NON-CLASSICAL POLYNOMIALS AND THE INVERSE THEOREM

AARON BERGER, ASHWIN SAH, MEHTAAB SAWHNEY, AND JONATHAN TIDOR

ABSTRACT. In this note we characterize when non-classical polynomials are necessary in the inverse theorem for the Gowers U^k -norm. We give a brief deduction of the fact that a bounded function on \mathbb{F}_p^n with large U^k -norm must correlate with a classical polynomial when $k \leq p+1$. To the best of our knowledge, this result is new for k=p+1 (when p>2). We then prove that non-classical polynomials are necessary in the inverse theorem for the Gowers U^k -norm over \mathbb{F}_p^n for all $k \geq p+2$, completely characterizing when classical polynomials suffice.

1. Introduction

The inverse theorem for the Gowers U^k -norm states that a bounded function $f\colon G\to \mathbb{C}$ has large U^k -norm if and only if f correlates with a certain structured object. When $G=\mathbb{Z}/N\mathbb{Z}$, these structured objects are quite complicated and need the theory of nilsequences to describe. When $G=\mathbb{F}_p^n$, the situation is somewhat simpler. When p>k, a bounded function $f\colon \mathbb{F}_p^n\to \mathbb{C}$ has large U^k -norm if and only if f has non-negligible correlation with a polynomial phase function, i.e., $e^{2\pi i P(x)/p}$ where $P\colon \mathbb{F}_p^n\to \mathbb{F}_p$ is a polynomial of degree at most k-1.

The situation when p is small compared to k is more delicate. Green and Tao [3] and independently Lovett, Meshulam, and Samorodnitsky [4] showed that the corresponding conjecture is false for k=4 and p=2. In other words, there exist bounded functions $f: \mathbb{F}_2^n \to \mathbb{C}$ with large U^4 -norm but with correlation $o_{n\to\infty}(1)$ with every cubic phase function. Tao and Ziegler [7] clarified this situation by proving that for all k and p, a bounded function $f: \mathbb{F}_p^n \to \mathbb{C}$ has large U^k -norm if and only if f has non-negligible correlation with a non-classical polynomial phase function, i.e., $e^{2\pi i P(x)}$ where $P: \mathbb{F}_p^n \to \mathbb{R}/\mathbb{Z}$ is a non-classical polynomial of degree at most k-1. (See Section 2 for the relevant definitions.)

A natural question which remains from the above discussion is to determine for which pairs p, k does the U^k -inverse theorem over \mathbb{F}_p^n hold with classical polynomials. In the positive direction, it is known due to Samorodnitsky [5] that the U^3 -inverse theorem over \mathbb{F}_2^n holds with classical polynomials. In the negative direction, Lovett, Meshulam, and Samorodnitsky [4] proved that the U^{p^ℓ} -inverse theorem over \mathbb{F}_p^n requires non-classical polynomials for all p and $\ell \geq 2$. (A curious feature of this problem is that it is not monotone in k, e.g., the Lovett-Meshulam-Samorodnitsky result does not imply that non-classical polynomials are necessary in the U^k -inverse theorem for all $k \geq p^2$.)

In this paper we completely characterize when classical polynomials suffice in the statement of the inverse theorem. We first prove the inverse theorem for the Gowers U^{p+1} -norm with classical polynomials. This result is proved via a short deduction from the usual inverse theorem for the U^{p+1} -norm that involves non-classical polynomials.

Theorem 1.1. Fix a prime p and $\delta > 0$. There exists $\epsilon > 0$ such that the following holds. Let V be a finite-dimensional \mathbb{F}_p -vector space. Given a function $f: V \to \mathbb{C}$ satisfying $||f||_{\infty} \leq 1$ and

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¹See Section 2 for the definitions and notation used in the statement of these results.

 $||f||_{U^{p+1}} > \delta$, there exists a classical polynomial $P \in \operatorname{Poly}_{\leq p}(V \to \mathbb{F}_p)$ such that

$$|\mathbb{E}_{x \in V} f(x) e_p(-P(x))| \ge \epsilon.$$

Second, we give an example showing that non-classical polynomials are necessary in the U^k -inverse theorem for all $k \geq p + 2$.

Theorem 1.2. Fix a prime p and an integer $k \geq p+2$. For all n, there exists a function $f: \mathbb{F}_p^n \to \mathbb{C}$ satisfying $||f||_{\infty} \leq 1$ and $||f||_{U^k} = 1$ such that for all (classical) polynomials $P \in \operatorname{Poly}_{\leq k-1}(\mathbb{F}_p^n \to \mathbb{F}_p)$,

$$|\mathbb{E}_{x \in \mathbb{F}_p^n} f(x) e_p(-P(x))| = o_{p,k;n \to \infty}(1).$$

Our example is fairly simple to write down. For $k \ge p+2$, we write $k-1=r+(p-1)\ell$ where $\ell \ge 1$ and 0 < r < p. Then our function is

$$f(x) = e^{2\pi i \frac{\sum_{i=1}^{n} |x_i|^r}{p^{\ell+1}}}$$

(where $|\cdot|: \mathbb{F}_p \to \{0, \dots, p-1\}$ is the standard map). Note that this function f is a non-classical polynomial phase function of degree k-1, so the content of this result is that it does not correlate with any classical polynomial phase functions of the same degree.

The o(1) correlation in Theorem 1.2 is fairly bad – the inverse of many iterated logarithms. This is due to our use of a Ramsey-theoretic argument inspired by a similar argument of Alon and Beigel. We conjecture that this bound on the correlation can be improved.

Conjecture 1.3. Fix a prime p and an integer $k \geq p+2$. For all n there exist $f: \mathbb{F}_p^n \to \mathbb{C}$ satisfying $||f||_{\infty} \leq 1$ and $||f||_{U^k} \geq c_{p,k} > 0$ such that for all (classical) polynomials $P \in \text{Poly}_{\leq k-1}(\mathbb{F}_p^n \to \mathbb{F}_p)$,

$$|\mathbb{E}_{x \in \mathbb{F}_p^n} f(x) e_p(-P(x))| \le \exp(-\Omega_{p,k}(n)).$$

In fact, we believe that this conjecture is true with the same functions that we use to prove Theorem 1.2.

