# NON-CLASSICAL POLYNOMIALS AND THE INVERSE THEOREM 

AARON BERGER, ASHWIN SAH, MEHTAAB SAWHNEY, AND JONATHAN TIDOR


#### Abstract

In this note we characterize when non-classical polynomials are necessary in the inverse theorem for the Gowers $U^{k}$-norm. We give a brief deduction of the fact that a bounded function on $\mathbb{F}_{p}^{n}$ with large $U^{k}$-norm must correlate with a classical polynomial when $k \leq p+1$. To the best of our knowledge, this result is new for $k=p+1$ (when $p>2$ ). We then prove that non-classical polynomials are necessary in the inverse theorem for the Gowers $U^{k}$-norm over $\mathbb{F}_{p}^{n}$ for all $k \geq p+2$, completely characterizing when classical polynomials suffice.


## 1. Introduction

The inverse theorem for the Gowers $U^{k}$-norm states that a bounded function $f: G \rightarrow \mathbb{C}$ has large $U^{k}$-norm if and only if $f$ correlates with a certain structured object. When $G=\mathbb{Z} / N \mathbb{Z}$, these structured objects are quite complicated and need the theory of nilsequences to describe. When $G=\mathbb{F}_{p}^{n}$, the situation is somewhat simpler. When $p>k$, a bounded function $f: \mathbb{F}_{p}^{n} \rightarrow \mathbb{C}$ has large $U^{k}$-norm if and only if $f$ has non-negligible correlation with a polynomial phase function, i.e., $e^{2 \pi i P(x) / p}$ where $P: \mathbb{F}_{p}^{n} \rightarrow \mathbb{F}_{p}$ is a polynomial of degree at most $k-1$.

The situation when $p$ is small compared to $k$ is more delicate. Green and Tao [3] and independently Lovett, Meshulam, and Samorodnitsky [4] showed that the corresponding conjecture is false for $k=4$ and $p=2$. In other words, there exist bounded functions $f: \mathbb{F}_{2}^{n} \rightarrow \mathbb{C}$ with large $U^{4}$-norm but with correlation $o_{n \rightarrow \infty}(1)$ with every cubic phase function. Tao and Ziegler [7] clarified this situation by proving that for all $k$ and $p$, a bounded function $f: \mathbb{F}_{p}^{n} \rightarrow \mathbb{C}$ has large $U^{k}$-norm if and only if $f$ has non-negligible correlation with a non-classical polynomial phase function, i.e., $e^{2 \pi i P(x)}$ where $P: \mathbb{F}_{p}^{n} \rightarrow \mathbb{R} / \mathbb{Z}$ is a non-classical polynomial of degree at most $k-1$. (See Section 2 for the relevant definitions.)

A natural question which remains from the above discussion is to determine for which pairs $p, k$ does the $U^{k}$-inverse theorem over $\mathbb{F}_{p}^{n}$ hold with classical polynomials. In the positive direction, it is known due to Samorodnitsky [5] that the $U^{3}$-inverse theorem over $\mathbb{F}_{2}^{n}$ holds with classical polynomials. In the negative direction, Lovett, Meshulam, and Samorodnitsky [4] proved that the $U^{p^{\ell}}$-inverse theorem over $\mathbb{F}_{p}^{n}$ requires non-classical polynomials for all $p$ and $\ell \geq 2$. (A curious feature of this problem is that it is not monotone in $k$, e.g., the Lovett-Meshulam-Samorodnitsky result does not imply that non-classical polynomials are necessary in the $U^{k}$-inverse theorem for all $k \geq p^{2}$.)

In this paper we completely characterize when classical polynomials suffice in the statement of the inverse theorem. We first prove the inverse theorem for the Gowers $U^{p+1}$-norm with classical polynomials. This result is proved via a short deduction from the usual inverse theorem for the $U^{p+1}$-norm that involves non-classical polynomials. ${ }^{1}$

Theorem 1.1. Fix a prime $p$ and $\delta>0$. There exists $\epsilon>0$ such that the following holds. Let $V$ be a finite-dimensional $\mathbb{F}_{p}$-vector space. Given a function $f: V \rightarrow \mathbb{C}$ satisfying $\|f\|_{\infty} \leq 1$ and

[^0]$\|f\|_{U^{p+1}}>\delta$, there exists a classical polynomial $P \in \operatorname{Poly}_{\leqslant p}\left(V \rightarrow \mathbb{F}_{p}\right)$ such that
$$
\left|\mathbb{E}_{x \in V} f(x) e_{p}(-P(x))\right| \geq \epsilon
$$

Second, we give an example showing that non-classical polynomials are necessary in the $U^{k}$-inverse theorem for all $k \geq p+2$.

Theorem 1.2. Fix a prime $p$ and an integer $k \geq p+2$. For all $n$, there exists a function $f: \mathbb{F}_{p}^{n} \rightarrow \mathbb{C}$ satisfying $\|f\|_{\infty} \leq 1$ and $\|f\|_{U^{k}}=1$ such that for all (classical) polynomials $P \in \operatorname{Poly}_{\leqslant k-1}\left(\mathbb{F}_{p}^{n} \rightarrow\right.$ $\mathbb{F}_{p}$ ),

$$
\left|\mathbb{E}_{x \in \mathbb{F}_{p}^{n}} f(x) e_{p}(-P(x))\right|=o_{p, k ; n \rightarrow \infty}(1) .
$$

Our example is fairly simple to write down. For $k \geq p+2$, we write $k-1=r+(p-1) \ell$ where $\ell \geq 1$ and $0<r<p$. Then our function is

$$
f(x)=e^{2 \pi i \frac{\sum_{i=1}^{n}\left|x_{i}\right|^{r}}{p^{p+1}}}
$$

(where $|\cdot|: \mathbb{F}_{p} \rightarrow\{0, \ldots, p-1\}$ is the standard map). Note that this function $f$ is a non-classical polynomial phase function of degree $k-1$, so the content of this result is that it does not correlate with any classical polynomial phase functions of the same degree.

The $o(1)$ correlation in Theorem 1.2 is fairly bad - the inverse of many iterated logarithms. This is due to our use of a Ramsey-theoretic argument inspired by a similar argument of Alon and Beigel. We conjecture that this bound on the correlation can be improved.
Conjecture 1.3. Fix a prime $p$ and an integer $k \geq p+2$. For all $n$ there exist $f: \mathbb{F}_{p}^{n} \rightarrow \mathbb{C}$ satisfying $\|f\|_{\infty} \leq 1$ and $\|f\|_{U^{k}} \geq c_{p, k}>0$ such that for all (classical) polynomials $P \in \operatorname{Poly}_{\leqslant k-1}\left(\mathbb{F}_{p}^{n} \rightarrow \mathbb{F}_{p}\right)$,

$$
\left|\mathbb{E}_{x \in \mathbb{F}_{p}^{n}} f(x) e_{p}(-P(x))\right| \leq \exp \left(-\Omega_{p, k}(n)\right) .
$$

In fact, we believe that this conjecture is true with the same functions that we use to prove Theorem 1.2.

Structure of the paper: In Section 2 we give the definition of the Gowers $U^{k}$-norm and of nonclassical polynomials. In Section 3 we prove Theorem 1.1. We prove Theorem 1.2 in the remainder of the paper. Section 4 develops the symmetrization tool that we use and Section 5 gives the full proof.

