ANTICONCENTRATION VERSUS THE NUMBER OF SUBSET SUMS

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Abstract. Let $\vec{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$. We show that for any $n^{-2} \le \epsilon \le 1$, if

$$\#\{\vec{\xi} \in \{0,1\}^n : \langle \vec{\xi}, \vec{w}\rangle = r\} \ge 2^{-\epsilon n} \cdot 2^n$$

for some $r \in \mathbb{R}$, then

$$\#\{\langle \vec{\xi}, \vec{w}\rangle : \vec{\xi} \in \{0,1\}^n\} \le 2^{O(\sqrt{\epsilon} n)}.$$

This exponentially improves a recent result of Nederlof, Pawlewicz, Swennenhuis, and Węgrzycki and leads to a similar improvement in the parameterized (by the number of bins) runtime of bin packing.

1. Introduction

For $\vec{w} := (w_1, \dots, w_n) \in \mathbb{R}^n$ and a real random variable ξ, recall that the Lévy concentration function of $\vec{w}$ with respect to ξ is defined for all $r \ge 0$ by

$$\mathcal{L}_\xi(\vec{w}, r) = \sup_{\tau \in \mathbb{R}} \mathbb{P}[|w_1\xi_1 + \dots + w_n\xi_n - \tau| \le r],$$

where $\xi_1, \dots, \xi_n$ are i.i.d. copies of ξ. In combinatorial settings (where $\vec{w} \in \mathbb{Z}^n$) a particularly natural and interesting case is when $r = 0$ and ξ is a Bernoulli random variable, i.e., $\xi = 0$ with probability $1/2$ and $\xi = 1$ with probability $1/2$. For lightness of notation, we will denote this special case by

$$\rho(\vec{w}) = \mathcal{L}_{\mathrm{Rad}}(\vec{w}, 0) = \sup_{\tau \in \mathbb{R}} \mathbb{P}[\langle \vec{w}, \vec{\xi}\rangle = \tau].$$

In this note, we study the following question.

Question 1.1. For a vector $\vec{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$ with $\rho(\vec{w}) \ge \rho$, how large can the range

$$\mathcal{R}(\vec{w}) = \{w_1\xi_1 + \dots + w_n\xi_n : \xi_i \in \{0,1\}\}$$

be?

The two extremal examples here are $\vec{w} = (0, 0, \dots, 0)$, which corresponds to $\rho(\vec{w}) = 1$, $|\mathcal{R}(\vec{w})| = 1$ and $\vec{w} = (1, 10, \dots, 10^{n-1})$, which corresponds to $\rho(\vec{w}) = 2^{-n}$, $|\mathcal{R}(\vec{w})| = 2^n$. Motivated by these examples, one may ask if there is a smooth trade-off between $\rho(\vec{w})$ and $|\mathcal{R}(\vec{w})|$. This turns out not to be the case. Indeed, for any $\epsilon > 0$, Wiman [6] gives an example of a $\vec{w} \in \mathbb{Z}^n$ for which $|\mathcal{R}(\vec{w})| \ge 2^{(1-\epsilon)n}$ and $\rho(\vec{w}) \ge 2^{-0.7447n}$. At the other end of the spectrum, when $\rho(\vec{w}) \ge 2^{-\epsilon n}$, the so-called inverse Littlewood–Offord theory [4,5] *heuristically* suggests that $\vec{w}$ is essentially contained in a low-rank generalized arithmetic progression of 'small' volume so that $|\mathcal{R}(\vec{w})|$ is also 'small'. However, the number of 'exceptional elements' in the inverse Littlewood–Offord theorems (cf. [3]) is unfortunately too large to be able to rigorously establish such a statement.

