OPTIMAL MINIMIZATION OF THE COVARIANCE LOSS

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ABSTRACT. Let X be a random vector valued in \mathbb{R}^m such that $||X||_2 \leq 1$ almost surely. For every $k \geq 3$, we show that there exists a sigma algebra \mathcal{F} generated by a partition of \mathbb{R}^m into k sets such that

$$\|\operatorname{Cov}(X) - \operatorname{Cov}(\mathbb{E}[X \mid \mathcal{F}])\|_{\mathrm{F}} \lesssim \frac{1}{\sqrt{\log k}}.$$

This is optimal up to the implicit constant and improves on a previous bound due to Boedihardjo, Strohmer, and Vershynin.

Our proof provides an efficient algorithm for constructing \mathcal{F} and leads to improved accuracy guarantees for k-anonymous or differentially private synthetic data. We also establish a connection between the above problem of minimizing the covariance loss and the pinning lemma from statistical physics, providing an alternate (and much simpler) algorithmic proof in the important case when $X \in \{\pm 1\}^m / \sqrt{m}$ almost surely.

1. INTRODUCTION

Let X be a random vector valued in \mathbb{R}^m . By slightly abusing notation, we identify X with its law, which is a probability measure on $(\mathbb{R}^m, \mathcal{G})$, where \mathcal{G} is a sigma-algebra on \mathbb{R}^m . Let \mathcal{F} be a sigma sub-algebra of \mathcal{G} and let $Y = \mathbb{E}[X | \mathcal{F}]$ denote the corresponding conditional expectation. In particular, $\mathbb{E}[X] = \mathbb{E}[Y]$. Let

$$\Sigma_X := \mathbb{E}[(X - \mathbb{E}X)(X - \mathbb{E}X)^T]$$

denote the covariance matrix of X and let Σ_Y denote the covariance matrix of Y. When m = 1, Σ_X is precisely the variance of X, which we denote by Var(X), and similarly for Σ_Y . The familiar law of total variance asserts that

$$\operatorname{Var}(X) - \operatorname{Var}(Y) = \mathbb{E}(X - Y)^2 \ge 0,$$

so that taking a conditional expectation results in a loss of variance. This phenomenon extends to higher dimensions as the law of total covariance:

$$\Sigma_X - \Sigma_Y = \mathbb{E}(X - Y)(X - Y)^T \succeq 0, \qquad (1.1)$$

where \succeq denotes the usual Loewner order on positive semi-definite matrices.

Recently, motivated by the design of privacy-preserving synthetic data (see the discussion in Section 1.1), Boedihardjo, Strohmer, and Vershynin [2] asked the following fundamental question: how much covariance is lost upon taking a conditional expectation? The answer to this clearly depends on the sigma sub-algebra \mathcal{F} (for instance, the choice $\mathcal{F} = \mathcal{G}$ loses no covariance, whereas the trivial sigma sub-algebra $\mathcal{F} = \{\emptyset, \mathbb{R}^m\}$ leads to the maximum possible covariance loss of Σ_X). This suggests restricting the 'complexity' of the sigma sub-algebra \mathcal{F} and investigating how much covariance is necessarily lost upon taking a conditional expectation with respect to a sigma subalgebra \mathcal{F} with a given complexity. Moreover, for applications, one would like to be able to find the best possible (at least asymptotically) sigma sub-algebra with a given complexity in an efficient manner.

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Since every finitely generated sigma-algebra \mathcal{F} may be viewed as the sigma-algebra generated by a partition of \mathbb{R}^m into k sets (for some finite k), a natural and useful measure of complexity of \mathcal{F} is the number of sets in the underlying partition, k. With this notion of complexity, and measuring covariance loss in the Frobenius norm, Boedihardjo, Strohmer, and Vershynin [2, Theorem 1.2] showed that there exists an absolute constant C > 0 such that for any random vector X valued in \mathbb{R}^m for which $||X||_2 \leq 1$ almost surely, and for every $k \geq 3$, there exists a partition of \mathbb{R}^m into at most k sets such that for the sigma-algebra \mathcal{F} generated by this partition, $Y = \mathbb{E}[X | \mathcal{F}]$ satisfies the dimension-independent bound

$$\|\Sigma_X - \Sigma_Y\|_{\mathbf{F}} \le C \sqrt{\frac{\log \log k}{\log k}},\tag{1.2}$$

where for $A \in \mathbb{R}^{m \times m}$, $||A||_{\mathrm{F}} = \sqrt{\sum_{i,j} A_{ij}^2}$ denotes its Frobenius norm. They noted [2, Proposition 3.14] that the upper bound is optimal up to the factor of $\sqrt{\log \log k}$.