Structure of the paper: In Section 2 we give the definition of the Gowers U^k -norm and of non-classical polynomials. In Section 3 we prove Theorem 1.1. We prove Theorem 1.2 in the remainder of the paper. Section 4 develops the symmetrization tool that we use and Section 5 gives the full proof.

Notation: We use $|\cdot|$ for the standard map $\mathbb{F}_p \to \{0, \dots, p-1\}$. We often treat \mathbb{F}_p as an additive subgroup of \mathbb{R}/\mathbb{Z} via the map $x \mapsto |x|/p$ and, by some abuse of notation, freely switch between these two viewpoints. We use $e \colon \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ for the function $e(x) = e^{2\pi i x}$ and $e_p \colon \mathbb{F}_p \to \mathbb{C}$ for the function $e_p(x) = e^{2\pi i |x|/p}$.

2. Background on non-classical polynomials

Definition 2.1. Fix a prime p, a finite-dimensional \mathbb{F}_p -vector space V, and an abelian group G. Given a function $f: V \to G$ and a shift $h \in V$, define the **additive derivative** $\Delta_h f: V \to G$ by

$$(\Delta_h f)(x) = f(x+h) - f(x).$$

Given a function $f: V \to \mathbb{C}$ and a shift $h \in V$, define the **multiplicative derivative** $\partial_h f: V \to \mathbb{C}$ by

$$(\partial_h f)(x) = f(x+h)\overline{f(x)}.$$

Definition 2.2. Fix a prime p and a finite-dimensional \mathbb{F}_p -vector space V. Given a function $f: V \to \mathbb{C}$ and $d \geq 1$, the **Gowers uniformity norm** $||f||_{U^d}$ is defined by

$$||f||_{U^d} = |\mathbb{E}_{x,h_1,\dots,h_d \in V}(\partial_{h_1} \cdots \partial_{h_d} f)(x)|^{1/2^d}.$$

See [7, Lemma B.1] for some basic facts about the Gowers uniformity norms.

Definition 2.3. Fix a prime p, and a non-negative integer $d \geq 0$. Let V be a finite-dimensional \mathbb{F}_p -vector space. A **non-classical polynomial** of degree at most d is a map $P \colon V \to \mathbb{R}/\mathbb{Z}$ that satisfies

$$(\Delta_{h_1} \cdots \Delta_{h_{d+1}} P)(x) = 0$$

for all $h_1, \ldots, h_{d+1}, x \in V$. We write $\operatorname{Poly}_{\leq d}(V \to \mathbb{R}/\mathbb{Z})$ for the set of non-classical polynomials of degree at most d.

A classical polynomial is a map $V \to \mathbb{F}_p$ satisfying the same property. By composing with the standard map $x \mapsto |x|/p$ we can view $\operatorname{Poly}_{\leq d}(V \to \mathbb{F}_p)$ as a subset of $\operatorname{Poly}_{\leq d}(V \to \mathbb{R}/\mathbb{Z})$.

See [7, Lemma 1.7] for some properties of non-classical polynomials. We give one property below which will be used several times in this paper.

Lemma 2.4 ([7, Lemma 1.7(iii)]). Fix a prime p and a finite-dimensional \mathbb{F}_p -vector space $V = \mathbb{F}_p^n$. Then $P: V \to \mathbb{R}/\mathbb{Z}$ is a non-classical polynomial of degree at most d if and only if it can be expressed in the form

$$P(x_1, \dots, x_n) = \alpha + \sum_{\substack{0 \le i_1, \dots, i_n < p, j \ge 0:\\0 < i_1 + \dots + i_n \le d - j(p - 1)}} \frac{c_{i_1, \dots, i_n, j} |x_1|^{i_1} \cdots |x_n|^{i_n}}{p^{j+1}} \pmod{1},$$

for some $\alpha \in \mathbb{R}/\mathbb{Z}$ and coefficients $c_{i_1,...,i_n,j} \in \{0,...,p-1\}$. Furthermore, this representation is unique.

Define $\mathbb{U}_k \subset \mathbb{R}/\mathbb{Z}$ to be $\{0, 1/p^k, \dots, (p^k - 1)/p^k\}$. As a corollary we see that in characteristic p, every non-classical polynomial of degree at most d takes values in a coset of $\mathbb{U}_{\lfloor (d-1)/(p-1)\rfloor+1}$. Finally we state the inverse theorem of Tao and Ziegler.

Theorem 2.5 ([7, Theorem 1.10]). Fix a prime p, a positive integer k, and a parameter $\delta > 0$. There exists $\epsilon > 0$ such that the following holds. Let V be a finite-dimensional \mathbb{F}_p -vector space. Given a function $f: V \to \mathbb{C}$ satisfying $||f||_{\infty} \leq 1$ and $||f||_{U^k} > \delta$, there exists a non-classical polynomial $P \in \operatorname{Poly}_{\leq k-1}(V \to \mathbb{R}/\mathbb{Z})$ such that

$$|\mathbb{E}_{x \in V} f(x) e^{-2\pi i P(x)}| \ge \epsilon.$$

Earlier works of Bergelson, Tao, and Ziegler [2] and Tao and Ziegler [6] show this result in the high-characteristic regime $p \geq k$, with the additional guarantee that P is a classical polynomial of degree at most k-1.

3. Classical polynomials for the U^{p+1} -inverse theorem

The inverse theorem for the U^k -norm does not require non-classical polynomials when $p \geq k$ for the simple reason that every non-classical polynomial of degree at most p-1 is a classical polynomial of the same degree (up to a constant shift). To prove Theorem 1.1, about the U^{p+1} -inverse theorem, we use the following fact. Every non-classical polynomial of degree p agrees with a classical polynomial on a codimension 1 hyperplane (up to a constant shift).

Proposition 3.1. Let $P \in \operatorname{Poly}_{\leq p}(V \to \mathbb{R}/\mathbb{Z})$ be a non-classical polynomial of degree at most p. Then there exists a codimension 1 hyperplane $U \leq V$, a classical polynomial $Q \in \operatorname{Poly}_{\leq p}(V \to \mathbb{F}_p)$, and $\alpha \in \mathbb{R}/\mathbb{Z}$ such that $P(x) = \alpha + |Q(x)|/p$ for all $x \in U$.