Notation: We use $|\cdot|$ for the standard map $\mathbb{F}_{p} \rightarrow\{0, \ldots, p-1\}$. We often treat $\mathbb{F}_{p}$ as an additive subgroup of $\mathbb{R} / \mathbb{Z}$ via the map $x \mapsto|x| / p$ and, by some abuse of notation, freely switch between these two viewpoints. We use $e: \mathbb{R} / \mathbb{Z} \rightarrow \mathbb{C}$ for the function $e(x)=e^{2 \pi i x}$ and $e_{p}: \mathbb{F}_{p} \rightarrow \mathbb{C}$ for the function $e_{p}(x)=e^{2 \pi i|x| / p}$.

## 2. Background on non-classical polynomials

Definition 2.1. Fix a prime $p$, a finite-dimensional $\mathbb{F}_{p}$-vector space $V$, and an abelian group $G$. Given a function $f: V \rightarrow G$ and a shift $h \in V$, define the additive derivative $\Delta_{h} f: V \rightarrow G$ by

$$
\left(\Delta_{h} f\right)(x)=f(x+h)-f(x) .
$$

Given a function $f: V \rightarrow \mathbb{C}$ and a shift $h \in V$, define the multiplicative derivative $\partial_{h} f: V \rightarrow \mathbb{C}$ by

$$
\left(\partial_{h} f\right)(x)=f(x+h) \overline{f(x)}
$$

Definition 2.2. Fix a prime $p$ and a finite-dimensional $\mathbb{F}_{p}$-vector space $V$. Given a function $f: V \rightarrow \mathbb{C}$ and $d \geq 1$, the Gowers uniformity norm $\|f\|_{U^{d}}$ is defined by

$$
\|f\|_{U^{d}}=\left|\mathbb{E}_{x, h_{1}, \ldots, h_{d} \in V}\left(\partial_{h_{1}} \cdots \partial_{h_{d}} f\right)(x)\right|^{1 / 2^{d}}
$$

See [7, Lemma B.1] for some basic facts about the Gowers uniformity norms.
Definition 2.3. Fix a prime $p$, and a non-negative integer $d \geq 0$. Let $V$ be a finite-dimensional $\mathbb{F}_{p}$-vector space. A non-classical polynomial of degree at most $d$ is a map $P: V \rightarrow \mathbb{R} / \mathbb{Z}$ that satisfies

$$
\left(\Delta_{h_{1}} \cdots \Delta_{h_{d+1}} P\right)(x)=0
$$

for all $h_{1}, \ldots, h_{d+1}, x \in V$. We write $\operatorname{Poly}_{\leqslant d}(V \rightarrow \mathbb{R} / \mathbb{Z})$ for the set of non-classical polynomials of degree at most $d$.

A classical polynomial is a map $V \rightarrow \mathbb{F}_{p}$ satisfying the same property. By composing with the standard map $x \mapsto|x| / p$ we can view $\operatorname{Poly}_{\leqslant d}\left(V \rightarrow \mathbb{F}_{p}\right)$ as a subset of $\mathrm{Poly}_{\leqslant d}(V \rightarrow \mathbb{R} / \mathbb{Z})$.

See [7, Lemma 1.7] for some properties of non-classical polynomials. We give one property below which will be used several times in this paper.

Lemma 2.4 ([7, Lemma 1.7(iii)]). Fix a prime $p$ and a finite-dimensional $\mathbb{F}_{p}$-vector space $V=\mathbb{F}_{p}^{n}$. Then $P: V \rightarrow \mathbb{R} / \mathbb{Z}$ is a non-classical polynomial of degree at most $d$ if and only if can be expressed in the form
for some $\alpha \in \mathbb{R} / \mathbb{Z}$ and coefficients $c_{i_{1}, \ldots, i_{n}, j} \in\{0, \ldots, p-1\}$. Furthermore, this representation is unique.

Define $\mathbb{U}_{k} \subset \mathbb{R} / \mathbb{Z}$ to be $\left\{0,1 / p^{k}, \ldots,\left(p^{k}-1\right) / p^{k}\right\}$. As a corollary we see that in characteristic $p$, every non-classical polynomial of degree at most $d$ takes values in a coset of $\mathbb{U}_{\lfloor(d-1) /(p-1)\rfloor+1}$.

Finally we state the inverse theorem of Tao and Ziegler.
Theorem 2.5 ([7, Theorem 1.10]). Fix a prime $p$, a positive integer $k$, and a parameter $\delta>0$. There exists $\epsilon>0$ such that the following holds. Let $V$ be a finite-dimensional $\mathbb{F}_{p}$-vector space. Given a function $f: V \rightarrow \mathbb{C}$ satisfying $\|f\|_{\infty} \leq 1$ and $\|f\|_{U^{k}}>\delta$, there exists a non-classical polynomial $P \in \operatorname{Poly}_{\leqslant k-1}(V \rightarrow \mathbb{R} / \mathbb{Z})$ such that

$$
\left|\mathbb{E}_{x \in V} f(x) e^{-2 \pi i P(x)}\right| \geq \epsilon .
$$

Earlier works of Bergelson, Tao, and Ziegler [2] and Tao and Ziegler [6] show this result in the high-characteristic regime $p \geq k$, with the additional guarantee that $P$ is a classical polynomial of degree at most $k-1$.

## 3. Classical polynomials for the $U^{p+1}$-Inverse theorem

The inverse theorem for the $U^{k}$-norm does not require non-classical polynomials when $p \geq k$ for the simple reason that every non-classical polynomial of degree at most $p-1$ is a classical polynomial of the same degree (up to a constant shift). To prove Theorem 1.1, about the $U^{p+1}$ inverse theorem, we use the following fact. Every non-classical polynomial of degree $p$ agrees with a classical polynomial on a codimension 1 hyperplane (up to a constant shift).
Proposition 3.1. Let $P \in \operatorname{Poly}_{\leqslant p}(V \rightarrow \mathbb{R} / \mathbb{Z})$ be a non-classical polynomial of degree at most $p$. Then there exists a codimension 1 hyperplane $U \leq V$, a classical polynomial $Q \in \operatorname{Poly}_{\leqslant p}\left(V \rightarrow \mathbb{F}_{p}\right)$, and $\alpha \in \mathbb{R} / \mathbb{Z}$ such that $P(x)=\alpha+|Q(x)| / p$ for all $x \in U$.

Proof. Pick an isomorphism $V \simeq \mathbb{F}_{p}^{n}$. By Lemma 2.4, we have

$$
P\left(x_{1}, \ldots, x_{n}\right)=\alpha+P^{\prime}\left(x_{1}, \ldots, x_{n}\right)+\frac{c_{1}\left|x_{1}\right|+\cdots+c_{n}\left|x_{n}\right|}{p^{2}} \quad(\bmod 1)
$$

for $\alpha \in \mathbb{R} / \mathbb{Z}$, a polynomial $P^{\prime}$ taking values in $\mathbb{U}_{1}=\{0,1 / p, \ldots,(p-1) / p\}$, and $c_{1}, \ldots, c_{n} \in$ $\{0, \ldots, p-1\}$. Define the codimension 1 hyperplane $U \leq V$ by $c_{1} x_{1}+\cdots+c_{n} x_{n}=0$. Note that for $\left(x_{1}, \ldots, x_{n}\right) \in U$, we have $c_{1}\left|x_{1}\right|+\cdots+c_{n}\left|x_{n}\right| \equiv 0(\bmod p)$. Thus $\left.P\right|_{U}$ takes values in $\alpha+\mathbb{U}_{1}$. Thus by our identification of $\mathbb{U}_{1}$ with $\mathbb{F}_{p},\left.P\right|_{U}-\alpha$ is a classical polynomial of degree at most $p$.