Nevertheless, in a recent work on the parameterized complexity of the bin packing problem (see Section 1.1), Nederlof, Pawlewics, Swennenhuis and Węgrzycki [2] showed that for any $\epsilon > 0$,

$$\rho(\vec{w}) \ge 2^{-\epsilon n} \implies |R(\vec{w})| \le 2^{\delta(\epsilon) n},$$

where

$$\delta(\epsilon) = O\left(\frac{\log\log(\epsilon^{-1})}{\sqrt{\log(\epsilon^{-1})}}\right). \tag{1.1}$$

In particular, $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. Moreover, we must have $\delta(\epsilon) \geq (2 - o(1))\epsilon$, as can be seen by considering

$$\vec{w} = (C_1, \ldots, C_1, C_2, \ldots, C_2, \ldots, C_{n/k}, \ldots, C_{n/k}) \in \mathbb{R}^n,$$

where each C_i is repeated k times, and C_i is sufficiently small compared to C_{i+1} for all i. Indeed, for such $\vec{w}$, we have $\rho(\vec{w}) = 2^{-(\frac{1}{2}+o_k(1))\frac{n}{k}\log_2 k}$ while $|\mathcal{R}(\vec{w})| \leq 2^{(1+o_k(1))\frac{n}{k}\log_2 k}$.

We conjecture that this example is essentially the worst possible, so that $\delta(\epsilon) \leq 2\epsilon$. We are able to show that

$$\delta(\epsilon) = O(\sqrt{\epsilon}), \tag{1.2}$$

thereby obtaining an exponential improvement over (1.1). More precisely,

Theorem 1.2. *Let $\epsilon > 0$. For any $n \geq \epsilon^{-1/2}$ and any $\vec{w} \in \mathbb{R}^n$ satisfying $\rho(\vec{w}) \geq \exp(-\epsilon n)$, we have*

$$|\mathcal{R}(\vec{w})| \leq \exp(C_{1.2}\epsilon^{1/2}n),$$

where $C_{1.2}$ is an absolute constant.

We prove this theorem in Section 2.

1.1. **Application to bin packing.** The bin packing problem is a classic NP-complete problem whose decision version may be stated as follows: given n items with weights $w_1, \ldots, w_n \in [0,1]$ and m bins, each of capacity 1, is there a way to assign the items to the bins without violating the capacity constraints? Formally, is there a map $f : [n] \to [m]$ such that $\sum_{i \in f^{-1}(j)} w_i \leq 1$ for all $j \in [m]$?

Björklund, Husfeldt, and Koivisto [1] provided an algorithm for solving bin packing in time $\tilde{O}(2^n)$ where the tilde hides polynomial factors in n. It is natural to ask whether the base of the exponent may be improved at all i.e. is there a (possibly randomized) algorithm to solve bin packing in time $\tilde{O}(2^{(1-\epsilon)n})$ for some absolute constant $\epsilon > 0$?

In recent work, Nederlof, Pawlewics, Swennenhuis and Węgrzycki [2] showed that this is true provided that the number of bins m is fixed. More precisely, they showed that there exists a function $\sigma : \mathbb{N} \to \mathbb{R}^{>0}$ and an algorithm for solving bin packing which, on instances with m bins, runs in time $\tilde{O}(2^{(1-\sigma(m))n})$, where $\tilde{O}$ hides polynomials in n as well as exponential factors in m. Their analysis, which crucially relies on (1.1), gives a very small value of $\sigma(m)$ satisfying

$$\sigma(m) \leq 2^{-1/m^9}. \tag{1.3}$$

Using Theorem 1.2 instead of (1.1) in a black-box manner in the analysis of [2], we exponentially improve the bound on $\sigma(m)$.

Corollary 1.3. *With notation as above, the algorithm of [2] solves bin packing instances with m bins in time $\tilde{O}(2^{(1-\sigma(m))n})$ for $\sigma\colon \mathbb{N} \to \mathbb{R}^{>0}$ satisfying*

$$\sigma(m) = \tilde{\Omega}(m^{-12}), \tag{1.4}$$

where $\tilde{\Omega}$ hides logarithmic factors in m.

We remark that the conjecturally optimal bound $\delta = O(\epsilon)$, plugged into the analysis of [2], would lead to $\sigma(m) = \tilde{\Omega}(m^{-6})$.