Proof. Pick an isomorphism $V \simeq \mathbb{F}_p^n$. By Lemma 2.4, we have

$$P(x_1, \dots, x_n) = \alpha + P'(x_1, \dots, x_n) + \frac{c_1|x_1| + \dots + c_n|x_n|}{p^2} \pmod{1}$$

for $\alpha \in \mathbb{R}/\mathbb{Z}$, a polynomial P' taking values in $\mathbb{U}_1 = \{0, 1/p, \dots, (p-1)/p\}$, and $c_1, \dots, c_n \in \{0, \dots, p-1\}$. Define the codimension 1 hyperplane $U \leq V$ by $c_1x_1 + \dots + c_nx_n = 0$. Note that for $(x_1, \dots, x_n) \in U$, we have $c_1|x_1| + \dots + c_n|x_n| \equiv 0 \pmod{p}$. Thus $P|_U$ takes values in $\alpha + \mathbb{U}_1$. Thus by our identification of \mathbb{U}_1 with \mathbb{F}_p , $P|_U - \alpha$ is a classical polynomial of degree at most p. \square

Proof of Theorem 1.1. By the usual inverse theorem, Theorem 2.5, there exists $P \in \operatorname{Poly}_{\leq p}(V \to \mathbb{R}/\mathbb{Z})$ such that $|\mathbb{E}_{x \in V} f(x) e(-P(x))| \geq \epsilon$. By Proposition 3.1, there exists a codimension 1 hyperplane U and $\alpha \in \mathbb{R}/\mathbb{Z}$ such that $P|_U$ takes values in $\alpha + \mathbb{U}_1$, i.e., $P|_U - \alpha$ is classical. Pick an isomorphism $V \simeq \mathbb{F}_p^n$ such that U is the hyperplane defined by $x_1 = 0$. In this basis, there exists a classical polynomial $Q \in \operatorname{Poly}_{\leq p}(\mathbb{F}_p^n \to \mathbb{F}_p)$ and $c \in \{0, \ldots, p-1\}$ so that

$$P(x_1, \dots, x_n) = \alpha + \frac{|Q(x_1, \dots, x_n)|}{p} + \frac{c|x_1|}{p^2} \pmod{1}.$$

Thus we have

$$\epsilon \leq |\mathbb{E}_{x \in \mathbb{F}_p^n} f(x) e(-P(x))|$$

$$\leq \mathbb{E}_{x_1 \in \mathbb{F}_p} \left| \mathbb{E}_{y \in \mathbb{F}_p^{n-1}} f(x_1, y) e(-P(x_1, y)) \right|$$

$$= \mathbb{E}_{x_1 \in \mathbb{F}_p} \left| \mathbb{E}_{y \in \mathbb{F}_p^{n-1}} f(x_1, y) e_p(-Q(x_1, y)) \right|$$

$$\leq \left(\mathbb{E}_{x_1 \in \mathbb{F}_p} \left| \mathbb{E}_{y \in \mathbb{F}_p^{n-1}} f(x_1, y) e_p(-Q(x_1, y)) \right|^2 \right)^{1/2}.$$

By Parseval and the pigeonhole principle, there exists $a \in \mathbb{F}_p$ such that

$$\left| \mathbb{E}_{x_1 \in \mathbb{F}_p} e(-ax_1) \mathbb{E}_{y \in \mathbb{F}_p^{n-1}} f(x_1, y) e_p(-Q(x_1, y)) \right| \ge \epsilon / \sqrt{p}.$$

Therefore f has correlation at least ϵ/\sqrt{p} with the classical polynomial $Q(x_1,\ldots,x_n)+ax_1$.

4. Symmetrization techniques

We now extend a symmetrization technique of Alon and Beigel [1] which will be needed to prove the non-correlation property of our example. At a high level, this technique use Ramsey theory to show that if a function correlates with a bounded degree polynomial, then some restriction of coordinates correlates with a symmetric polynomial. Unfortunately, as stated this only holds for multilinear polynomials, and otherwise one can only reduce to the class of so-called quasisymmetric polynomials. These are a generalization of the notion of symmetric polynomials which have found extensive use in enumerative and algebraic combinatorics.

Definition 4.1. For a prime p and a tuple $(\alpha_1, \ldots, \alpha_s)$ of positive integers satisfying $\alpha_i < p$ for all i, the **elementary quasisymmetric polynomial associated to** $(\alpha_1, \ldots, \alpha_s)$ in n variables is the polynomial $Q_{\alpha} : \mathbb{F}_p^n \to \mathbb{F}_p$ defined by

$$Q_{\alpha}(x_1, \dots, x_n) = \sum_{1 \le i_1 < \dots < i_s \le n} \prod_{j=1}^s x_{i_j}^{\alpha_j}.$$

We additionally note the total degree is $|\alpha| := \alpha_1 + \cdots + \alpha_s$.

Theorem 4.2. Fix a prime p and an integer $d \ge 1$. For any n, there exists $m = \omega_{p,d;n\to\infty}(1)$ such that the following holds. Let $P(x_1,\ldots,x_n)$ be a polynomial of degree at most d with coefficients in \mathbb{F}_p . There exists $I \subseteq [n]$ of size |I| = m such that for any $y_{[n]\setminus I} \in \mathbb{F}_p^{[n]\setminus I}$, the function $P(x_I,y_{[n]\setminus I})$

(viewed as a polynomial in the x_I) can be written as a quasisymmetric polynomial of degree d plus an arbitrary polynomial of degree at most d-1.

Proof. We can uniquely express P in the form

$$P(x_1, \dots, x_n) = \sum_{s=0}^{d} \sum_{1 \le i_1 < \dots < i_s \le n} \sum_{\substack{\alpha \in \{1, \dots, p-1\}^s \\ |\alpha| \le d}} c_{i_1, \dots, i_s, \alpha} \prod_{j=1}^{s} x_{i_j}^{\alpha_j}$$

where the $c_{i_1,...,i_s,\alpha} \in \mathbb{F}_p$ are arbitrary.

Define $\Lambda = \{\lambda \in \{0, \dots, p-1\}^{d'} : |\lambda| = d\}$. We define a coloring of the complete d-uniform hypergraph on n vertices where the set of colors is \mathbb{F}_p^{Λ} . For an edge $\{i_1, \dots, i_d\}$ with $1 \leq i_1 < \dots < i_d \leq n$, let the color of this edge be given by $c_{i_1,\dots,i_d} \colon \Lambda \to \mathbb{F}_p$. We define $c_{i_1,\dots,i_d}(\lambda) = c_{j_1,\dots,j_s,\alpha}$ where α is formed by removing the 0's from the tuple λ and (j_1, \dots, j_s) is formed from (i_1, \dots, i_d) by removing the coordinate i_k if $\lambda_k = 0$.