Proof of Theorem 1.1. By the usual inverse theorem, Theorem 2.5, there exists $P \in$ Poly $_{\leqslant p}(V \rightarrow$ $\mathbb{R} / \mathbb{Z})$ such that $\left|\mathbb{E}_{x \in V} f(x) e(-P(x))\right| \geq \epsilon$. By Proposition 3.1, there exists a codimension 1 hyperplane $U$ and $\alpha \in \mathbb{R} / \mathbb{Z}$ such that $\left.P\right|_{U}$ takes values in $\alpha+\mathbb{U}_{1}$, i.e., $\left.P\right|_{U}-\alpha$ is classical. Pick an isomorphism $V \simeq \mathbb{F}_{p}^{n}$ such that $U$ is the hyperplane defined by $x_{1}=0$. In this basis, there exists a classical polynomial $Q \in \operatorname{Poly}_{\leqslant p}\left(\mathbb{F}_{p}^{n} \rightarrow \mathbb{F}_{p}\right)$ and $c \in\{0, \ldots, p-1\}$ so that

$$
P\left(x_{1}, \ldots, x_{n}\right)=\alpha+\frac{\left|Q\left(x_{1}, \ldots, x_{n}\right)\right|}{p}+\frac{c\left|x_{1}\right|}{p^{2}} \quad(\bmod 1) .
$$

Thus we have

$$
\begin{aligned}
\epsilon & \leq\left|\mathbb{E}_{x \in \mathbb{F}_{p}^{n}} f(x) e(-P(x))\right| \\
& \leq \mathbb{E}_{x_{1} \in \mathbb{F}_{p}}\left|\mathbb{E}_{y \in \mathbb{F}_{p}^{n-1}} f\left(x_{1}, y\right) e\left(-P\left(x_{1}, y\right)\right)\right| \\
& =\mathbb{E}_{x_{1} \in \mathbb{F}_{p}}\left|\mathbb{E}_{y \in \mathbb{F}_{p}^{n-1}} f\left(x_{1}, y\right) e_{p}\left(-Q\left(x_{1}, y\right)\right)\right| \\
& \leq\left(\mathbb{E}_{x_{1} \in \mathbb{F}_{p}}\left|\mathbb{E}_{y \in \mathbb{F}_{p}^{n-1}} f\left(x_{1}, y\right) e_{p}\left(-Q\left(x_{1}, y\right)\right)\right|^{2}\right)^{1 / 2} .
\end{aligned}
$$

By Parseval and the pigeonhole principle, there exists $a \in \mathbb{F}_{p}$ such that

$$
\left|\mathbb{E}_{x_{1} \in \mathbb{F}_{p}} e\left(-a x_{1}\right) \mathbb{E}_{y \in \mathbb{F}_{p}^{n-1}} f\left(x_{1}, y\right) e_{p}\left(-Q\left(x_{1}, y\right)\right)\right| \geq \epsilon / \sqrt{p}
$$

Therefore $f$ has correlation at least $\epsilon / \sqrt{p}$ with the classical polynomial $Q\left(x_{1}, \ldots, x_{n}\right)+a x_{1}$.

## 4. Symmetrization techniques

We now extend a symmetrization technique of Alon and Beigel [1] which will be needed to prove the non-correlation property of our example. At a high level, this technique use Ramsey theory to show that if a function correlates with a bounded degree polynomial, then some restriction of coordinates correlates with a symmetric polynomial. Unfortunately, as stated this only holds for multilinear polynomials, and otherwise one can only reduce to the class of so-called quasisymmetric polynomials. These are a generalization of the notion of symmetric polynomials which have found extensive use in enumerative and algebraic combinatorics.

Definition 4.1. For a prime $p$ and a tuple $\left(\alpha_{1}, \ldots, \alpha_{s}\right)$ of positive integers satisfying $\alpha_{i}<p$ for all $i$, the elementary quasisymmetric polynomial associated to $\left(\alpha_{1}, \ldots, \alpha_{s}\right)$ in $n$ variables is the polynomial $Q_{\alpha}: \mathbb{F}_{p}^{n} \rightarrow \mathbb{F}_{p}$ defined by

$$
Q_{\alpha}\left(x_{1}, \ldots, x_{n}\right)=\sum_{1 \leq i_{1}<\cdots<i_{s} \leq n} \prod_{j=1}^{s} x_{i_{j}}^{\alpha_{j}}
$$

We additionally note the total degree is $|\alpha|:=\alpha_{1}+\cdots+\alpha_{s}$.
Theorem 4.2. Fix a prime $p$ and an integer $d \geq 1$. For any $n$, there exists $m=\omega_{p, d ; n \rightarrow \infty}(1)$ such that the following holds. Let $P\left(x_{1}, \ldots, x_{n}\right)$ be a polynomial of degree at most d with coefficients in $\mathbb{F}_{p}$. There exists $I \subseteq[n]$ of size $|I|=m$ such that for any $y_{[n] \backslash I} \in \mathbb{F}_{p}^{[n] \backslash I}$, the function $P\left(x_{I}, y_{[n] \backslash I}\right)$
(viewed as a polynomial in the $x_{I}$ ) can be written as a quasisymmetric polynomial of degree $d$ plus an arbitrary polynomial of degree at most $d-1$.

Proof. We can uniquely express $P$ in the form

$$
P\left(x_{1}, \ldots, x_{n}\right)=\sum_{s=0}^{d} \sum_{1 \leq i_{1}<\cdots<i_{s} \leq n} \sum_{\substack{\alpha \in\{1, \ldots, p-1\}^{s} \\|\alpha| \leq d}} c_{i_{1}, \ldots, i_{s}, \alpha} \prod_{j=1}^{s} x_{i_{j}}^{\alpha_{j}}
$$

where the $c_{i_{1}, \ldots, i_{s}, \alpha} \in \mathbb{F}_{p}$ are arbitrary.
Define $\Lambda=\left\{\lambda \in\{0, \ldots, p-1\}^{d}:|\lambda|=d\right\}$. We define a coloring of the complete $d$-uniform hypergraph on $n$ vertices where the set of colors is $\mathbb{F}_{p}^{\Lambda}$. For an edge $\left\{i_{1}, \ldots, i_{d}\right\}$ with $1 \leq i_{1}<\cdots<$ $i_{d} \leq n$, let the color of this edge be given by $c_{i_{1}, \ldots, i_{d}}: \Lambda \rightarrow \mathbb{F}_{p}$. We define $c_{i_{1}, \ldots, i_{d}}(\lambda)=c_{j_{1}, \ldots, j_{s}, \alpha}$ where $\alpha$ is formed by removing the 0 's from the tuple $\lambda$ and $\left(j_{1}, \ldots, j_{s}\right)$ is formed from $\left(i_{1}, \ldots, i_{d}\right)$ by removing the coordinate $i_{k}$ if $\lambda_{k}=0$.

Applying the hypergraph Ramsey theorem, there exists a subset $I \subseteq[n]$ such that the induced subhypergraph on vertex set $I$ is colored monochromatically with color $c: \Lambda \rightarrow \mathbb{F}_{p}$ and $|I|=$ $\omega_{p, d ; n \rightarrow \infty}(1)$. Unwinding the definitions, we see that

$$
P\left(x_{I}, y_{[n] \backslash I}\right)=\sum_{s=0}^{d} \sum_{\substack{\alpha \in\{1, \ldots, p-1\}^{s} \\|\alpha|=d}} c(\alpha, \underbrace{0, \ldots, 0}_{d-s}) Q_{\alpha}\left(x_{I}\right)+\text { mixed terms. }
$$

Now the mixed terms involve at least one factor of $y_{[n] \backslash I}$, so their total $x_{I}$-degree is strictly smaller than $d$.