Note that in the case when X is the uniform distribution over $x_1, \ldots, x_n \in \mathbb{R}^m$ with $\max_i ||x_i||_2 \leq 1$, and \mathcal{F} is generated by a partition into k sets, the dimension-independence of (1.2) stands in starkcontrast to (a variation of) the k-means objective

$$\inf_{y_1, \dots, y_k \in \mathbb{R}^m, I_1 \sqcup \dots \sqcup I_k = [n]} \sum_{i=1}^k \sum_{j \in I_i} \|x_j - y_i\|_2,$$

which bounds $\inf_{\mathcal{F}} \|\Sigma_X - \Sigma_Y\|_{\mathrm{F}}$ from above (via a direct application of Jensen's inequality) and, in general, can decay as slowly as $\Omega(k^{-1/m})$, which is significantly worse in the high-dimensional regime of interest here.

As our main result, we remove the gap between the upper bound in Theorem 1.1 and the lower bound in [2, Proposition 3.14], thereby obtaining an optimal and algorithmic answer to the problem of minimizing covariance loss raised by Boedihardjo, Strohmer, and Vershynin.

Theorem 1.1. Let X be a random vector valued in \mathbb{R}^m which satisfies $||X||_2 \leq 1$ almost surely. Then for every $k \geq 3$, there exists a partition of \mathbb{R}^m into at most k sets such that for the associated σ -algebra \mathcal{F} , the conditional expectation $Y = \mathbb{E}[X | \mathcal{F}]$ satisfies

$$\|\Sigma_X - \Sigma_Y\|_{\mathrm{F}} \le \frac{C}{\sqrt{\log k}},$$

where C is an absolute constant.

As noted earlier, our bound is optimal up to the value of the absolute constant C. We prove Theorem 1.1 in Section 3. Before doing so, in Section 2, we provide a completely different proof of Theorem 1.1 in the case when $X \in \frac{1}{\sqrt{m}} \cdot \{\pm 1\}^m$ based on the pinning lemma from statistical physics; this case is especially important for applications, since it corresponds to the case of Boolean 'true' data in the setting of Section 1.1. The proof in Section 2 is much simpler than the general proof in Section 3 and provides a significantly faster and simpler algorithm for finding \mathcal{F} .

Remark. By following exactly the same procedure as in [2, Section 3.6], if the probability space has no atoms, then the partition can be made with exactly k sets, all of which have the same probability 1/k.

Remark. By combining Theorem 1.1 with the tensorization principle [2, Theorem 3.10], we immediately obtain an analog of Theorem 1.1 for higher moments, which improves [2, Corollary 3.12] by a factor of $\sqrt{\log \log k}$: for all $d \ge 2$,

$$\|\mathbb{E}X^{\otimes d} - \mathbb{E}Y^{\otimes d}\|_{\mathrm{F}} \le 4^{d} \cdot \frac{C}{\sqrt{\log k}},\tag{1.3}$$

where C is the absolute constant appearing in Theorem 1.1. Here, $X^{\otimes d} \in \mathbb{R}^{m \times m \cdots \times m}$ is defined by $X^{\otimes d}(i_1, \ldots, i_d) := X(i_1) \cdots X(i_d)$, where $i_1, \ldots, i_d \in [m]$ (and similarly for $Y^{\otimes d}$), and for $A \in \mathbb{R}^{m \times \ldots m}$, $||A||_{\mathrm{F}} := \sqrt{\sum_{i_1, \ldots, i_d \in [m]} A(i_1, \ldots, i_d)^2}$.

1.1. Applications to the design of privacy-preserving synthetic data. As mentioned earlier, the problem of minimizing covariance loss was studied in [2] with a view towards designing privacy-preserving synthetic data. Here, one is given 'true' data points $x_1, \ldots, x_n \in \mathbb{R}^m$ and would like to construct a map $\mathcal{A} : \{x_1, \ldots, x_n\} \to \mathbb{R}^m$ such that the set of 'synthetic' data $\{\mathcal{A}(x_1), \ldots, \mathcal{A}(x_n)\}$ is both 'private' and 'accurate'. We refer the reader to [2] for a much more detailed discussion of these notions and further references, limiting ourselves here to the most basic application of Theorem 1.1.

A popular notion of preserving privacy is k-anonymity [7]; for synthetic data, this is the requirement that for any $y \in \{\mathcal{A}(x_1), \ldots, \mathcal{A}(x_n)\}$, the preimage $\mathcal{A}^{-1}(y)$ has cardinality at least k. In words, the true data is transformed into synthetic data in such a manner that the information of each person in the dataset cannot be distinguished from that of at least k - 1 other individuals in the dataset.