Applying the hypergraph Ramsey theorem, there exists a subset $I \subseteq [n]$ such that the induced subhypergraph on vertex set I is colored monochromatically with color $c \colon \Lambda \to \mathbb{F}_p$ and $|I| = \omega_{p,d;n\to\infty}(1)$. Unwinding the definitions, we see that

$$P(x_I, y_{[n]\setminus I}) = \sum_{s=0}^{d} \sum_{\substack{\alpha \in \{1, \dots, p-1\}^s \\ |\alpha| = d}} c(\alpha, \underbrace{0, \dots, 0}_{d-s \ 0's}) Q_{\alpha}(x_I) + \text{mixed terms.}$$

Now the mixed terms involve at least one factor of $y_{[n]\setminus I}$, so their total x_I -degree is strictly smaller than d.

5. Non-classical polynomials are necessary

In this section we prove Theorem 1.2, namely that non-classical polynomials are necessary in the U^{k+1} inverse theorem when k > p. To do this, we use the function $f_n^{(k)} : \mathbb{F}_p^n \to \mathbb{R}/\mathbb{Z}$ defined by

$$f_n^{(k)}(x) = \frac{1}{p^{\ell+1}} \sum_{i=1}^n |x_i|^r \tag{5.1}$$

where $k = r + (p-1)\ell$ with $\ell \ge 1$ and 0 < r < p. Note that $f_n^{(k)}$ is a non-classical polynomial of degree k, so $||e(f_n^{(k)})||_{U^{k+1}} = 1$.

In order to motivate our proof, suppose for the sake of contradiction that $f_n^{(k)}$ has correlation at least ϵ with some classical polynomial of degree at most k. By Theorem 4.2, we will be able to reduce to the situation

$$\epsilon \le \left| \mathbb{E}_{x \sim \mathbb{F}_p^n} e(f_n^{(k)}(x) + g(x) + h(x)) \right|$$

where g is a homogeneous quasisymmetric polynomial of degree k and h is a classical polynomial of degree at most k-1. By the monotonicity of the Gowers norms (alternatively by the Gowers-Cauchy-Schwarz inequality), we deduce

$$\epsilon^{2^{k}} \leq \left| \mathbb{E}_{x} e(f_{n}^{(k)}(x) + g(x) + h(x)) \right|^{2^{k}}$$

$$= \mathbb{E}_{x,h_{1},\dots,h_{k}}(\partial_{h_{1}} \cdots \partial_{h_{k}} e(f_{n}^{(k)} + g + h))(x)$$

$$= \mathbb{E}_{h_{1},\dots,h_{k}} e(\Delta_{h_{1}} \cdots \Delta_{h_{k}}(f_{n}^{(k)} + g)).$$

Since $f_n^{(k)}$ and g are polynomials of degree k, the iterated derivatives $(\Delta_{h_1} \cdots \Delta_{h_k} f_n^{(k)})(x)$ and $(\Delta_{h_1} \cdots \Delta_{h_k} g)(x)$ are constants independent of x. Furthermore, they take values in \mathbb{U}_1 which (with

some abuse of notation) we identify with \mathbb{F}_p . Many results on these objects are known, including the fact that in general they are multilinear functions of h_1, \ldots, h_k (see [7, Section 4]). For the purposes of this paper, it is sufficient to do the following explicit computation.

Lemma 5.1. For $k = r + (p-1)\ell$ with $\ell \ge 0$ and 0 < r < p,

$$\iota_k(h_1,\ldots,h_k) := \Delta_{h_1}\cdots\Delta_{h_k}f_n^{(k)} = (-1)^{\ell}r!\sum_{i=1}^n(h_1)_i\cdots(h_k)_i,$$

and for $\alpha = (\alpha_1, \dots, \alpha_s)$ with $\alpha_1 + \dots + \alpha_s = k$ and $0 \le \alpha_i < p$,

$$\tau_{\alpha}(h_1,\ldots,h_k) := \Delta_{h_1}\cdots\Delta_{h_k}Q_{\alpha} = \sum_{\pi\in\mathfrak{S}_k}\sum_{\vec{i}}(h_1)_{i_{\pi(1)}}\cdots(h_k)_{i_{\pi(k)}}$$

where the sum is over sequences $1 \le i_1 \le \cdots \le i_k \le n$ that satisfy $i_{\alpha_1 + \cdots + \alpha_j + 1} = \cdots = i_{\alpha_1 + \cdots + \alpha_j + 1}$ and $i_{\alpha_1 + \cdots + \alpha_j} < i_{\alpha_1 + \cdots + \alpha_j + 1}$ for all j.

Proof of Lemma 5.1. For classical polynomials $P,Q:\mathbb{F}_p^n\to\mathbb{F}_p$, of degrees d_1,d_2 , the discrete Leibniz rule, $\Delta_h(PQ)=(\Delta_hP)Q+P(\Delta_hQ)+(\Delta_hP)(\Delta_hQ)$, can be easily verified. This implies the more convenient $\Delta_h(PQ)\equiv(\Delta_hP)Q+P(\Delta_hQ)\pmod{\operatorname{Poly}_{\leqslant d_1+d_2-2}(\mathbb{F}_p^n\to\mathbb{F}_p)}$. Note that taking d discrete derivatives kills $\operatorname{Poly}_{\leqslant d}(\mathbb{F}_p^n\to\mathbb{F}_p)$, so if $P\equiv Q\pmod{\operatorname{Poly}_{\leqslant d}(\mathbb{F}_p^n\to\mathbb{F}_p)}$, then $\Delta_{h_1}\cdots\Delta_{h_d}P(x)=\Delta_{h_1}\cdots\Delta_{h_d}Q(x)$.

We compute τ_{α} first. Define $f \in \operatorname{Poly}_{\leq k}(V \to \mathbb{F}_p)$ by $f(x) = x_{i_1} \cdots x_{i_k}$. By many applications of the discrete Leibniz rule, we see that

$$\Delta_{h_1} \cdots \Delta_{h_k} f(x) = \sum_{\pi \in \mathfrak{S}_k} \Delta_{h_1}(x_{i_{\pi(1)}}) \cdots \Delta_{h_k}(x_{i_{\pi(k)}}) = \sum_{\pi \in \mathfrak{S}_k} (h_1)_{i_{\pi(1)}} \cdots (h_k)_{i_{\pi(k)}}.$$

Extending this result by linearity gives the formula for τ_{α} .