## 5. Non-Classical polynomials are necessary

In this section we prove Theorem 1.2, namely that non-classical polynomials are necessary in the $U^{k+1}$ inverse theorem when $k>p$. To do this, we use the function $f_{n}^{(k)}: \mathbb{F}_{p}^{n} \rightarrow \mathbb{R} / \mathbb{Z}$ defined by

$$
\begin{equation*}
f_{n}^{(k)}(x)=\frac{1}{p^{\ell+1}} \sum_{i=1}^{n}\left|x_{i}\right|^{r} \tag{5.1}
\end{equation*}
$$

where $k=r+(p-1) \ell$ with $\ell \geq 1$ and $0<r<p$. Note that $f_{n}^{(k)}$ is a non-classical polynomial of degree $k$, so $\left\|e\left(f_{n}^{(k)}\right)\right\|_{U^{k+1}}=1$.

In order to motivate our proof, suppose for the sake of contradiction that $f_{n}^{(k)}$ has correlation at least $\epsilon$ with some classical polynomial of degree at most $k$. By Theorem 4.2, we will be able to reduce to the situation

$$
\epsilon \leq\left|\mathbb{E}_{x \sim \mathbb{F}_{p}^{n}} e\left(f_{n}^{(k)}(x)+g(x)+h(x)\right)\right|
$$

where $g$ is a homogeneous quasisymmetric polynomial of degree $k$ and $h$ is a classical polynomial of degree at most $k-1$. By the monotonicity of the Gowers norms (alternatively by the Gowers-Cauchy-Schwarz inequality), we deduce

$$
\begin{aligned}
\epsilon^{2^{k}} & \leq\left|\mathbb{E}_{x} e\left(f_{n}^{(k)}(x)+g(x)+h(x)\right)\right|^{2^{k}} \\
& =\mathbb{E}_{x, h_{1}, \ldots, h_{k}}\left(\partial_{h_{1}} \cdots \partial_{h_{k}} e\left(f_{n}^{(k)}+g+h\right)\right)(x) \\
& =\mathbb{E}_{h_{1}, \ldots, h_{k}} e\left(\Delta_{h_{1}} \cdots \Delta_{h_{k}}\left(f_{n}^{(k)}+g\right)\right) .
\end{aligned}
$$

Since $f_{n}^{(k)}$ and $g$ are polynomials of degree $k$, the iterated derivatives $\left(\Delta_{h_{1}} \cdots \Delta_{h_{k}} f_{n}^{(k)}\right)(x)$ and $\left(\Delta_{h_{1}} \cdots \Delta_{h_{k}} g\right)(x)$ are constants independent of $x$. Furthermore, they take values in $\mathbb{U}_{1}$ which (with
some abuse of notation) we identify with $\mathbb{F}_{p}$. Many results on these objects are known, including the fact that in general they are multilinear functions of $h_{1}, \ldots, h_{k}$ (see [7, Section 4]). For the purposes of this paper, it is sufficient to do the following explicit computation.

Lemma 5.1. For $k=r+(p-1) \ell$ with $\ell \geq 0$ and $0<r<p$,

$$
\iota_{k}\left(h_{1}, \ldots, h_{k}\right):=\Delta_{h_{1}} \cdots \Delta_{h_{k}} f_{n}^{(k)}=(-1)^{\ell} r!\sum_{i=1}^{n}\left(h_{1}\right)_{i} \cdots\left(h_{k}\right)_{i},
$$

and for $\alpha=\left(\alpha_{1}, \ldots, \alpha_{s}\right)$ with $\alpha_{1}+\cdots+\alpha_{s}=k$ and $0 \leq \alpha_{i}<p$,

$$
\tau_{\alpha}\left(h_{1}, \ldots, h_{k}\right):=\Delta_{h_{1}} \cdots \Delta_{h_{k}} Q_{\alpha}=\sum_{\pi \in \mathfrak{S}_{k}} \sum_{\vec{\imath}}\left(h_{1}\right)_{i_{\pi(1)}} \cdots\left(h_{k}\right)_{i_{\pi(k)}}
$$

where the sum is over sequences $1 \leq i_{1} \leq \cdots \leq i_{k} \leq n$ that satisfy $i_{\alpha_{1}+\cdots+\alpha_{j}+1}=\cdots=$ $i_{\alpha_{1}+\cdots+\alpha_{j}+\alpha_{j+1}}$ and $i_{\alpha_{1}+\cdots+\alpha_{j}}<i_{\alpha_{1}+\cdots+\alpha_{j}+1}$ for all $j$.
Proof of Lemma 5.1. For classical polynomials $P, Q: \mathbb{F}_{p}^{n} \rightarrow \mathbb{F}_{p}$, of degrees $d_{1}, d_{2}$, the discrete Leibniz rule, $\Delta_{h}(P Q)=\left(\Delta_{h} P\right) Q+P\left(\Delta_{h} Q\right)+\left(\Delta_{h} P\right)\left(\Delta_{h} Q\right)$, can be easily verified. This implies the more convenient $\Delta_{h}(P Q) \equiv\left(\Delta_{h} P\right) Q+P\left(\Delta_{h} Q\right)\left(\bmod\right.$ Poly $\left._{\leqslant d_{1}+d_{2}-2}\left(\mathbb{F}_{p}^{n} \rightarrow \mathbb{F}_{p}\right)\right)$. Note that taking $d$ discrete derivatives kills Poly ${ }_{\leqslant d}\left(\mathbb{F}_{p}^{n} \rightarrow \mathbb{F}_{p}\right)$, so if $P \equiv Q\left(\bmod\right.$ Poly $_{\leqslant d}\left(\mathbb{F}_{p}^{n} \rightarrow \mathbb{F}_{p}\right)$ ), then $\Delta_{h_{1}} \cdots \Delta_{h_{d}} P(x)=\Delta_{h_{1}} \cdots \Delta_{h_{d}} Q(x)$.

We compute $\tau_{\alpha}$ first. Define $f \in \operatorname{Poly}_{\leqslant k}\left(V \rightarrow \mathbb{F}_{p}\right)$ by $f(x)=x_{i_{1}} \cdots x_{i_{k}}$. By many applications of the discrete Leibniz rule, we see that

$$
\Delta_{h_{1}} \cdots \Delta_{h_{k}} f(x)=\sum_{\pi \in \mathfrak{S}_{k}} \Delta_{h_{1}}\left(x_{i_{\pi(1)}}\right) \cdots \Delta_{h_{k}}\left(x_{i_{\pi(k)}}\right)=\sum_{\pi \in \mathfrak{S}_{k}}\left(h_{1}\right)_{i_{\pi(1)}} \cdots\left(h_{k}\right)_{i_{\pi(k)}}
$$

Extending this result by linearity gives the formula for $\tau_{\alpha}$.
Now for $\iota_{k}$. For any $k \geq 1$, write $k=r+(p-1) \ell$ with $0<r<p$ and $\ell \geq 0$. Define $Q_{k} \in$ Poly $_{\leqslant k}\left(\mathbb{F}_{p} \rightarrow \mathbb{R} / \mathbb{Z}\right)$ by $Q_{k}(x)=|x|^{r} / p^{\ell+1}$. By linearity, it suffices to prove that $\Delta_{h_{1}} \cdots \Delta_{h_{k}} Q_{k}(x)=$ $r!(-1)^{\ell} h_{1} \cdots h_{k}$. We prove that $\Delta_{h} Q_{k}(x) \equiv r|h| Q_{k-1}(x)\left(\bmod \operatorname{Poly}_{\leqslant k-2}\left(\mathbb{F}_{p} \rightarrow \mathbb{R} / \mathbb{Z}\right)\right)$ for $k \geq 2$, while obviously $\Delta_{h} Q_{1}(x)=|h| / p$. Iterating (and applying the fact that $\left.(p-1)!\equiv-1(\bmod p)\right)$ gives the desired result.