1.2. **Notation.** We use $\mathrm{Ber}(1/2)$ to denote the balanced $\{0,1\}$ Bernoulli distribution and $\mathrm{Bin}(k)$ to denote the binomial distribution on k trials with parameter $1/2$. Recall that $\mathrm{Bin}(k)$ is the sum of k independent $\mathrm{Ber}(1/2)$ random variables. We also use the following standard additive combinatorics notation: $C + D = \{c + d : c \in C, d \in D\}$ is the sumset (if C, D are subsets of the same abelian group), and for a positive integer k, $k \cdot C = C + \cdots + C$ (k times) is the iterated sumset.

2. Proof of Theorem 1.2

We begin by recording the following key comparison bound, which will be proved at the end of this section.

Lemma 2.1. *Let $n \ge k \ge C_{2.1}$, where $C_{2.1}$ is an absolute constant and let $\delta > 0$. For any $A \subseteq \{0,1\}^n$ with $|A| \le \exp(\delta n)$, the following holds. Let $\vec{x}, \vec{b} \sim \mathrm{Bin}(k)^{\otimes n}$ be independent n-dimensional random vectors. Then,*

$$\mathbb{E}_{\vec{x}}\left[\sup_{\vec{a}\in A}\frac{\mathbb{P}_{\vec{b}}[\vec{b} = \vec{x} - \vec{a}]}{\mathbb{P}_{\vec{b}}[\vec{b} = \vec{x}]}\right] \le \exp\left(C_{2.1}\left(\frac{1}{k} + \sqrt{\frac{\delta}{k}}\right) n\right).$$

Let $\vec{w}$ be as in Theorem 1.2. Let τ be such that $\mathbb{P}[\langle \vec{w}, \vec{\xi}\rangle = \tau] = \rho(\vec{w})$, where $\vec{\xi}$ is a random vector with i.i.d. $\mathrm{Ber}(1/2)$ components. Let

$$B = \{\vec{\xi} \in \{0,1\}^n : \langle \vec{w}, \vec{\xi}\rangle = \tau\}.$$

In particular, $|B| \ge \exp(-\epsilon n) \cdot 2^n$. Let $|\mathcal{R}(\vec{w})| = \exp(\delta n)$. For each $r \in \mathcal{R}(\vec{w})$, let $\vec{\xi}(r)$ be a fixed (but otherwise arbitrary) element of $\{0,1\}^n$ such that $\langle \vec{w}, \vec{\xi}(r)\rangle = r$. Let

$$A = \{\vec{\xi}(r) \in \{0,1\}^n : r \in \mathcal{R}(\vec{w})\}.$$

Note that, by definition, for any distinct $\vec{a}_1, \vec{a}_2 \in A$, we have that $\langle \vec{w}, \vec{a}_1\rangle \ne \langle \vec{w}, \vec{a}_2\rangle$ and that $|A| = |\mathcal{R}(\vec{w})| = \exp(\delta n)$.

We will make use of the simple, but crucial, observation from [2] that A and $k \cdot B$ have a full sumset for all $k \ge 1$.

Lemma 2.2 ([2, Lemma 4.2])**.** *The map $(\vec{a}, \vec{c}) \mapsto \vec{a} + \vec{c}$ from $A \times (k \cdot B)$ to $A + k \cdot B$ is injective.*

Proof. Indeed, if $\vec{a}_1 + (\vec{b}_1^{(1)} + \cdots + \vec{b}_k^{(1)}) = \vec{a}_2 + (\vec{b}_1^{(2)} + \cdots + \vec{b}_k^{(2)})$, where $\vec{a}_i \in A$ and $\vec{b}_j^{(i)} \in B$, then taking the inner product of both sides with $\vec{w}$ and using $\langle \vec{w}, \vec{b}\rangle = \tau$ for all $b \in B$, we see that $\langle \vec{w}, \vec{a}_1\rangle = \langle \vec{w}, \vec{a}_2\rangle$, which implies that $\vec{a}_1 = \vec{a}_2$ by the definition of A. $\square$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $k \ge 2$ be a parameter which will be chosen later depending on ϵ. By Lemma 2.2, for each $\vec{x} \in \{0, \dots, k+1\}^n$ for which there exist $\vec{a} \in A$ and $\vec{c} \in k \cdot B$ with $\vec{a} + \vec{c} = \vec{x}$, there exists a unique such choice $\vec{a} = \vec{a}(\vec{x}) \in A$.