Now for ι_k . For any $k \geq 1$, write $k = r + (p-1)\ell$ with 0 < r < p and $\ell \geq 0$. Define $Q_k \in \operatorname{Poly}_{\leq k}(\mathbb{F}_p \to \mathbb{R}/\mathbb{Z})$ by $Q_k(x) = |x|^r/p^{\ell+1}$. By linearity, it suffices to prove that $\Delta_{h_1} \cdots \Delta_{h_k} Q_k(x) = r!(-1)^\ell h_1 \cdots h_k$. We prove that $\Delta_h Q_k(x) \equiv r|h|Q_{k-1}(x) \pmod{\operatorname{Poly}_{\leq k-2}(\mathbb{F}_p \to \mathbb{R}/\mathbb{Z})}$ for $k \geq 2$, while obviously $\Delta_h Q_1(x) = |h|/p$. Iterating (and applying the fact that $(p-1)! \equiv -1 \pmod{p}$) gives the desired result.

We break into two cases. First, if r=1 (and $\ell \geq 1$) then

$$\begin{split} \Delta_h Q_k(x) &= \frac{|x+h| - |x|}{p^{\ell+1}} \\ &= \frac{|h|}{p^{\ell+1}} - \frac{\mathbb{1}(|x| + |h| \ge p)}{p^{\ell}} \\ &= \frac{|h|}{p^{\ell+1}} - \sum_{c=p-|h|}^{p-1} \frac{\mathbb{1}(|x| = c)}{p^{\ell}} \\ &= \frac{|h|}{p^{\ell+1}} + |h| \frac{|x|^{p-1}}{p^{\ell}} - \sum_{c=p-|h|}^{p-1} \frac{\mathbb{1}(|x| = c) + |x|^{p-1}}{p^{\ell}}. \end{split}$$

Now $\mathbb{1}(|x| = c) \equiv 1 - (|x| - c)^{p-1} \pmod{p}$, say $\mathbb{1}(|x| = c) = 1 - (|x| - c)^{p-1} + pE_{c,p}(x)$ for some function $E_{c,p}$. Then we see

$$\Delta_h Q_k(x) - |h| \frac{|x|^{p-1}}{p^{\ell}} = \frac{|h|}{p^{\ell+1}} - \sum_{c=p-|h|}^{p-1} \frac{1 - (|x| - c)^{p-1} + |x|^{p-1}}{p^{\ell}} - \sum_{c=p-|h|}^{p-1} \frac{E_{c,p}(x)}{p^{\ell-1}}.$$

We know that every term in this equation is a non-classical polynomial of degree at most k-1 except for the last term. Thus we conclude that the last term is also a non-classical polynomial of degree k-1. Furthermore, of the three terms on the right-hand side, the first is a constant, the second is a non-classical polynomial of degree at most $(p-1)(\ell-1)+(p-2)=k-2$ (since the $|x|^{p-1}/p^{\ell}$ terms cancel), and the third has degree at most $(p-1)(\ell-1)=k-p$ (since it takes values in $\mathbb{U}_{\ell-1}$). Thus the right-hand side lies in $\operatorname{Poly}_{\leqslant k-2}(\mathbb{F}_p \to \mathbb{R}/\mathbb{Z})$, proving the desired result in the r=1 case.

Now assume that $k = r + (p-1)\ell$ where $r \ge 2$. We compute

$$\Delta_h Q_k(x) = \frac{|x+h|^r - |x|^r}{p^{\ell+1}}$$

$$= \frac{(|x|+|h|)^r - |x|^r}{p^{\ell+1}} - \mathbb{1}(|x|+|h| \ge p) \frac{(|x|+|h|)^r - (|x|+|h|-p)^r}{p^{\ell+1}}$$

$$= \frac{\sum_{i=1}^r {r \choose i} |h|^i |x|^{r-i}}{p^{\ell+1}} - \mathbb{1}(|x|+|h| \ge p) \left(\sum_{i=1}^r \frac{(-1)^{i-1} {r \choose i} (|x|+|h|)^{r-i}}{p^{\ell+1-i}}\right).$$

We rewrite this as

$$\Delta_h Q_k(x) - r|h| \frac{|x|^{r-1}}{p^{\ell+1}} = \frac{\sum_{i=2}^r {r \choose i} |h|^i |x|^{r-i}}{p^{\ell+1}} - \mathbb{1}(|x| + |h| \ge p) \left(\sum_{i=1}^r \frac{(-1)^{i-1} {r \choose i} (|x| + |h|)^{r-i}}{p^{\ell+1-i}} \right).$$

We know that every term in this equation is a non-classical polynomial of degree at most k-1 except for the last term, implying that the last term is also a non-classical polynomial of degree k-1. Furthermore, of the two terms on the right-hand side, the first is a non-classical polynomial of degree at most $(p-1)\ell + r - 2 = k-2$ and the second has degree at most $(p-1)\ell = k-r$ (since it takes values in $\mathbb{U}_{\ell-1}$). Thus the right-hand side lies in $\operatorname{Poly}_{\leq k-2}(\mathbb{F}_p \to \mathbb{R}/\mathbb{Z})$, proving the desired result in the $r \geq 2$ case.

Define the maps $I_k, T_\alpha \colon (\mathbb{F}_p^n)^{k-1} \to \mathbb{F}_p^n$ by the equations $\iota_k(h_1, \ldots, h_k) = I_k(h_1, \ldots, h_{k-1}) \cdot h_k$ and $\tau_\alpha(h_1, \ldots, h_k) = T_\alpha(h_1, \ldots, h_{k-1}) \cdot h_k$. From Lemma 5.1, clearly $I_k(h_1, \ldots, h_{k-1})_i = (-1)^\ell r!(h_1)_i \cdots (h_{k-1})_i$. To continue the argument we will need to show that T_α can be expressed in a particularly convenient form.