Let us quickly discuss how Theorem 1.1 may be used to obtain accurate $\lfloor n/k \rfloor$ -anonymous synthetic data. Given true data $x_1, \ldots, x_n \in \mathbb{R}^m$, we consider the random vector X which takes on each value x_i with probability 1/n each. Given $k \geq 3$, Theorem 1.1 gives a partition of \mathbb{R}^m into k sets, which induces a partition $[n] = I_1 \cup \cdots \cup I_k$ and a sigma algebra \mathcal{F} on $\{x_1, \ldots, x_n\}$. Moreover, by a slight variation of the remark following Theorem 1.1, we may assume that $|I_i| \geq \lfloor n/k \rfloor$ for all $i \in [k]$. For $j \in [n]$, let $I(j) \in \{I_1, \ldots, I_k\}$ denote the unique subset of [n] such that $j \in I(j)$. Then, the conditional expectation $Y = \mathbb{E}[X \mid \mathcal{F}]$ corresponds to the synthetic data map

$$x_j \mapsto y_{I(j)} := \frac{1}{|I(j)|} \sum_{i \in I(j)} x_i.$$

This map is $\lfloor n/k \rfloor$ -anonymous, by construction. As for accuracy, it follows from Theorem 1.1 that, with Y the random vector which takes on each value y_{ℓ} with probability 1/k,

$$\|\Sigma_X - \Sigma_Y\|_{\rm F} \lesssim \frac{1}{\sqrt{\log k}},$$

so that the synthetic data is accurate in the sense that it approximately preserves, on average, the second order marginals of the true data. This can be extended to higher-order marginals using (1.3).

The above idea is adapted in [2] to extract additional guarantees for anonymous, synthetic data (see [2, Theorems 4.4, 4.6]). In both cases, replacing (1.2) with our Theorem 1.1 leads to quantitative improvements by a factor of $\log \log k$.

Finally, we remark that in [2, Theorems 5.9-5.11], a generalization of (1.2) is used with additional arguments to design differentially-private synthetic data. Our proof of Theorem 1.1 in Section 3 can also be generalized using similar arguments as in [2] to yield versions of [2, Theorems 5.9-5.11] without the log log n factor there; we leave the details to the interested reader.

2. Proof of Theorem 1.1 for Boolean Data

In this section, we provide a proof of Theorem 1.1 in the case when X is valued in $\{\pm 1\}^m/\sqrt{m}$ almost surely. In the setting of Section 1.1, this corresponds to the case when the true data is Boolean and hence is particularly relevant for applications. Our proof relies on the so-called pinning lemma from statistical physics, discovered independently by Montanari [1] and by Raghavendra and Tan [6]. The statement below follows by combining [6, Lemma 4.5] with Pinsker's inequality (cf. the proofs of [3, Lemmas 4.2, A.2]).

Lemma 2.1. Let X_1, \ldots, X_m be a collection of $\{\pm 1\}$ -valued random variables. Then, for any $\ell \in [m]$, we have that

$$\mathbb{E}_{t \sim \{0,1,\dots,\ell\}} \mathbb{E}_{S \sim \binom{[m]}{t}} \left[\mathbb{E}_{X_S} \left(\sum_{i \neq j \in [m]} \operatorname{Cov}(X_i, X_j \mid X_S)^2 \right) \right] \le \frac{8m^2 \log 2}{\ell}.$$

Roughly speaking, the intuition behind the pinning lemma is the following: either the average (pairwise) covariance between the random variables X_1, \ldots, X_m is already small (in which case, we're done) or the average covariance is not small. In the latter case, we expect a random coordinate X_i to contain substantial information about many of the other coordinates X_1, \ldots, X_m , so that conditioning on a small random subset of the coordinates makes the average conditional covariance sufficiently small.

Given Lemma 2.1, we can quickly deduce Theorem 1.1 for Boolean data.

Proof of Theorem 1.1 for Boolean data. Recall that X is valued in $\{\pm 1\}^m / \sqrt{m}$ almost surely. Note that we may assume that $m \ge \log_2 k$; otherwise X takes on at most $2^m \le k$ values, so that the sigma algebra \mathcal{F} generated by the partition of $\{\pm 1\}^m / \sqrt{m}$ which assigns each point to its own part has at most k parts and satisfies $Y := \mathbb{E}[X \mid \mathcal{F}] = X$.