Now, let $\vec{a}$ be uniform on A, let $\vec{b}_1, \dots, \vec{b}_k$ be uniform on B, and let $\vec{v}_1, \dots, \vec{v}_k$ be uniform on $\{0,1\}^n$. Let $C_i \subseteq \{0, \dots, k+1\}^n$ be the set of vectors with i coordinates equal to $k+1$. For $\vec{x} \in \{0, \dots, k+1\}^n$, we let $\vec{x}^* \in \{0, \dots, k\}^n$ denote the vector obtained by setting every occurrence of $k+1$ in $\vec{x}$ to k. We have

$$\begin{aligned}1 &= \mathbb{P}[\vec{a} + \vec{b}_1 + \cdots + \vec{b}_k \in \{0, \dots, k+1\}^n] \\ &= \sum_{i=0}^{n}\sum_{\vec{x}\in C_i}\mathbb{P}[\vec{a} + \vec{b}_1 + \cdots + \vec{b}_k = \vec{x}]\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{n} \sum_{\vec{x}\in C_i} \mathbb{P}[\vec{a} = \vec{a}(\vec{x})]\mathbb{P}[\vec{b}_1 + \cdots + \vec{b}_k = \vec{x} - \vec{a}(\vec{x})] \\
&\leq \frac{1}{|A|} \sum_{i=0}^{n} \sum_{\vec{x}\in C_i} \left(\frac{2^n}{|B|}\right)^k \mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x} - \vec{a}(\vec{x})] \\
&\leq \frac{e^{k\epsilon n}}{|A|} \sum_{i=0}^{n} \sum_{\vec{x}\in C_i} \mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x}^*] \sup_{\vec{a}\in A} \frac{\mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x} - \vec{a}]}{\mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x}^*]} \\
&= \frac{e^{k\epsilon n}}{|A|} \sum_{i=0}^{n} (1/2^k)^i \sum_{S\in\binom{[n]}{i}} \mathbb{E}_{\vec{x}\sim \mathrm{Bin}(k)^{\otimes([n]\setminus S)}\times\{k+1\}^S} \left[\sup_{\vec{a}\in A} \frac{\mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x} - \vec{a}]}{\mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x}^*]}\right].
\end{aligned}
$$

Let A_S be the set of elements in $A \subseteq \{0,1\}^n$ whose support contains S. Let

$$
A'_S = \{\vec{a}' \in \{0,1\}^{[n]\setminus S} : \exists \vec{a} \in A_S \text{ with } \vec{a}|_{[n]\setminus S} = \vec{a}'\}.
$$

Recall that $|A| = \exp(\delta n)$. Abusing notation so that the supremum of an empty set is 0, can continue the above chain of inequalities to get that