We break into two cases. First, if $r=1$ (and $\ell \geq 1$ ) then

$$
\begin{aligned}
\Delta_{h} Q_{k}(x) & =\frac{|x+h|-|x|}{p^{\ell+1}} \\
& =\frac{|h|}{p^{\ell+1}}-\frac{\mathbb{1}(|x|+|h| \geq p)}{p^{\ell}} \\
& =\frac{|h|}{p^{\ell+1}}-\sum_{c=p-|h|}^{p-1} \frac{\mathbb{1}(|x|=c)}{p^{\ell}} \\
& =\frac{|h|}{p^{\ell+1}}+|h| \frac{|x|^{p-1}}{p^{\ell}}-\sum_{c=p-|h|}^{p-1} \frac{\mathbb{1}(|x|=c)+|x|^{p-1}}{p^{\ell}} .
\end{aligned}
$$

Now $\mathbb{1}(|x|=c) \equiv 1-(|x|-c)^{p-1}(\bmod p)$, say $\mathbb{1}(|x|=c)=1-(|x|-c)^{p-1}+p E_{c, p}(x)$ for some function $E_{c, p}$. Then we see

$$
\Delta_{h} Q_{k}(x)-|h| \frac{|x|^{p-1}}{p^{\ell}}=\frac{|h|}{p^{\ell+1}}-\sum_{c=p-|h|}^{p-1} \frac{1-(|x|-c)^{p-1}+|x|^{p-1}}{p^{\ell}}-\sum_{c=p-|h|}^{p-1} \frac{E_{c, p}(x)}{p^{\ell-1}} .
$$

We know that every term in this equation is a non-classical polynomial of degree at most $k-1$ except for the last term. Thus we conclude that the last term is also a non-classical polynomial of degree $k-1$. Furthermore, of the three terms on the right-hand side, the first is a constant, the second is a non-classical polynomial of degree at most $(p-1)(\ell-1)+(p-2)=k-2$ (since the $|x|^{p-1} / p^{\ell}$ terms cancel), and the third has degree at most $(p-1)(\ell-1)=k-p$ (since it takes values in $\left.\mathbb{U}_{\ell-1}\right)$. Thus the right-hand side lies in $\operatorname{Poly}_{\leqslant k-2}\left(\mathbb{F}_{p} \rightarrow \mathbb{R} / \mathbb{Z}\right)$, proving the desired result in the $r=1$ case.

Now assume that $k=r+(p-1) \ell$ where $r \geq 2$. We compute

$$
\begin{aligned}
\Delta_{h} Q_{k}(x) & =\frac{|x+h|^{r}-|x|^{r}}{p^{\ell+1}} \\
& =\frac{(|x|+|h|)^{r}-|x|^{r}}{p^{\ell+1}}-\mathbb{1}(|x|+|h| \geq p) \frac{(|x|+|h|)^{r}-(|x|+|h|-p)^{r}}{p^{\ell+1}} \\
& =\frac{\sum_{i=1}^{r}\binom{r}{i}|h|^{i}|x|^{r-i}}{p^{\ell+1}}-\mathbb{1}(|x|+|h| \geq p)\left(\sum_{i=1}^{r} \frac{(-1)^{i-1}\binom{r}{i}(|x|+|h|)^{r-i}}{p^{\ell+1-i}}\right) .
\end{aligned}
$$

We rewrite this as

$$
\Delta_{h} Q_{k}(x)-r|h| \frac{|x|^{r-1}}{p^{\ell+1}}=\frac{\sum_{i=2}^{r}\binom{r}{i}|h|^{i}|x|^{r-i}}{p^{\ell+1}}-\mathbb{1}(|x|+|h| \geq p)\left(\sum_{i=1}^{r} \frac{(-1)^{i-1}\binom{r}{i}(|x|+|h|)^{r-i}}{p^{\ell+1-i}}\right)
$$

We know that every term in this equation is a non-classical polynomial of degree at most $k-1$ except for the last term, implying that the last term is also a non-classical polynomial of degree $k-1$. Furthermore, of the two terms on the right-hand side, the first is a non-classical polynomial of degree at most $(p-1) \ell+r-2=k-2$ and the second has degree at most $(p-1) \ell=k-r$ (since it takes values in $\left.\mathbb{U}_{\ell-1}\right)$. Thus the right-hand side lies in Poly ${ }_{\leqslant k-2}\left(\mathbb{F}_{p} \rightarrow \mathbb{R} / \mathbb{Z}\right)$, proving the desired result in the $r \geq 2$ case.

Define the maps $I_{k}, T_{\alpha}:\left(\mathbb{F}_{p}^{n}\right)^{k-1} \rightarrow \mathbb{F}_{p}^{n}$ by the equations $\iota_{k}\left(h_{1}, \ldots, h_{k}\right)=I_{k}\left(h_{1}, \ldots, h_{k-1}\right)$. $h_{k}$ and $\tau_{\alpha}\left(h_{1}, \ldots, h_{k}\right)=T_{\alpha}\left(h_{1}, \ldots, h_{k-1}\right) \cdot h_{k}$. From Lemma 5.1, clearly $I_{k}\left(h_{1}, \ldots, h_{k-1}\right)_{i}=$ $(-1)^{\ell} r!\left(h_{1}\right)_{i} \cdots\left(h_{k-1}\right)_{i}$. To continue the argument we will need to show that $T_{\alpha}$ can be expressed in a particularly convenient form.

Lemma 5.2. For $\alpha=\left(\alpha_{1}, \ldots, \alpha_{s}\right)$ with $\alpha_{1}+\cdots+\alpha_{s}=k$ and $0<\alpha_{i}<p$ for all $i$,

$$
T_{\alpha}\left(h_{1}, \ldots, h_{k-1}\right)_{i}=\sum_{J \subseteq[k-1]} C_{i, \alpha, J}\left(h_{[k-1],<i},\left(\tau_{\beta}\left(h_{I}\right)\right)_{\beta, I}\right) \prod_{j \in J}\left(h_{j}\right)_{i}
$$

for some functions $C_{i, \alpha, J}(\cdot, \cdot)$, evaluated at the tuple of $\left(h_{j}\right)_{i^{\prime}}$ for all $j \in[k-1]$ and $i^{\prime}<i$ and the tuple of $\tau_{\beta}\left(h_{I}\right)$ for all $I \subseteq[k-1]$ and $\beta=\left(\beta_{1}, \ldots, \beta_{t}\right)$ with $\beta_{1}+\cdots+\beta_{t}=|I|$ and $0<\beta_{i}<p$ for all i. Furthermore,

$$
C_{i, \alpha,[k-1]}=(-1)^{s-1} \alpha_{1}(k-1)!
$$

Proof. Fix $i \in[n]$ for the rest of the proof. We introduce some notation. We have $h_{1}, \ldots, h_{k-1} \in \mathbb{F}_{p}^{n}$. For $I \subseteq[k-1]$, we write $h_{I}=\left(h_{j}\right)_{j \in I}$. We use

$$
h_{I,<}=\left(\left(h_{j}\right)_{1}, \ldots,\left(h_{j}\right)_{i-1}\right)_{j \in I} \in\left(\mathbb{F}_{p}^{i-1}\right)^{I} \quad \text { and } \quad h_{I,>}=\left(\left(h_{j}\right)_{i+1}, \ldots,\left(h_{j}\right)_{n}\right)_{j \in I} \in\left(\mathbb{F}_{p}^{n-i}\right)^{I}
$$

For $\alpha=\left(\alpha_{1}, \ldots, \alpha_{s}\right)$ we write $\alpha_{<\ell}=\left(\alpha_{1}, \ldots, \alpha_{\ell-1}\right)$ and $\alpha_{>\ell}=\left(\alpha_{\ell+1}, \ldots, \alpha_{s}\right)$. We define $\alpha_{\leqslant \ell}$ analogously. Recall that we use $|\alpha|=\alpha_{1}+\cdots+\alpha_{s}$.