Now, let t be chosen uniformly from $\{0, 1, \ldots, \log_2 k\}$ and let S be chosen uniformly from $\binom{[m]}{t}$. This provides a decomposition of $\{\pm 1\}^m/\sqrt{m}$ into at most $2^t \leq k$ clusters, where each cluster consists of all points of $\{\pm 1\}^m/\sqrt{m}$ which agree on the coordinates in S. In other words, each cluster corresponds to a setting of $X_S := (X_i)_{i \in S} \in \{\pm 1\}^S/\sqrt{m}$. Let \mathcal{F} denote the sigma algebra generated by these clusters and let $Y = \mathbb{E}[X \mid \mathcal{F}] = \mathbb{E}[X \mid X_S]$. Let Σ_X and Σ_Y denote the covariance matrices of X and Y respectively. Then,

$$\mathbb{E}_{S} \| \Sigma_{X} - \Sigma_{Y} \|_{\mathrm{F}} = \mathbb{E}_{S} \| \mathbb{E}_{X} (X - \mathbb{E}[X \mid X_{S}])(X - \mathbb{E}[X \mid X_{S}])^{T} \|_{\mathrm{F}} \qquad (\text{from (1.1)})$$

$$\leq \mathbb{E}_{S} \mathbb{E}_{X_{S}} \| \mathbb{E}_{X|X_{S}} (X - \mathbb{E}[X \mid X_{S}])(X - \mathbb{E}[X \mid X_{S}])^{T} \|_{\mathrm{F}} \qquad (\text{norm convexity})$$

$$= \mathbb{E}_{S} \mathbb{E}_{X_{S}} \sqrt{\sum_{i \neq j \in [m]} \operatorname{Cov}(X_{i}, X_{j} \mid X_{S})^{2} + \sum_{i \in [m]} \operatorname{Var}(X_{i} \mid X_{S})^{2}} \qquad (\text{norm convexity})$$

$$\leq \sqrt{\mathbb{E}_{S} \mathbb{E}_{X_{S}} \left(\sum_{i \neq j \in [m]} \operatorname{Cov}(X_{i}, X_{j} \mid X_{S})^{2} + \sum_{i \in [m]} \operatorname{Var}(X_{i} \mid X_{S})^{2}\right)} \qquad (\text{Jensen})$$

$$= \sqrt{\frac{8m^{2} \log 2}{1}} \frac{1}{1} = \frac{3}{1}$$

$$\leq \sqrt{\frac{8m^2\log 2}{\log_2 k} \cdot \frac{1}{m^2} + m \cdot \frac{1}{m^2}} \leq \frac{3}{\sqrt{\log_2 k}},$$

where the first term in the penultimate inequality follows by applying Lemma 2.1 with $\ell = \log_2 k$ and rescaling by a factor of m^{-2} (since each X_i is valued in $\{\pm 1\}/\sqrt{m}$) and the second term in the penultimate inequality follows by noting that $\operatorname{Var}(X_i \mid X_S) \leq 1/m$ (again, since $X_i \in \{\pm 1\}/\sqrt{m}$).

Finally, by Markov's inequality,

$$\mathbb{P}_{S}\left[\|\Sigma_{X} - \Sigma_{Y}\|_{\mathrm{F}} \ge \frac{9}{\sqrt{\log_{2} k}}\right] \le \frac{1}{3},$$

so we have a very simple randomized algorithm for finding (with probability at least 2/3) a sigma algebra \mathcal{F} obtaining the desired guarantee: first choose t uniformly from $\{0, 1, \ldots, \log_2 k\}$, then choose S uniformly from $\binom{[m]}{t}$, and finally decompose $\{\pm 1\}^m/\sqrt{m}$ based on the values of the coordinates in S.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 for general random vectors $X \in \mathbb{R}^m$ satisfying $\|X\|_2 \leq 1$ almost surely. As in [2], we use principal component analysis to reduce to the case where m = $c \log k$, for a sufficiently small absolute constant c > 0. However, our treatment of the dimensionreduced problem is rather different from [2]. Indeed, whereas [2] partitions the dimension-reduced random vector according to the closest point in a volumetric epsilon-net (thereby, only exploiting the information that $||X||_2 \leq 1$ almost surely), our clustering scheme also takes into account the distributional profile of the dimension-reduced random vector; briefly, we place each 'heavy' point into its own cluster, place nearby points, which are 'collectively light' into a single cluster, and for the intermediate case, adopt a randomized rounding scheme to cluster the points. In particular, our proof provides another instance where nets based on randomized rounding provide better control than volumetric nets (see [4, 5, 8] for some other recent examples).

This section is organized as follows: in Section 3.1, we show how to appropriately cluster points in the most challenging 'intermediate' case, mentioned above (Proposition 3.2). Given this, the proof of Theorem 1.1 is completed in Section 3.2 by following the aforementioned decomposition into heavy, collectively light, and intermediate cases.