$$
\begin{aligned}
1 &\leq \frac{e^{k\epsilon n}}{|A|} \sum_{i=0}^{n} (1/2^k)^i \sum_{S\in\binom{[n]}{i}} \mathbb{E}_{\vec{x}\sim \mathrm{Bin}(k)^{\otimes([n]\setminus S)}\times\{k+1\}^S} \left[\sup_{\vec{a}\in A} \frac{\mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x} - \vec{a}]}{\mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x}^*]}\right] \\
&= \frac{e^{k\epsilon n}}{|A|} \sum_{i=0}^{n} (1/2^k)^i \sum_{S\in\binom{[n]}{i}} \mathbb{E}_{\vec{x}\sim \mathrm{Bin}(k)^{\otimes([n]\setminus S)}} \left[\sup_{\vec{a}\in A_S} \frac{\mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x} - \vec{a}\times\{0\}^S]}{\mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x}\times\{k\}^S]}\right] \\
&= \frac{e^{k\epsilon n}}{|A|} \sum_{i=0}^{n} (1/2^k)^i \sum_{S\in\binom{[n]}{i}} \mathbb{E}_{\vec{x}\sim \mathrm{Bin}(k)^{\otimes([n]\setminus S)}} \left[\sup_{\vec{a}\in A_S} \frac{\mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x} - \vec{a}]}{\mathbb{P}[\vec{v}_1 + \cdots + \vec{v}_k = \vec{x}]}\right] \\
&\leq \frac{e^{k\epsilon n}}{|A|} \left(\sum_{i=0}^{n/2} \cdot + \sum_{i=n/2}^{n} 2^{-ki}\cdot 2^n \cdot \left(\max_{\ell} \frac{\binom{k}{\ell\pm1}}{\binom{k}{\ell}}\right)^i\right) \\
&\leq \frac{e^{k\epsilon n}}{|A|} \left(\sum_{i=0}^{n/2} \cdot + n\cdot 2^{-kn/2}\cdot 2^n \cdot k^n\right) \\
&\leq \frac{e^{k\epsilon n}}{|A|} \left(\sum_{i=0}^{n/2} 2^{-ki} \exp(C_{2.1}(k^{-1} + \delta^{1/2}k^{-1/2})n) + 2^{-kn/4}\right) \\
&\leq \exp(-\delta n) \exp\Big(O(k\epsilon + k^{-1} + \delta^{1/2}k^{-1/2})n\Big)
\end{aligned}
$$

by Lemma 2.1 applied to A_S, as long as $n/2 \geq k \geq C_{2.1} \geq 10$. Hence,

$$
\delta \leq C(k\epsilon + k^{-1} + \delta^{1/2}k^{-1/2})
$$

for some absolute constant $C > 0$. Now letting $k = \epsilon^{-1/2}/2$ (note that this satisfies $2k = \epsilon^{-1/2} \leq n$), we find that

$$
\delta = O(\epsilon^{1/2}),
$$

as desired. □

The proof of Lemma 2.1 relies on the following preliminary estimate.

Lemma 2.3. *If $1 \le s \le k/(16\pi)$, then*

$$\mathbb{E}_{x\sim\mathrm{Bin}(k)}\left(\frac{x}{k+1-x}\right)^s \le \exp(10\pi s^2/k) + 2k^s(4/5)^k.$$

Proof. We let $x \sim \mathrm{Bin}(k)$ and $y = x - k/2 \sim \mathrm{Bin}(k) - k/2$ throughout. We let $z \sim \mathcal{N}(0, k\pi/8)$. We have

$$\begin{aligned}
\mathbb{E}_{x\sim\mathrm{Bin}(k)}\left(\frac{x}{k+1-x}\right)^s &= \mathbb{E}_y\left(1+\frac{2y-1}{k/2+1-y}\right)^s \\
&\le \mathbb{E}_y\left(1+\frac{2y}{k/2+1-y}\right)^s \mathbb{1}_{|y|\le k/3} + k^s\mathbb{P}[|y|\ge k/3] \\
&\le \mathbb{E}_y\left(1+\frac{2y}{k/2+1-y}\right)^s \mathbb{1}_{|y|\le k/3} + 2k^s(4/5)^k.
\end{aligned}$$

Since for $|y| \le k/3$,

$$\frac{2y}{(k/2+1-y)} \le \frac{2y}{k/2+1} + \frac{8y^2}{(k/2+1)^2},$$

and using $(1+x) \le \exp(x)$. we can continue the previous inequality as

$$\begin{aligned}
&\le \mathbb{E}_y\left(1+\frac{2y}{k/2+1}+\frac{8y^2}{(k/2+1)^2}\right)^s \mathbb{1}_{|y|\le k/3} + 2k^s(4/5)^k \\
&\le \mathbb{E}_y \exp\left(\frac{4sy}{k+2}+\frac{32sy^2}{k^2}\right) + 2k^s(4/5)^k.
\end{aligned}$$

Now, let $z_1, \dots, z_k$ be i.i.d. $\mathcal{N}(0,1)$ random variables. Then,

$$y \sim \frac{1}{2}\left(\mathrm{sgn}\, z_1 + \cdots + \mathrm{sgn}\, z_k\right).$$