By inspection, we can write

$$
\begin{equation*}
T_{\alpha}\left(h_{[k-1]}\right)_{i}=\sum_{\ell=1}^{s} \alpha_{\ell}!\sum_{\substack{I \sqcup J \cup K=[k-1]: \\|I|=\left|\alpha_{\ll l}\right|,|J|=\alpha \ell-1,|K|=\left|\alpha_{>\ell}\right|}}\left(\prod_{j \in J}\left(h_{j}\right)_{i}\right) \tau_{\alpha_{<\ell}}\left(h_{I,<}\right) \tau_{\alpha>\ell}\left(h_{K,>}\right) . \tag{5.2}
\end{equation*}
$$

We now remove the terms depending on $h_{[k-1],>}$. Take $\beta=\left(\beta_{1}, \ldots, \beta_{t}\right)$ with $0<\beta_{i}<p$ for all $i$ and $L \subseteq[k-1]$ with $|L|=|\beta|$. We have the identity

$$
\begin{align*}
& \tau_{\beta}\left(h_{L,>}\right)=\tau_{\beta}\left(h_{L}\right)-\sum_{\substack{ \\
\ell}} \sum_{\substack{I \cup K=L \dot{1} \\
|I|=|\beta \leqslant \ell\\
| K\left|=\left|\beta_{>}\right|\right.}} \tau_{\beta \leqslant \ell}\left(h_{I,<}\right) \tau_{\beta_{>\ell}}\left(h_{K,>}\right) \\
& -\sum_{\ell=1}^{t} \beta_{\ell}!\sum_{\substack{I \cup J \cup K=L: \\
|I|=\left|\beta_{\ell}\right|,|J|=\beta_{\ell},|K|=\left|\beta_{>}\right|}}\left(\prod_{j \in J}\left(h_{j}\right)_{i}\right) \tau_{\beta_{<\ell}}\left(h_{I,<}\right) \tau_{\beta_{>\ell}}\left(h_{K,>}\right) . \tag{5.3}
\end{align*}
$$

Repeatedly applying this identity eventually puts $T_{\alpha}\left(h_{[k-1]}\right)_{i}$ into the desired form. To see this, note that applying this identity to $\tau_{\beta}\left(h_{L,>}\right)$ produces many terms of the form $\tau_{\beta>\ell}\left(h_{K,>}\right)$ but all of these satisfy $\left|\beta_{>\ell}\right|=|K|<|\beta|=|L|$, so we always make progress.

Finally we need to compute $C_{i, \alpha,[k-1]}$. Obviously this coefficient is a constant since the final decomposition that we produce is multilinear in the $h_{1}, \ldots, h_{k-1}$. Furthermore, the only way to produce a term that is a multiple of $\left(h_{1}\right)_{i} \cdots\left(h_{k-1}\right)_{i}$ is to have no factors of $\tau_{\beta_{<\ell}}\left(h_{I, \ll}\right)$ in that term. (However, we have a choice of $I, J, K$ in (5.2).) This means that in the initial decomposition we need to be in the $\ell=1$ case of the sum and every time we use the identity (5.3) we need to be in the $\ell=1$ case of the second sum. Again, in every subsequent choice although $\ell=1$ is fixed, we have a choice of $I, J, K$. Tracing through all of these reductions, we see that we produce the coefficient

$$
\alpha_{1}!\binom{k-1}{\alpha_{1}-1} \prod_{j=2}^{s}\left(-\left(\alpha_{j}!\right)\binom{k-\alpha_{1}-\cdots-\alpha_{j-1}}{\alpha_{j}}\right)=(-1)^{s-1} \alpha_{1}(k-1)!.
$$

So far we have developed the tools to, starting with the assumption of correlation with a classical polynomial, reduce to a situation in which

$$
\begin{aligned}
\epsilon^{2^{k}} & \leq \mathbb{E}\left[e_{p}\left(\iota_{k}-\sum_{\alpha} c_{\alpha} \tau_{\alpha}\right)\right]=\mathbb{P}\left(I_{k}+\sum_{\alpha} c_{\alpha} T_{\alpha}=0\right) \\
& =\mathbb{P}_{h_{1}, \ldots, h_{k-1} \sim \mathbb{F}_{p}^{n}}\left(\forall i \in[n], I_{k}\left(h_{1}, \ldots, h_{k-1}\right)_{i}+\sum_{\alpha} c_{\alpha} T_{\alpha}\left(h_{1}, \ldots, h_{k-1}\right)_{i}=0\right) .
\end{aligned}
$$

Therefore, we need a bound on the probability that a multiaffine function equals zero.
Lemma 5.3. Let $L: \mathbb{F}_{p}^{r} \rightarrow \mathbb{F}_{p}$ be a multiaffine function whose leading coefficient (i.e., coefficient of $x_{1} \cdots x_{r}$ ) is non-zero. Then

$$
\mathbb{P}_{x_{1}, \ldots, x_{r} \sim \mathbb{F}_{p}}\left(L\left(x_{1}, \ldots, x_{r}\right)=0\right) \leq 1-\left(1-\frac{1}{p}\right)^{r}=: 1-c_{p, r}
$$

Proof. We prove this result by induction on $r$. The bound is trivially true for $r=0$.

For $r \geq 1$, we can write

$$
L\left(x_{1}, \ldots, x_{r}\right)=x_{r} M\left(x_{1}, \ldots, x_{r-1}\right)+N\left(x_{1}, \ldots, x_{r-1}\right)
$$

where $M$ and $N$ are multiaffine and the leading coefficient of $M$ is non-zero. Then for each fixed $x_{1}, \ldots, x_{r-1}$, there is at most 1 choice of $x_{r}$ that makes $L$ vanish unless $M\left(x_{1}, \ldots, x_{r}\right)=0$. Then

$$
\mathbb{P}_{x_{1}, \ldots, x_{r} \sim \mathbb{F}_{p}}\left(L\left(x_{1}, \ldots, x_{r}\right) \neq 0\right) \geq\left(1-\frac{1}{p}\right) \mathbb{P}_{x_{1}, \ldots, x_{r-1} \sim \mathbb{F}_{p}}\left(M\left(x_{1}, \ldots, x_{r-1}\right) \neq 0\right)
$$

The second term can be handled by the inductive hypothesis.
We now have the tools to prove the main theorem. The probability we are considering is the probability that $n$ multiaffine functions vanish simultaneously. If these were independent, by the above lemma, we could bound the probability by $\left(1-c_{p, k}\right)^{n}=o_{p, k ; n \rightarrow \infty}(1)$.

In order to introduce such independence, we can take a union bound over all possible $\tau_{\beta}\left(h_{I}\right)$. Then Lemma 5.2 shows that our multiaffine forms have the following property: if we plug in values for $\left(\left(h_{1}\right)_{i^{\prime}}, \ldots,\left(h_{k-1}\right)_{i^{\prime}}\right)_{i^{\prime}<i}$ and $\tau_{\beta}\left(h_{I}\right)$ for all $\beta, I$, then $T_{\alpha}\left(h_{1}, \ldots, h_{k-1}\right)_{i}$ is multiaffine in $\left(h_{1}\right)_{i}, \ldots,\left(h_{k-1}\right)_{i}$ with non-zero leading coefficient. Then we may reveal $\left(\left(h_{1}\right)_{i}, \ldots,\left(h_{k-1}\right)_{i}\right)$ one-by-one for $i \in[n]$, and find that the total probability is bounded by $\left(1-c_{p, k}\right)^{n}$. As the number of possible choices in the union bound is $O_{p, k}(1)$ we will be able to prove the desired bound.

