  $\mu_k = 2\cos(2\pi k/p)$ . As p is odd,  $\mu_k \neq 0$ . Write  $\chi_k(x) = \exp(2\pi i k x/p)$  for  $x \in \mathbb{Z}/p\mathbb{Z}$  and let  $E_k$  be the span of  $\chi_k$  and  $\chi_{-k}$  in  $\mathbb{C}^{\mathbb{Z}/p\mathbb{Z}}$ . Then, dim $(E_0) = 1$  and dim $(E_k) = 2$  for  $k \in \{1, ..., (p-1)/2\}$ , and  $E_k$  is the eigenspace of  $A(\mathbb{Z}/p\mathbb{Z}; \{-1, 1\})$ whose eigenvalue is  $\mu_k$ . As p is odd, the eigenspace decomposition of  $A(\mathbb{Z}/p\mathbb{Z}; \{-1, 1\})$  is

$$\mathbb{C}^{\mathbb{Z}/p\mathbb{Z}} = \bigoplus_{k=0}^{\frac{p-1}{2}} E_k.$$

The nonzero eigenvalues of  $A(G, \mathfrak{S} \times \{-1, 1\})$  are  $\{\lambda_j \mu_k : (j, k) \in \{1, ..., s\} \times \{0, ..., (p-1)/2\}\}$ ; we claim that these numbers are distinct, so that the eigenspace decomposition of  $A(G, \mathfrak{S} \times \{-1, 1\})$  is

$$\mathbb{C}^G \cong \mathbb{C}^H \otimes \mathbb{C}^{\mathbb{Z}/p\mathbb{Z}} = \left(\Lambda_0 \otimes \mathbb{C}^{\mathbb{Z}/p\mathbb{Z}}\right) \bigoplus \left(\bigoplus_{j=1}^s \bigoplus_{k=0}^{\frac{p-1}{2}} \Lambda_j \otimes E_k\right).$$

Indeed, if  $j, j' \in \{1, ..., s\}$  and  $k, k' \in \{1, ..., (p-1)/2\}$  are such that  $\lambda_j \mu_k = \lambda_{j'} \mu_{k'}$ , then  $\mu_k / \mu_{k'} = \lambda_{j'} / \lambda_j \in \mathbb{K}$ , so k = k' by Lemma 12 and therefore also j = j'.

Fix  $j \in \{1, ..., s\}$  at which M is attained, namely  $\|\psi\|_{\infty} \ge M \|\psi\|_2$  for every  $\psi \in \Lambda_j$ . Let  $\phi : G \to \mathbb{C}$  be an eigenfunction of  $A(G, \mathfrak{S} \times \{-1, 1\})$  whose eigenvalue is  $\lambda_j \mu_k$  for some  $k \in \{0, ..., (p-1)/2\}$ . So,  $\phi \in \Lambda_j \otimes E_k$  and therefore there exist  $\psi_+, \psi_- \in \Lambda_j$  with  $\|\psi_+\|_2^2 + \|\psi_-\|_2^2 = \|\phi\|_2^2$  such that  $\phi = \psi_+ \otimes \chi_k + \psi_- \otimes \chi_{-k}$ . There is  $\psi \in \{\psi_+, \psi_-\}$  with  $\|\psi\|_2^2 \ge \|\phi\|_2^2/2$ . Fix  $h_\phi \in H$  for which  $|\psi(h_\phi)| = \|\psi\|_\infty$ . Then,

$$\mathbb{E}_{x \in \mathbb{Z}/p\mathbb{Z}} \left[ |\phi(h_{\phi}, x)|^{2} \right] = \mathbb{E}_{x \in \mathbb{Z}/p\mathbb{Z}} \left[ \left| \psi_{+}(h_{\phi})e^{\frac{2\pi i k x}{p}} + \psi_{-}(h_{\phi})e^{-\frac{2\pi i k x}{p}} \right|^{2} \right] \\ = |\psi_{+}(h_{\phi})|^{2} + |\psi_{-}(h_{\phi})|^{2} \ge |\psi(h_{\phi})|^{2} = \|\psi\|_{\infty}^{2} \ge M^{2} \|\psi\|_{2}^{2} \ge \frac{M^{2}}{2} \|\phi\|_{2}^{2}.$$