3.1. Key estimate. Let $p = c \log k$, where c is a sufficiently small positive universal constant (for instance, $c \in (0, 1/120)$ is certainly sufficient). Let

$$\gamma := \frac{e^{-(\log k)/(4p)}}{\sqrt{p}} = \frac{e^{-1/(4c)}}{\sqrt{c \log k}}$$

Let X be a random vector valued in $x_0 + [-\gamma/2, \gamma/2]^p$, supported on finitely many points, such that for any $x \in \operatorname{supp}(X) =: \mathcal{X}$, we have $\mathbb{P}[X = x] \leq k^{-1/3}$. Let $\mathcal{W} := x_0 + \{\pm 3\gamma/2\}^p$. For each $x \in \mathcal{X}$, let $w_x \in \mathcal{W}$ be a random vector defined as follows: $\mathbb{E}[(w_x)_i] = x_i$, and the random variables $(w_x)_i$ are independent. In words, the vector w_x is obtained by randomly rounding x to a point in \mathcal{W} so that w_x has mean x; it is easily seen that such a distribution w_x is unique. Moreover, for distinct $x \in \mathcal{X}$, the random vectors w_x are independent.

Now, given a realisation of the random vectors w_x , for each $w \in \mathcal{W}$, let

$$C_w := \{ x \in \mathcal{X} : w_x = w \},\$$

so that C_w consists of those points in \mathcal{X} which are rounded to w. Let \mathcal{F} denote the sigma-algebra corresponding to the partition $(C_w)_{w \in \mathcal{W}}$. Note that \mathcal{F} is random, depending on the realisation of w_x .

In our analysis, we will also require the following random vector, which should be viewed as an idealised version of $\mathbb{E}[X \mid \mathcal{F}]$; this random vector, which we denote by Z, takes on the value

$$z_w := \frac{\sum_{x \in \mathcal{X}} x \mathbb{P}[w_x = w] \mathbb{P}[X = x]}{\sum_{x \in \mathcal{X}} \mathbb{P}[w_x = w] \mathbb{P}[X = x]}$$

with probability

$$q_w := \sum_{x \in \mathcal{X}} \mathbb{P}[w_x = w] \mathbb{P}[X = x]$$

for each $w \in \mathcal{W}$. We begin with the following preliminary, but key, lemma.

Lemma 3.1. With notation as above,

$$\|\Sigma_X - \Sigma_Z\|_{\mathbf{F}} = \|\mathbb{E}[XX^T] - \mathbb{E}[ZZ^T]\|_{\mathbf{F}} \le \frac{36e^{-1/(2c)}}{\sqrt{c\log k}}$$

Proof. The first equality follows from the observation that $\mathbb{E}[Z] = \mathbb{E}[X]$. We proceed to prove the inequality. For convenience of notation, let

$$\mu_{x,w} := \mathbb{P}[X=x]\mathbb{P}[w_x=w]$$

We have

$$\begin{split} \|\mathbb{E}[XX^{T}] - \mathbb{E}[ZZ^{T}]\|_{\mathcal{F}} &= \|\sum_{x,w} \mu_{x,w}(xx^{T} - z_{w}z_{w}^{T})\|_{\mathcal{F}} = \|\sum_{x,w} \mu_{x,w}(xx^{T} - xz_{w}^{T} - z_{w}x^{T} + z_{w}z_{w}^{T})\|_{\mathcal{F}} \\ &= \|\sum_{x,w} \mu_{x,w}(x - z_{w})(x - z_{w})^{T}\|_{\mathcal{F}} \\ &\leq^{(1)} 2\|\sum_{x,w} \mu_{x,w}(x - w)(x - w)^{T}\|_{\mathcal{F}} + 2\|\sum_{x,w} \mu_{x,w}(w - z_{w})(w - z_{w})^{T}\|_{\mathcal{F}} \\ &\leq^{(2)} 2\|\sum_{x,w} \mu_{x,w}(x - w)(x - w)^{T}\|_{\mathcal{F}} + 2\|\sum_{x,w} \mu_{x,w}\sum_{x'\in\mathcal{X}} \frac{\mu_{x',w}}{q_{w}}(w - x')(w - x')^{T}\|_{\mathcal{F}} \\ &= 2\|\sum_{x,w} \mu_{x,w}(x - w)(x - w)^{T}\|_{\mathcal{F}} + 2\|\sum_{w}\sum_{x'\in\mathcal{X}} \mu_{x',w}(w - x')(w - x')^{T}\|_{\mathcal{F}} \\ &= 4\|\sum_{x,w} \mu_{x,w}(x - w)(x - w)^{T}\|_{\mathcal{F}} \\ &\leq 4\max_{x\in\mathcal{X}} \|\mathbb{E}_{wx}[(x - w_{x})(x - w_{x})^{T}]\|_{\mathcal{F}} \\ &\leq^{(3)} 4\max_{x\in\mathcal{X}} \|9\gamma^{2} \cdot \mathrm{Id}_{p\times p}\|_{\mathcal{F}} \\ &\leq 36\gamma^{2}\sqrt{p} = \frac{36e^{-1/(2c)}}{\sqrt{c\log k}}, \end{split}$$

as desired.