Proof of Theorem 1.2. Take $k \geq p+1$. Consider $f_{n}^{(k)}: \mathbb{F}_{p}^{n} \rightarrow \mathbb{R} / \mathbb{Z}$ defined in (5.1). Since $f_{n}^{(k)}$ is a non-classical polynomial of degree $k$, we know that $\left\|e\left(f_{n}^{(k)}\right)\right\|_{U^{k+1}}=1$. For a classical polynomial $P \in \operatorname{Poly}_{\leqslant k}\left(\mathbb{F}_{p}^{n} \rightarrow \mathbb{F}_{p}\right)$, set $\epsilon=\left|\mathbb{E}_{x} e\left(f_{n}^{(k)}(x)+|P(x)| / p\right)\right|$. We will prove that $\epsilon=o_{p, k ; n \rightarrow \infty}(1)$.

By Theorem 4.2, there exists $m=\omega_{p, k ; n \rightarrow \infty}(1)$ and a subset $I \subseteq[n]$ such that for all $y_{[n] \backslash I} \in \mathbb{F}_{p}^{[n] \backslash I}$,

$$
P\left(x_{I}, y_{[n] \backslash I}\right)=Q\left(x_{I}\right)+P_{y_{[n] \backslash I}}\left(x_{I}\right)
$$

where $Q$ is a homogeneous quasisymmetric polynomial of degree $k$ and $P_{y_{[n] \backslash I}}$ is a polynomial of degree at most $k-1$.

Without loss of generality, assume that $I=[m]$. Then

$$
\epsilon=\left|\mathbb{E}_{x \sim \mathbb{F}_{p}^{n}} e\left(f_{n}^{(k)}(x)+|P(x)| / p\right)\right| \leq \mathbb{E}_{y \sim \mathbb{F}_{p}^{n-m}}\left|\mathbb{E}_{x \sim \mathbb{F}_{p}^{m}} e\left(f_{n}^{(k)}(x, y)+|P(x, y)| / p\right)\right|
$$

Then by the pigeonhole principle, there exists $y \in \mathbb{F}_{p}^{n-m}$ such that the inner expectation is at least $\epsilon$. Fix this choice of $y$ for the rest of the proof. Note that $f_{n}^{(k)}(x, y)=f_{m}^{(k)}(x)+c_{y}$ where $c_{y}=f_{n-m}^{(k)}(y) \in \mathbb{R} / \mathbb{Z}$ is a constant. Thus we have

$$
\epsilon \leq\left|\mathbb{E}_{x \sim \mathbb{F}_{p}^{m}} e\left(f_{m}^{(k)}(x)+|Q(x)| / p+\left|P_{y}(x)\right| / p+c_{y}\right)\right| .
$$

Note that the right-hand side is a $U^{1}$-norm. Using the monotonicity of the Gowers norms (see, e.g., [7, Lemma B.1.(ii)]) we deduce

$$
\begin{aligned}
\epsilon^{2^{k}} & \leq\left\|e\left(f_{m}^{(k)}+|Q| / p+\left|P_{y}\right| / p+c_{y}\right)\right\|_{U^{1}}^{2^{k}} \\
& \leq\left\|e\left(f_{m}^{(k)}+|Q| / p+\left|P_{y}\right| / p+c_{y}\right)\right\|_{U^{k}}^{2^{k}} \\
& =\mathbb{E}_{x, h_{1}, \ldots, h_{k}} \partial_{h_{1}} \cdots \partial_{h_{k}} e\left(f_{m}^{(k)}+|Q| / p+\left|P_{y}\right| / p+c_{y}\right)(x) \\
& =\mathbb{E}_{x, h_{1}, \ldots, h_{k}} e\left(\Delta_{h_{1}} \cdots \Delta_{h_{k}}\left(f_{m}^{(k)}+|Q| / p+\left|P_{y}\right| / p+c_{y}\right)(x)\right) .
\end{aligned}
$$

Taking $k$ discrete derivatives kills (non-classical) polynomials of degree at most $k-1$ and turns those of degree $k$ into constants. Thus the final expression is equal to

$$
\mathbb{E}_{h_{1}, \ldots, h_{k}} e\left(\Delta_{h_{1}} \cdots \Delta_{9}\left(f_{m}^{(k)}+|Q| / p\right)\right) .
$$

Since $Q$ is a homogeneous quasisymmetric polynomial of degree $k$, it can be written as $\sum_{\alpha} c_{\alpha} Q_{\alpha}$ where $\alpha$ ranges over all tuples $\left(\alpha_{1}, \ldots, \alpha_{s}\right)$ with $|\alpha|=k$ and $0<\alpha_{i}<p$ for all $i$ and the $c_{\alpha} \in \mathbb{F}_{p}$ are arbitrary coefficients.

We computed $\Delta_{h_{1}} \cdots \Delta_{h_{k}} f_{m}^{(k)}$ and $\Delta_{h_{1}} \cdots \Delta_{h_{k}} Q_{\alpha}$ in Lemma 5.1. These are the $k$-linear forms denoted $\iota_{k}, \tau_{\alpha}:\left(\mathbb{F}_{p}^{m}\right)^{k} \rightarrow \mathbb{F}_{p}$ respectively. Thus so far we have shown that

$$
\epsilon^{2^{k}} \leq \mathbb{E}_{h_{1}, \ldots, h_{k}} e_{p}\left(\iota_{k}\left(h_{1}, \ldots, h_{k}\right)+\sum_{\alpha} c_{\alpha} \tau_{\alpha}\left(h_{1}, \ldots, h_{k}\right)\right) .
$$

For an arbitrary $k$-linear form $\sigma:\left(\mathbb{F}_{p}^{m}\right)^{k} \rightarrow \mathbb{F}_{p}$, there is a unique $(k-1)$-linear function $S:\left(\mathbb{F}_{p}^{m}\right)^{k-1} \rightarrow$ $\mathbb{F}_{p}^{m}$ that satisfies $\sigma\left(h_{1}, \ldots, h_{k}\right)=S\left(h_{1}, \ldots, h_{k-1}\right) \cdot h_{k}$. Furthermore, we have

$$
\mathbb{E}_{h_{1}, \ldots, h_{k}} e_{p}\left(\sigma\left(h_{1}, \ldots, h_{k}\right)\right)=\mathbb{E}_{h_{1}, \ldots, h_{k}} e_{p}\left(S\left(h_{1}, \ldots, h_{k-1}\right) \cdot h_{k}\right)=\mathbb{P}_{h_{1}, \ldots, h_{k-1}}\left(S\left(h_{1}, \ldots, h_{k-1}\right)=0\right) .
$$

From this we conclude

$$
\epsilon^{2^{k}} \leq \mathbb{P}_{h_{1}, \ldots, h_{k-1}}\left(\forall i \in[m], I_{k}\left(h_{1}, \ldots, h_{k-1}\right)_{i}+\sum_{\alpha} c_{\alpha} T_{\alpha}\left(h_{1}, \ldots, h_{k-1}\right)_{i}=0\right)
$$