Lemma 5.2. For $\alpha = (\alpha_1, \dots, \alpha_s)$ with $\alpha_1 + \dots + \alpha_s = k$ and $0 < \alpha_i < p$ for all i,

$$T_{\alpha}(h_1, \dots, h_{k-1})_i = \sum_{J \subseteq [k-1]} C_{i,\alpha,J}(h_{[k-1], < i}, (\tau_{\beta}(h_I))_{\beta,I}) \prod_{j \in J} (h_j)_i$$

for some functions $C_{i,\alpha,J}(\cdot,\cdot)$, evaluated at the tuple of $(h_j)_{i'}$ for all $j \in [k-1]$ and i' < i and the tuple of $\tau_{\beta}(h_I)$ for all $I \subseteq [k-1]$ and $\beta = (\beta_1, \ldots, \beta_t)$ with $\beta_1 + \cdots + \beta_t = |I|$ and $0 < \beta_i < p$ for all i. Furthermore,

$$C_{i,\alpha,[k-1]} = (-1)^{s-1}\alpha_1(k-1)!.$$

Proof. Fix $i \in [n]$ for the rest of the proof. We introduce some notation. We have $h_1, \ldots, h_{k-1} \in \mathbb{F}_p^n$. For $I \subseteq [k-1]$, we write $h_I = (h_i)_{i \in I}$. We use

$$h_{I,<} = ((h_j)_1, \dots, (h_j)_{i-1})_{j \in I} \in (\mathbb{F}_p^{i-1})^I$$
 and $h_{I,>} = ((h_j)_{i+1}, \dots, (h_j)_n)_{j \in I} \in (\mathbb{F}_p^{n-i})^I$.

For $\alpha=(\alpha_1,\ldots,\alpha_s)$ we write $\alpha_{<\ell}=(\alpha_1,\ldots,\alpha_{\ell-1})$ and $\alpha_{>\ell}=(\alpha_{\ell+1},\ldots,\alpha_s)$. We define $\alpha_{\leqslant\ell}$ analogously. Recall that we use $|\alpha|=\alpha_1+\cdots+\alpha_s$.

By inspection, we can write

$$T_{\alpha}(h_{[k-1]})_{i} = \sum_{\ell=1}^{s} \alpha_{\ell}! \sum_{\substack{I \cup J \cup K = [k-1]: \\ |I| = |\alpha_{<\ell}|, \\ |J| = \alpha_{\ell} - 1, \\ |K| = |\alpha_{>\ell}|}} \left(\prod_{j \in J} (h_{j})_{i} \right) \tau_{\alpha_{<\ell}}(h_{I,<}) \tau_{\alpha_{>\ell}}(h_{K,>}).$$
 (5.2)

We now remove the terms depending on $h_{[k-1],>}$. Take $\beta = (\beta_1, \ldots, \beta_t)$ with $0 < \beta_i < p$ for all i and $L \subseteq [k-1]$ with $|L| = |\beta|$. We have the identity

$$\tau_{\beta}(h_{L,>}) = \tau_{\beta}(h_{L}) - \sum_{\ell=1}^{t} \sum_{\substack{I \sqcup K = L: \\ |I| = |\beta_{\leqslant \ell}|, \\ |K| = |\beta_{> \ell}|}} \tau_{\beta_{\leqslant \ell}}(h_{I,<}) \tau_{\beta_{> \ell}}(h_{K,>})$$

$$- \sum_{\ell=1}^{t} \beta_{\ell}! \sum_{\substack{I \sqcup J \sqcup K = L: \\ |I| = |\beta_{< \ell}|, \\ |K| = |\beta_{> \ell}|}} \left(\prod_{j \in J} (h_{j})_{i} \right) \tau_{\beta_{< \ell}}(h_{I,<}) \tau_{\beta_{> \ell}}(h_{K,>}). \tag{5.3}$$

Repeatedly applying this identity eventually puts $T_{\alpha}(h_{[k-1]})_i$ into the desired form. To see this, note that applying this identity to $\tau_{\beta}(h_{L,>})$ produces many terms of the form $\tau_{\beta>\ell}(h_{K,>})$ but all of these satisfy $|\beta>\ell| = |K| < |\beta| = |L|$, so we always make progress.

Finally we need to compute $C_{i,\alpha,[k-1]}$. Obviously this coefficient is a constant since the final decomposition that we produce is multilinear in the h_1,\ldots,h_{k-1} . Furthermore, the only way to produce a term that is a multiple of $(h_1)_i\cdots(h_{k-1})_i$ is to have no factors of $\tau_{\beta<\ell}(h_{I,<})$ in that term. (However, we have a choice of I,J,K in (5.2).) This means that in the initial decomposition we need to be in the $\ell=1$ case of the sum and every time we use the identity (5.3) we need to be in the $\ell=1$ case of the second sum. Again, in every subsequent choice although $\ell=1$ is fixed, we have a choice of I,J,K. Tracing through all of these reductions, we see that we produce the coefficient

$$\alpha_1! \binom{k-1}{\alpha_1 - 1} \prod_{j=2}^s \left(-(\alpha_j!) \binom{k - \alpha_1 - \dots - \alpha_{j-1}}{\alpha_j} \right) = (-1)^{s-1} \alpha_1 (k-1)!. \quad \Box$$

So far we have developed the tools to, starting with the assumption of correlation with a classical polynomial, reduce to a situation in which

$$\epsilon^{2^k} \leq \mathbb{E}\left[e_p\left(\iota_k - \sum_{\alpha} c_{\alpha} \tau_{\alpha}\right)\right] = \mathbb{P}\left(I_k + \sum_{\alpha} c_{\alpha} T_{\alpha} = 0\right)$$

$$= \mathbb{P}_{h_1,\dots,h_{k-1} \sim \mathbb{F}_p^n}\left(\forall i \in [n], \ I_k(h_1,\dots,h_{k-1})_i + \sum_{\alpha} c_{\alpha} T_{\alpha}(h_1,\dots,h_{k-1})_i = 0\right).$$

Therefore, we need a bound on the probability that a multiaffine function equals zero.

Lemma 5.3. Let $L: \mathbb{F}_p^r \to \mathbb{F}_p$ be a multiaffine function whose leading coefficient (i.e., coefficient of $x_1 \cdots x_r$) is non-zero. Then

$$\mathbb{P}_{x_1,\dots,x_r \sim \mathbb{F}_p}(L(x_1,\dots,x_r) = 0) \le 1 - \left(1 - \frac{1}{p}\right)^r =: 1 - c_{p,r}.$$

Proof. We prove this result by induction on r. The bound is trivially true for r=0.