Inequality (1) follows since $(x + y)(x + y)^T \preceq 2xx^T + 2yy^T$ for any vectors x, y and $0 \preceq A \preceq B$ for symmetric matrices A, B implies that $||A||_F \leq ||B||_F$. (To see this inequality note that $||B||_F^2 - ||A||_F^2 = \operatorname{tr}(B^2 - A^2) = \operatorname{tr}((B - A)(B + A)) = \operatorname{tr}((B + A)^{1/2}(B - A)(B + A)^{1/2}) \geq 0$.) Inequality (2) follows since for any collection of vectors y_1, \ldots, y_m and for any $p_1 \geq 0, \ldots, p_m \geq 0$ such that $\sum_i p_i = 1$, we have $(\sum_i p_i y_i)(\sum_i p_i y_i)^T \preceq \sum_i p_i y_i y_i^T$, as is verified by noting that for any vector u,

$$u^{T}(\sum_{i} p_{i}y_{i})(\sum_{i} p_{i}y_{i})^{T}u = \left(\sum_{i} p_{i}(u^{T}y_{i})\right)^{2}$$

$$\leq \left(\sum_{i} p_{i}\right) \cdot \left(\sum_{i} p_{i}(u^{T}y_{i})^{2}\right) \qquad (\text{Cauchy-Schwarz})$$

$$= u^{T}\left(\sum_{i} p_{i}y_{i}y_{i}^{T}\right)u.$$

Finally, inequality (3) uses that $\mathbb{E}[(w_x)_i] = x_i$, the independence of $(w_x)_i$ and $(w_x)_j$, and the crude estimate $|(x - w_x)_i| \leq 3\gamma$.

The following is the main result of this subsection.

Proposition 3.2. There exists an absolute constant K > 0 such that for all $k \ge K$, and with $Y = \mathbb{E}[X \mid \mathcal{F}]$ (notation as above), we have

$$\|\Sigma_X - \Sigma_Y\|_{\mathbf{F}} = \|\mathbb{E}[XX^T] - \mathbb{E}[YY^T]\|_{\mathbf{F}} \le \frac{36e^{-1/(2c)}}{\sqrt{c\log k}} + k^{-1/48},$$

with probability (over the realisation of \mathcal{F}) at least $1 - \exp(-k^{1/7})$.

Proof. Without loss of generality we may assume that $x_0 = 0$. By Lemma 3.1 and the triangle inequality, it suffices to show that for all sufficiently large k, except with probability at most $\exp(-k^{1/7})$,

$$\|\mathbb{E}[ZZ^T] - \mathbb{E}[YY^T]\|_{\mathrm{F}} \le k^{-1/48}$$

For convenience of notation, for $w \in \mathcal{W}$ let

$$p_w := \sum_{x \in \mathcal{X}} \mathbb{P}[X = x] \mathbb{1}[w_x = w]$$

and for $w \in \mathcal{W}, i \in [p]$, let

$$(y_w)_i := \sum_{x \in \mathcal{X}} x_i \mathbb{P}[X = x] \mathbb{1}[w_x = w]$$

By Hoeffding's inequality, for a given $w \in \mathcal{W}$,

$$\mathbb{P}\left[|p_w - q_w| \ge k^{-1/12}\right] \le \exp(-2k^{-1/6} / \sum_x \mathbb{P}[X = x]^2)$$
$$\le^{(1)} \exp(-2k^{-1/6} / k^{-1/3})$$
$$= \exp(-2k^{1/6}),$$

where inequality (1) uses $\sum_{x} \mathbb{P}[X = x]^2 \leq \max_{x} \mathbb{P}[X = x] \leq k^{-1/3}$, by assumption. Similarly, for a given $w \in \mathcal{W}$ and $i \in [p]$, we have

$$\mathbb{P}\left[|(y_w)_i - q_w \cdot (z_w)_i| \ge \gamma \cdot k^{-1/12} \right] \le \exp(-2\gamma^2 k^{-1/6} / \sum_x x_i^2 \mathbb{P}[X=x]^2) \le \exp(-2k^{1/6}).$$