Recall that $I_{k}\left(h_{1}, \ldots, h_{k-1}\right)_{i}=(-1)^{\ell} r!\left(h_{1}\right)_{i} \cdots\left(h_{k-1}\right)_{i}$ where $k=r+(p-1) \ell$ with $\ell \geq 1$ and $0<r<p$. Note that $(-1)^{\ell} r!\neq 0$ in $\mathbb{F}_{p}$. Furthermore, Lemma 5.2 states that

$$
T_{\alpha}\left(h_{1}, \ldots, h_{k-1}\right)_{i}=\sum_{J \subseteq[k-1]} C_{i, \alpha, J}\left(h_{[k-1],<i},\left(\tau_{\beta}\left(h_{I}\right)\right)_{\beta, I}\right) \prod_{j \in J}\left(h_{j}\right)_{i} .
$$

In other words $T_{\alpha}\left(h_{1}, \ldots, h_{k-1}\right)_{i}$, viewed just as a function of $\left(h_{1}\right)_{i}, \ldots,\left(h_{k-1}\right)_{i}$ is multiaffine with coefficients given by $C_{i, \alpha, J}$. Additionally, Lemma 5.2 also gives the critical fact that the leading coefficient, $C_{i, \alpha,[k-1]}$, is equal to $(-1)^{s-1} \alpha_{1}(k-1)$ ! for all $i$. Since $k \geq p+1$, we have that $C_{i, \alpha,[k-1]}=0$ (recall that the coefficients live in $\mathbb{F}_{p}$ ).

This implies that $I_{k}\left(h_{1}, \ldots, h_{k-1}\right)_{i}+\sum_{\alpha} c_{\alpha} T_{\alpha}\left(h_{1}, \ldots, h_{k-1}\right)_{i}$, viewed just as a function of $\left(h_{1}\right)_{i}, \ldots,\left(h_{k-1}\right)_{i}$ is multiaffine with non-zero leading coefficient, say

$$
I_{k}\left(h_{1}, \ldots, h_{k-1}\right)_{i}+\sum_{\alpha} c_{\alpha} T_{\alpha}\left(h_{1}, \ldots, h_{k-1}\right)_{i}=\sum_{J \subseteq[k-1]} C_{i, J}\left(h_{[k-1],<i},\left(\tau_{\beta}\left(h_{I}\right)\right)_{\beta, I}\right) \prod_{j \in J}\left(h_{j}\right)_{i}
$$

where $C_{i,[k-1]}=(-1)^{\ell} r!\neq 0$ for all $i$.
By Lemma 5.3, if the coefficients are fixed then this function vanishes with probability at most $1-c_{p, k}<1$. To complete the proof, we need to show that we can approximately decouple these events. Formally,

$$
\begin{aligned}
\epsilon^{2^{k}} & \leq \mathbb{P}_{h_{1}, \ldots, h_{k-1}}\left(\forall i \in[m], \sum_{J \subseteq[k-1]} C_{i, J}\left(h_{[k-1],<i},\left(\tau_{\beta}\left(h_{I}\right)\right)_{\beta, I}\right) \prod_{j \in J}\left(h_{j}\right)_{i}=0\right) \\
& =\sum_{\left(A_{\beta, I}\right)_{\beta, I}} \mathbb{P}_{h_{1}, \ldots, h_{k-1}}\left(\forall \beta, I, \tau_{\beta}\left(h_{I}\right)=A_{\beta, I} \cap \sum_{J \subseteq[k-1]} C_{i, J}\left(h_{[k-1],<i},\left(\tau_{\beta}\left(h_{I}\right)\right)_{\beta, I}\right) \prod_{j \in J}\left(h_{j}\right)_{i}=0\right) \\
& \leq \sum_{\left(A_{\beta, I}\right)_{\beta, I}} \mathbb{P}_{h_{1}, \ldots, h_{k-1}}\left(\sum_{J \subseteq[k-1]} C_{i, J}\left(h_{[k-1],<i},\left(A_{\beta, I}\right)_{\beta, I}\right) \prod_{j \in J}\left(h_{j}\right)_{i}=0\right) .
\end{aligned}
$$

The final replacement simply comes by substituting in the values $A_{\beta, I}$.

Now for each $i \in[m]$, let $E_{i}$ be the event that $\sum_{J \subseteq[k-1]} C_{i, J}\left(h_{[k-1],<i},\left(A_{\beta, I}\right)_{\beta, I}\right) \prod_{j \in J}\left(h_{j}\right)_{i}=0$. We wish to bound

$$
\mathbb{P}_{h_{1}, \ldots, h_{k-1}}\left(E_{i} \mid \forall i^{\prime}<i, E_{i^{\prime}}\right) .
$$

Since the event we are conditioning on only depends on $h_{[k-1],<i}$, the conditional distribution of $\left(h_{1}\right)_{i}, \ldots,\left(h_{k-1}\right)_{i}$ is still uniform. Thus we can upper bound the above probability by

$$
\sup _{h_{[k-1],<i}} \mathbb{P}_{\left(h_{1}\right)_{i}, \ldots,\left(h_{k-1}\right)_{i} \sim \mathbb{F}_{p}}\left(\sum_{J \subseteq[k-1]} C_{i, J}\left(h_{[k-1],<i},\left(A_{\beta, I}\right)_{\beta, I}\right) \prod_{j \in J}\left(h_{j}\right)_{i}=0\right) .
$$

By Lemma 5.3, and the fact that $C_{i,[k-1]}=(-1)^{\ell} r!\neq 0$ always, this probability is upper-bounded by $1-c_{p, k}<1$. Putting everything together, we have shown that $\epsilon^{2^{k}} \leq O_{p, k}\left(\left(1-c_{p, k}\right)^{m}\right)$. (The hidden constant is the number of terms in the sum over $\left(A_{\beta, I}\right)_{\beta, I}$, which depends on $p, k$ but not on $m, n$. It can be bounded by $p^{4^{k}}$.) We showed that $m=\omega_{p, k ; n \rightarrow \infty}(1)$, implying that $\epsilon=o_{p, k ; n \rightarrow \infty}(1)$.

## References

[1] Noga Alon and Richard Beigel, Lower bounds for approximations by low degree polynomials over $\mathbb{Z} / m \mathbb{Z}$, Proceedings 16th Annual IEEE Conference on Computational Complexity, IEEE, 2001, pp. 184-187.
[2] Vitaly Bergelson, Terence Tao, and Tamar Ziegler, An inverse theorem for the uniformity seminorms associated with the action of $\mathbb{F}_{p}^{\infty}$, Geom. Funct. Anal. 19 (2010), 1539-1596.
[3] Ben Green and Terence Tao, The distribution of polynomials over finite fields, with applications to the Gowers norms, Contrib. Discrete Math. 4 (2009), 1-36.
[4] Shachar Lovett, Roy Meshulam, and Alex Samorodnitsky, Inverse conjecture for the Gowers norm is false, Theory Comput. 7 (2011), 131-145.
[5] Alex Samorodnitsky, Low-degree tests at large distances, STOC'07-Proceedings of the 39th Annual ACM Symposium on Theory of Computing, ACM, New York, 2007, pp. 506-515.
[6] Terence Tao and Tamar Ziegler, The inverse conjecture for the gowers norm over finite fields via the correspondence principle, Analysis \& PDE 3 (2010), 1-20.
[7] Terence Tao and Tamar Ziegler, The inverse conjecture for the Gowers norm over finite fields in low characteristic, Ann. Comb. 16 (2012), 121-188.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
Email address: \{bergera, asah,msawhney,jtidor\}@mit.edu


[^0]:    Berger, Sah, Sawhney, and Tidor were supported by NSF Graduate Research Fellowship Program DGE-1745302.
    ${ }^{1}$ See Section 2 for the definitions and notation used in the statement of these results.