If  $\mathcal{B} \subseteq \mathbb{C}^G$  is an orthonormal eigenbasis of  $A(G, \mathfrak{S} \times \{-1, 1\})$ , then let  $\mathcal{B}' \subseteq \mathcal{B}$  be the elements of  $\mathcal{B}$  whose eigenvalue is  $\lambda_j \mu_k$  for some  $k \in \{0, \dots, (p-1)/2\}$ . Thus,  $|\mathcal{B}'| = \dim(\Lambda_j) p \ge p$ . By the pigeonhole principle there are  $\mathcal{B}'' \subseteq \mathcal{B}'$  and  $h \in H$  such that  $|\mathcal{B}''| \ge |\mathcal{B}'|/|H| \ge p/|H|$  and  $h_{\phi} = h$  for every  $\phi \in \mathcal{B}''$ . Consequently,

$$\mathbb{E}_{\phi\in\mathcal{B}}\left[\left|\mathbb{E}_{x\in G}\left[\mathbf{1}_{\{h\}\times\mathbb{Z}/p\mathbb{Z}}(x)|\phi(x)|^{2}\right]-\mathbb{E}\mathbf{1}_{\{h\}\times\mathbb{Z}/p\mathbb{Z}}\right|\right] = \frac{1}{p|H|^{2}}\sum_{\phi\in\mathcal{B}}\left|\mathbb{E}_{x\in\mathbb{Z}/p\mathbb{Z}}\left[|\phi(h_{\phi},x)|^{2}\right]-1\right|$$

$$\geq \frac{1}{p|H|^{2}}\sum_{\phi\in\mathcal{B}''}\left|\mathbb{E}_{x\in\mathbb{Z}/p\mathbb{Z}}\left[|\phi(h_{\phi},x)|^{2}\right]-1\right| \geq \frac{|\mathcal{B}''|}{p|H|^{2}}\left(\frac{M^{2}}{2}-1\right) \geq \frac{1}{|H|^{3}}\left(\frac{M^{2}}{2}-1\right).$$

$$(17)$$

If  $\mathcal{B}$  is  $\varepsilon$ -quantum ergodic, then the first term in (17) is at most  $\varepsilon$ , and therefore  $M \leq \sqrt{2(1+|H|^3 \varepsilon)}$ .