Let \mathcal{E} denote the event that $|\sum_{x \in \mathcal{X}} \mathbb{P}[X = x]\mathbb{1}[w_x = w] - q_w| \leq k^{-1/12}$ and $|(y_w)_i - q_w \cdot (z_w)_i| \leq \gamma \cdot k^{-1/12}$ for all $w \in \mathcal{W}, i \in [p]$. By the preceding discussion,

$$\mathbb{P}[\mathcal{E}^c] \le 2 \cdot 2^p \cdot p \cdot \exp(-2k^{1/6}) \le \exp(-k^{1/7})$$

for all sufficiently large k. Moreover, for every $i \in [p]$, $x \in \mathcal{X}$, and $\varepsilon \in \{\pm 1\}$, we have $\mathbb{P}[(w_x)_i = \varepsilon \cdot 3\gamma/2] \ge 1/3$, so that for every $w \in \mathcal{W}$,

$$q_w \ge 3^{-p} \ge k^{-3c/2}$$

and hence, on the event \mathcal{E} , we have for all $w \in \mathcal{W}$ that

$$p_w = q_w \pm k^{-1/12} = q_w (1 \pm k^{-1/24}),$$

assuming that c < 1/36. Finally, we see that on the event \mathcal{E} ,

$$\begin{split} \|\mathbb{E}[ZZ^{T}] - \mathbb{E}[YY^{T}]\|_{\mathrm{F}} &= \|\sum_{w \in \mathcal{W}} (q_{w} z_{w} z_{w}^{T} - y_{w} y_{w}^{T} / p_{w})\|_{\mathrm{F}} \\ &\leq \|\sum_{w} (q_{w} z_{w} - y_{w}) z_{w}^{T}\|_{\mathrm{F}} + \|\sum_{w} y_{w} (z_{w}^{T} - y_{w}^{T} / p_{w})\|_{\mathrm{F}} \\ &\leq 2^{p} \cdot (\max_{w} \|q_{w} z_{w} - y_{w}\|_{2} \|z_{w}\|_{2} + \max_{w} q_{w}^{-1} \|y_{w}\|_{2} \|q_{w} z_{w}^{T} - y_{w} \cdot q_{w} / p_{w}\|_{2}) \end{split}$$

$$\leq 2^{p} \left(\gamma^{2} p \cdot k^{-1/12} + k^{3c/2} \gamma \sqrt{p} \max_{w} (\|q_{w} z_{w}^{T} - y_{w}\|_{2} + \|y_{w}(1 - q_{w}/p_{w})\|_{2}) \right)$$

$$\leq \gamma^{2} p k^{c} \cdot k^{-1/12} + k^{5c/2} \gamma^{2} p \cdot k^{-1/12} + k^{5c/2} \gamma^{2} p \cdot \max_{w} |1 - q_{w}/p_{w}|$$

$$\leq 3 \cdot k^{5c/2} \gamma^{2} p \cdot k^{-1/24} \leq k^{-1/48},$$

provided that c < 1/120.

3.2. Finishing the proof. With Proposition 3.2, we are ready to prove Theorem 1.1 through a sequence of reductions. Recall that in the statement of Theorem 1.1, X is a random vector valued in \mathbb{R}^m which satisfies $||X||_2 \leq 1$ almost surely. Without loss of generality, we may assume that X is finitely supported, by rounding the points in the support to a sufficiently fine ε -net with respect to the Euclidean metric (see, e.g., [2, Lemma 3.6]).

Next, we show that it suffices to assume that X is valued in \mathbb{R}^p , for $p = c \log k$, where c is as in Section 3.1. The following lemma is a slight modification of [2, Lemmas 3.2,3.3].

Lemma 3.3. Suppose that X is a random vector with $||X||_2 \leq 1$ almost surely. Let $S = \mathbb{E}[XX^T]$ and let P the projection onto the subspace corresponding to the largest $t \geq 1$ eigenvectors of S. Let $Y = \mathbb{E}[X \mid PX]$. Then,

$$\|\Sigma_X - \Sigma_Y\|_{\mathbf{F}} = \|\mathbb{E}[XX^T] - \mathbb{E}[YY^T]\|_{\mathbf{F}} \le \frac{1}{\sqrt{t}}.$$

Proof. The equality holds since $\mathbb{E}[X] = \mathbb{E}[Y]$. For the inequality, we note that, with $A := \mathbb{E}[XX^T] - \mathbb{E}[YY^T]$,

$$\begin{split} \|A\|_{\rm F} &\leq^{(1)} \|PAP\|_{\rm F} + \|(I-P)\mathbb{E}[XX^T](I-P)\|_{\rm F} \\ &\leq^{(2)} \|PAP\|_{\rm F} + \frac{1}{\sqrt{t}} \\ &= \|\mathbb{E}(PX-PY)(PX-PY)^T\|_{\rm F} + \frac{1}{\sqrt{t}} \\ &=^{(3)} \frac{1}{\sqrt{t}}, \end{split}$$

where (1) follows from the proof of [2, Lemma 3.2], (2) follows from [2, Lemma 3.3], and (3) follows since $PY = P\mathbb{E}[X \mid PX] = PX$.