For $r \geq 1$, we can write

$$L(x_1,\ldots,x_r) = x_r M(x_1,\ldots,x_{r-1}) + N(x_1,\ldots,x_{r-1})$$

where M and N are multiaffine and the leading coefficient of M is non-zero. Then for each fixed x_1, \ldots, x_{r-1} , there is at most 1 choice of x_r that makes L vanish unless $M(x_1, \ldots, x_r) = 0$. Then

$$\mathbb{P}_{x_1,\dots,x_r \sim \mathbb{F}_p}(L(x_1,\dots,x_r) \neq 0) \ge \left(1 - \frac{1}{p}\right) \mathbb{P}_{x_1,\dots,x_{r-1} \sim \mathbb{F}_p}(M(x_1,\dots,x_{r-1}) \neq 0).$$

The second term can be handled by the inductive hypothesis.

We now have the tools to prove the main theorem. The probability we are considering is the probability that n multiaffine functions vanish simultaneously. If these were independent, by the above lemma, we could bound the probability by $(1 - c_{p,k})^n = o_{p,k;n\to\infty}(1)$.

In order to introduce such independence, we can take a union bound over all possible $\tau_{\beta}(h_I)$. Then Lemma 5.2 shows that our multiaffine forms have the following property: if we plug in values for $((h_1)_{i'}, \ldots, (h_{k-1})_{i'})_{i' < i}$ and $\tau_{\beta}(h_I)$ for all β, I , then $T_{\alpha}(h_1, \ldots, h_{k-1})_i$ is multiaffine in $(h_1)_i, \ldots, (h_{k-1})_i$ with non-zero leading coefficient. Then we may reveal $((h_1)_i, \ldots, (h_{k-1})_i)$ one-by-one for $i \in [n]$, and find that the total probability is bounded by $(1 - c_{p,k})^n$. As the number of possible choices in the union bound is $O_{p,k}(1)$ we will be able to prove the desired bound.

Proof of Theorem 1.2. Take $k \geq p+1$. Consider $f_n^{(k)} \colon \mathbb{F}_p^n \to \mathbb{R}/\mathbb{Z}$ defined in (5.1). Since $f_n^{(k)}$ is a non-classical polynomial of degree k, we know that $\|e(f_n^{(k)})\|_{U^{k+1}} = 1$. For a classical polynomial $P \in \operatorname{Poly}_{\leq k}(\mathbb{F}_p^n \to \mathbb{F}_p)$, set $\epsilon = |\mathbb{E}_x e(f_n^{(k)}(x) + |P(x)|/p)|$. We will prove that $\epsilon = o_{p,k;n\to\infty}(1)$.

By Theorem 4.2, there exists $m=\omega_{p,k;n\to\infty}(1)$ and a subset $I\subseteq [n]$ such that for all $y_{[n]\setminus I}\in \mathbb{F}_p^{[n]\setminus I}$,

$$P(x_I, y_{[n]\setminus I}) = Q(x_I) + P_{y_{[n]\setminus I}}(x_I)$$

where Q is a homogeneous quasisymmetric polynomial of degree k and $P_{y_{[n]\setminus I}}$ is a polynomial of degree at most k-1.

Without loss of generality, assume that I = [m]. Then

$$\epsilon = |\mathbb{E}_{x \sim \mathbb{F}_n^n} e(f_n^{(k)}(x) + |P(x)|/p)| \le \mathbb{E}_{y \sim \mathbb{F}_n^{n-m}} |\mathbb{E}_{x \sim \mathbb{F}_n^m} e(f_n^{(k)}(x,y) + |P(x,y)|/p)|.$$

Then by the pigeonhole principle, there exists $y \in \mathbb{F}_p^{n-m}$ such that the inner expectation is at least ϵ . Fix this choice of y for the rest of the proof. Note that $f_n^{(k)}(x,y) = f_m^{(k)}(x) + c_y$ where $c_y = f_{n-m}^{(k)}(y) \in \mathbb{R}/\mathbb{Z}$ is a constant. Thus we have

$$\epsilon \le |\mathbb{E}_{x \sim \mathbb{F}_n^m} e(f_m^{(k)}(x) + |Q(x)|/p + |P_y(x)|/p + c_y)|.$$

Note that the right-hand side is a U^1 -norm. Using the monotonicity of the Gowers norms (see, e.g., [7, Lemma B.1.(ii)]) we deduce

$$\epsilon^{2^{k}} \leq \|e(f_{m}^{(k)} + |Q|/p + |P_{y}|/p + c_{y})\|_{U^{1}}^{2^{k}}
\leq \|e(f_{m}^{(k)} + |Q|/p + |P_{y}|/p + c_{y})\|_{U^{k}}^{2^{k}}
= \mathbb{E}_{x,h_{1},...,h_{k}} \partial_{h_{1}} \cdots \partial_{h_{k}} e(f_{m}^{(k)} + |Q|/p + |P_{y}|/p + c_{y})(x)
= \mathbb{E}_{x,h_{1},...,h_{k}} e(\Delta_{h_{1}} \cdots \Delta_{h_{k}} (f_{m}^{(k)} + |Q|/p + |P_{y}|/p + c_{y})(x)).$$

Taking k discrete derivatives kills (non-classical) polynomials of degree at most k-1 and turns those of degree k into constants. Thus the final expression is equal to

$$\mathbb{E}_{h_1,\ldots,h_k}e(\Delta_{h_1}\cdots\Delta_{h_k}(f_m^{(k)}+|Q|/p)).$$

Since Q is a homogeneous quasisymmetric polynomial of degree k, it can be written as $\sum_{\alpha} c_{\alpha} Q_{\alpha}$ where α ranges over all tuples $(\alpha_1, \ldots, \alpha_s)$ with $|\alpha| = k$ and $0 < \alpha_i < p$ for all i and the $c_{\alpha} \in \mathbb{F}_p$ are arbitrary coefficients.