#### REFERENCES

- Shigeki Aida, Takao Masuda, and Ichirō Shigekawa, Logarithmic Sobolev inequalities and exponential integrability, J. Funct. Anal. 126 (1994), 83–101. <sup>†5</sup>
- [2] Nalini Anantharaman, Quantum ergodicity on regular graphs, Comm. Math. Phys. 353 (2017), 633–690. 1
- [3] Nalini Anantharaman, Delocalization of Schrödinger eigenfunctions, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 341–375. <sup>1</sup>
- [4] Nalini Anantharaman and Etienne Le Masson, Quantum ergodicity on large regular graphs, Duke Math. J. 164 (2015), 723–765.
- [5] Nalini Anantharaman and Mostafa Sabri, Quantum ergodicity for the Anderson model on regular graphs, J. Math. Phys. 58 (2017), 091901, 10. <sup>1</sup>1
- [6] Alexander Barvinok, Thrifty approximations of convex bodies by polytopes, Int. Math. Res. Not. IMRN (2014), 4341–4356. <sup>†</sup>4
- [7] Károly Böröczky, Jr. and Gergely Wintsche, *Covering the sphere by equal spherical balls*, Discrete and computational geometry, Algorithms Combin., vol. 25, Springer, Berlin, 2003, pp. 235–251. <sup>†</sup>4
- [8] Y. Colin de Verdière, Ergodicité et fonctions propres du laplacien, Comm. Math. Phys. 102 (1985), 497–502. †1
- [9] Michael J. Collins, Bounds for finite primitive complex linear groups, J. Algebra **319** (2008), 759–776. <sup>†</sup>2
- [10] E. B. Davies and B. Simon, Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians, J. Funct. Anal. 59 (1984), 335–395. <sup>†5</sup>
- [11] T. Figiel, J. Lindenstrauss, and V. D. Milman, The dimension of almost spherical sections of convex bodies, Acta Math. 139 (1977), 53–94. 14
- [12] Gerald B. Folland, How to integrate a polynomial over a sphere, Amer. Math. Monthly 108 (2001), 446–448. <sup>†</sup>6
- [13] W. T. Gowers, Quasirandom groups, Combin. Probab. Comput. 17 (2008), 363–387. <sup>†</sup>2
- [14] Leonard Gross, Logarithmic Sobolev inequalities, Amer. J. Math. 97 (1975), 1061–1083. <sup>†5</sup>
- [15] Edwin Hewitt and Kenneth A. Ross, Abstract harmonic analysis. Vol. II: Structure and analysis for compact groups. Analysis on locally compact Abelian groups, Die Grundlehren der mathematischen Wissenschaften, Band 152, Springer-Verlag, New York-Berlin, 1970. <sup>1</sup>1
- [16] Serge Lang, Cyclotomic fields I and II, second ed., Graduate Texts in Mathematics, vol. 121, Springer-Verlag, New York, 1990, With an appendix by Karl Rubin. <sup>†7</sup>
- [17] M. Ledoux, Remarks on logarithmic Sobolev constants, exponential integrability and bounds on the diameter, J. Math. Kyoto Univ. 35 (1995), 211–220. 15
- [18] Michel Ledoux, *The concentration of measure phenomenon*, Mathematical Surveys and Monographs, vol. 89, American Mathematical Society, Providence, RI, 2001. <sup>†5</sup>
- [19] Martin W. Liebeck and László Pyber, *Upper bounds for the number of conjugacy classes of a finite group*, J. Algebra **198** (1997), 538–562. <sup>†</sup>2
- [20] Elizabeth S. Meckes, The random matrix theory of the classical compact groups, Cambridge Tracts in Mathematics, vol. 218, Cambridge University Press, Cambridge, 2019. <sup>†5</sup>
- [21] Elizabeth S. Meckes and Mark W. Meckes, Spectral measures of powers of random matrices, Electron. Commun. Probab. 18 (2013), no. 78, 13. 15
- [22] Márton Naszódi, Fedor Nazarov, and Dmitry Ryabogin, *Fine approximation of convex bodies by polytopes*, Amer. J. Math. **142** (2020), 809–820. <sup>†4</sup>
- [23] Ashwin Sah, Mehtaab Sawhney, and Yufei Zhao, Cayley graphs without a bounded eigenbasis, Int. Math. Res. Not. IMRN (2022), 6157–6185. <sup>†</sup>2, <sup>†</sup>7
- [24] Barry Simon, *Representations of finite and compact groups*, Graduate Studies in Mathematics, vol. 10, American Mathematical Society, Providence, RI, 1996. <sup>1</sup>
- [25] Terence Tao, *Expansion in finite simple groups of Lie type*, Graduate Studies in Mathematics, vol. 164, American Mathematical Society, Providence, RI, 2015. <sup>†</sup>2
- [26] A. I. Šnirel'man, Ergodic properties of eigenfunctions, Uspehi Mat. Nauk 29 (1974), 181–182. 1
- [27] Steven Zelditch, Uniform distribution of eigenfunctions on compact hyperbolic surfaces, Duke Math. J. 55 (1987), 919–941.

DEPARTMENT OF MATHEMATICS, PRINCETON NJ 08544-1000 *Email address:* naor@math.princeton.edu

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139, USA *Email address*: {asah,msawhney,yufeiz}@mit.edu