By taking $t = c \log k$ in Lemma 3.3 and using the triangle inequality, we see that it suffices to prove Theorem 1.1 for $X \in \mathbb{R}^p$, with $p = c \log k$ (the clustering in the original problem corresponds to applying the map P^{-1} to the clustering in the dimension-reduced problem). Therefore, consider such an X, and recall that we may assume that X is finitely supported, denoting the support by \mathcal{X} . Let

$$\mathcal{X}^{(1)} = \{ x \in \mathcal{X} : \mathbb{P}[X = x] \ge 3/k \}.$$

Note that $|\mathcal{X}^{(1)}| \leq k/3$. By assigning each point in $\mathcal{X}^{(1)}$ to its own cluster, it suffices to find a clustering of the points in $\mathcal{X} \setminus \mathcal{X}^{(1)}$ into fewer than 2k/3 clusters.

For this, we begin by writing $\mathcal{B} := \{x \in \mathbb{R}^p : ||x||_2 \leq 1\}$ as a disjoint union of cubes, denoted by \mathfrak{C} , each with side length $\gamma = e^{-\log k/(4p)}/\sqrt{p}$. By a standard volumetric estimate (see, e.g., [2, Proposition 3.7]), the number of cubes in \mathfrak{C} is at most $k^{1/3}$ (if c < 1/120, say, and k is sufficiently large). Therefore, it suffices to cluster the points in each cube into at most $(2/3)k^{2/3}$ clusters. We have two cases:

• Case I: $\mathcal{C} \in \mathfrak{C}$ satisfies $\mathbb{P}[X \in \mathcal{C} \setminus \mathcal{X}^{(1)}] \leq k^{-1/2}$. Let us denote all such cubes by \mathfrak{C}_1 . In this case, we assign all the points in $\mathcal{C} \setminus \mathcal{X}^{(1)}$ to a single cluster (say, corresponding to the midpoint of \mathcal{C}).

• Case II: $\mathcal{C} \in \mathfrak{C}$ satisfies $\mathbb{P}[X \in \mathcal{C} \setminus \mathcal{X}^{(1)}] \geq k^{-1/2}$. Let us denote all such cubes by \mathfrak{C}_2 . In this case, consider the random vector $X_{\mathcal{C}}$, which takes on each value $x \in \mathcal{C} \setminus \mathcal{X}^{(1)}$ with probability $\mathbb{P}[X = x]/\mathbb{P}[X \in \mathcal{C} \setminus \mathcal{X}^{(1)}]$. Note that $X_{\mathcal{C}}$ is supported on a *p*-dimensional cube of side length γ , and for any $x \in X_{\mathcal{C}}$, we have that $\mathbb{P}[X_{\mathcal{C}} = x] \leq (3/k)/k^{-1/2} \leq 3k^{-1/2} \leq k^{-1/3}$. We partition the points in $\mathcal{C} \setminus \mathcal{X}^{(1)}$ according to the clusters coming from Proposition 3.2 applied to $X_{\mathcal{C}}$, noting that there are at most $2^p < k^{1/2}$ clusters for each cube $\mathcal{C} \in \mathfrak{C}_2$ (provided that c < 1/2). Denote the corresponding sigma algebra by $\mathcal{F}_{\mathcal{C}}$.

At this point, we have partitioned the points in \mathcal{X} into at most $k/3 + k^{1/3} \cdot k^{1/2} < k/2$ clusters. To complete the proof, we check that the sigma algebra \mathcal{F} generated by this clustering satisfies the conclusion of Theorem 1.1. Letting $Y = \mathbb{E}[X \mid \mathcal{F}]$, we have

$$\begin{split} \|\Sigma_X - \Sigma_Y\|_{\mathrm{F}} &= \|\mathbb{E}[XX^T] - \mathbb{E}[YY^T]\|_{\mathrm{F}} \\ &\leq \sum_{\mathcal{C} \in \mathfrak{C}_1} \mathbb{P}[X \in \mathcal{C} \setminus \mathcal{X}^{(1)}] \cdot \gamma^2 p + \sum_{\mathcal{C} \in \mathcal{C}_2} \mathbb{P}[X \in \mathcal{C} \setminus \mathcal{X}^{(1)}] \cdot \|\Sigma_{X_{\mathcal{C}}} - \Sigma_{\mathbb{E}[X_{\mathcal{C}}|\mathcal{F}_{\mathcal{C}}]}\|_{\mathrm{F}} \\ &\leq k^{1/3} \cdot k^{-1/2} + \sum_{C \in \mathcal{C}_2} \mathbb{P}[X \in \mathcal{C} \setminus \mathcal{X}^{(1)}] \cdot \left(\frac{36e^{-1/(2c)}}{\sqrt{c \log k}} + k^{-1/48}\right) \qquad (\text{Proposition 3.2}) \\ &\leq \frac{40}{\sqrt{c \log k}}, \end{split}$$

provided that c < 1/120 and k is sufficiently large.

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