We computed $\Delta_{h_1} \cdots \Delta_{h_k} f_m^{(k)}$ and $\Delta_{h_1} \cdots \Delta_{h_k} Q_\alpha$ in Lemma 5.1. These are the *k*-linear forms denoted $\iota_k, \tau_\alpha \colon (\mathbb{F}_p^m)^k \to \mathbb{F}_p$ respectively. Thus so far we have shown that

$$\epsilon^{2^k} \leq \mathbb{E}_{h_1,\dots,h_k} e_p \left(\iota_k(h_1,\dots,h_k) + \sum_{\alpha} c_{\alpha} \tau_{\alpha}(h_1,\dots,h_k) \right).$$

For an arbitrary k-linear form $\sigma \colon (\mathbb{F}_p^m)^k \to \mathbb{F}_p$, there is a unique (k-1)-linear function $S \colon (\mathbb{F}_p^m)^{k-1} \to \mathbb{F}_p^m$ that satisfies $\sigma(h_1, \ldots, h_k) = S(h_1, \ldots, h_{k-1}) \cdot h_k$. Furthermore, we have

$$\mathbb{E}_{h_1,\dots,h_k} e_p(\sigma(h_1,\dots,h_k)) = \mathbb{E}_{h_1,\dots,h_k} e_p(S(h_1,\dots,h_{k-1})\cdot h_k) = \mathbb{P}_{h_1,\dots,h_{k-1}}(S(h_1,\dots,h_{k-1})=0).$$

From this we conclude

$$\epsilon^{2^k} \le \mathbb{P}_{h_1,\dots,h_{k-1}} \left(\forall i \in [m], \ I_k(h_1,\dots,h_{k-1})_i + \sum_{\alpha} c_{\alpha} T_{\alpha}(h_1,\dots,h_{k-1})_i = 0 \right).$$

Recall that $I_k(h_1, \ldots, h_{k-1})_i = (-1)^{\ell} r! (h_1)_i \cdots (h_{k-1})_i$ where $k = r + (p-1)\ell$ with $\ell \geq 1$ and 0 < r < p. Note that $(-1)^{\ell} r! \neq 0$ in \mathbb{F}_p . Furthermore, Lemma 5.2 states that

$$T_{\alpha}(h_1,\ldots,h_{k-1})_i = \sum_{J\subseteq[k-1]} C_{i,\alpha,J}(h_{[k-1],< i},(\tau_{\beta}(h_I))_{\beta,I}) \prod_{j\in J} (h_j)_i.$$

In other words $T_{\alpha}(h_1, \ldots, h_{k-1})_i$, viewed just as a function of $(h_1)_i, \ldots, (h_{k-1})_i$ is multiaffine with coefficients given by $C_{i,\alpha,J}$. Additionally, Lemma 5.2 also gives the critical fact that the leading coefficient, $C_{i,\alpha,[k-1]}$, is equal to $(-1)^{s-1}\alpha_1(k-1)!$ for all i. Since $k \geq p+1$, we have that $C_{i,\alpha,[k-1]} = 0$ (recall that the coefficients live in \mathbb{F}_p).

This implies that $I_k(h_1, \ldots, h_{k-1})_i + \sum_{\alpha} c_{\alpha} T_{\alpha}(h_1, \ldots, h_{k-1})_i$, viewed just as a function of $(h_1)_i, \ldots, (h_{k-1})_i$ is multiaffine with non-zero leading coefficient, say

$$I_k(h_1, \dots, h_{k-1})_i + \sum_{\alpha} c_{\alpha} T_{\alpha}(h_1, \dots, h_{k-1})_i = \sum_{J \subseteq [k-1]} C_{i,J}(h_{[k-1], < i}, (\tau_{\beta}(h_I))_{\beta, I}) \prod_{j \in J} (h_j)_i$$

where $C_{i,[k-1]} = (-1)^{\ell} r! \neq 0$ for all *i*.

By Lemma 5.3, if the coefficients are fixed then this function vanishes with probability at most $1 - c_{p,k} < 1$. To complete the proof, we need to show that we can approximately decouple these events. Formally,

$$\epsilon^{2^{k}} \leq \mathbb{P}_{h_{1},\dots,h_{k-1}} \left(\forall i \in [m], \sum_{J \subseteq [k-1]} C_{i,J}(h_{[k-1],< i}, (\tau_{\beta}(h_{I}))_{\beta,I}) \prod_{j \in J} (h_{j})_{i} = 0 \right) \\
= \sum_{(A_{\beta,I})_{\beta,I}} \mathbb{P}_{h_{1},\dots,h_{k-1}} \left(\forall \beta, I, \ \tau_{\beta}(h_{I}) = A_{\beta,I} \cap \sum_{J \subseteq [k-1]} C_{i,J}(h_{[k-1],< i}, (\tau_{\beta}(h_{I}))_{\beta,I}) \prod_{j \in J} (h_{j})_{i} = 0 \right) \\
\leq \sum_{(A_{\beta,I})_{\beta,I}} \mathbb{P}_{h_{1},\dots,h_{k-1}} \left(\sum_{J \subseteq [k-1]} C_{i,J}(h_{[k-1],< i}, (A_{\beta,I})_{\beta,I}) \prod_{j \in J} (h_{j})_{i} = 0 \right).$$

The final replacement simply comes by substituting in the values $A_{\beta,I}$.

Now for each $i \in [m]$, let E_i be the event that $\sum_{J \subseteq [k-1]} C_{i,J}(h_{[k-1],< i}, (A_{\beta,I})_{\beta,I}) \prod_{j \in J} (h_j)_i = 0$. We wish to bound

$$\mathbb{P}_{h_1, \dots, h_{k-1}} \left(E_i \mid \forall i' < i, E_{i'} \right).$$

Since the event we are conditioning on only depends on $h_{[k-1],< i}$, the conditional distribution of $(h_1)_i, \ldots, (h_{k-1})_i$ is still uniform. Thus we can upper bound the above probability by

$$\sup_{h_{[k-1],$$

By Lemma 5.3, and the fact that $C_{i,[k-1]} = (-1)^{\ell} r! \neq 0$ always, this probability is upper-bounded by $1 - c_{p,k} < 1$. Putting everything together, we have shown that $\epsilon^{2^k} \leq O_{p,k}((1 - c_{p,k})^m)$. (The hidden constant is the number of terms in the sum over $(A_{\beta,I})_{\beta,I}$, which depends on p,k but not on m,n. It can be bounded by p^{4^k} .) We showed that $m = \omega_{p,k;n\to\infty}(1)$, implying that $\epsilon = o_{p,k;n\to\infty}(1)$. \square

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DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139, USA *Email address*: {bergera,asah,msawhney,jtidor}@mit.edu