Moreover, for any $-k \le \ell \le k$,

$$\mathbb{E}[z_1 + \cdots + z_k \mid \mathrm{sgn}(z_1) + \cdots + \mathrm{sgn}(z_k) = \ell] = \sqrt{\frac{2}{\pi}}\ell.$$

In particular, under this coupling of $y, z_1, \dots, z_k$, we have

$$\mathbb{E}[z_1 + \cdots + z_k \mid y] = \sqrt{\frac{8}{\pi}}y.$$

Let $z = z_1 + \cdots + z_k$, so that $z \sim \mathcal{N}(0,k)$. Then, by the convexity of

$$f(r) = \exp\left(\frac{4sy}{k+2}+\frac{32sy^2}{k^2}\right),$$

and using Jensen's inequality, we have

$$\begin{aligned}
\mathbb{E}_y f(y) &= \mathbb{E}_{y,z_1,\dots,z_k} f(y) \\
&= \mathbb{E}_{y,z_1,\dots,z_k} f\left(\sqrt{\frac{\pi}{8}}\mathbb{E}[z \mid y]\right) \\
&\le \mathbb{E}_z f(\sqrt{\pi}z/\sqrt{8}) \\
&= \mathbb{E}_{w\sim\mathcal{N}(0,1)} \exp\left(\frac{s\sqrt{2k\pi}}{k+2}w + \frac{4s\pi}{k}w^2\right) \\
&= \left(1-\frac{8\pi s}{k}\right)^{-1/2} \exp\left(\frac{\pi s^2 k^2}{(k+2)^2(k-8\pi s)}\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \exp\left(\frac{8\pi s}{k} + \frac{2\pi s^2}{k}\right) \\
&\le \exp(10\pi s^2/k). \qquad \square
\end{aligned}
$$

Finally, we can prove Lemma 2.1

Proof of Lemma 2.1. We may assume that $\delta \ge 2000/k$ since the statement for $\delta < 2000/k$ follows from the statement for $\delta = 2000/k$. Also, note that $\delta \le \log 2$. For any $t \in \mathbb{R}$, we have

$$
\begin{aligned}
\mathbb{P}_{\vec{x}}\left[\sup_{\vec{a}\in A} \frac{\mathbb{P}_{\vec{b}}[\vec{b} = \vec{x} - \vec{a}]}{\mathbb{P}_{\vec{b}}[\vec{b} = \vec{x}]} \ge e^{tn}\right] &\le |A| \sup_{\vec{a}\in A} \mathbb{P}_{\vec{x}}\left[\frac{\mathbb{P}_{\vec{b}}[\vec{b} = \vec{x} - \vec{a}]}{\mathbb{P}_{\vec{b}}[\vec{b} = \vec{x}]} \ge e^{tn}\right] \\
&\le |A| \sup_{\vec{a}\in A} \inf_{s\ge 2} \exp(-stn) \mathbb{E}_{\vec{x}}\left[\left(\frac{\mathbb{P}_{\vec{b}}[\vec{b} = \vec{x} - \vec{a}]}{\mathbb{P}_{\vec{b}}[\vec{b} = \vec{x}]}\right)^s\right] \\
&= |A| \sup_{\vec{a}\in A} \inf_{s\ge 2} \exp(-stn) \prod_{i=1}^{n} \mathbb{E}_{x\sim \mathrm{Bin}(k)}\left[\left(\frac{\mathbb{P}[\mathrm{Bin}(k) = x - a_i]}{\mathbb{P}[\mathrm{Bin}(k) = x]}\right)^s\right] \\
&\le |A| \inf_{s\ge 2} \exp(-stn) \left(\mathbb{E}_{x\sim \mathrm{Bin}(k)}\left(\frac{x}{k+1-x}\right)^s\right)^n.
\end{aligned}
$$

In the last line, we have used that

$$
\begin{aligned}
\mathbb{E}_{x\sim \mathrm{Bin}(k)}\left[\left(\frac{x}{k+1-x}\right)^s\right] &\ge \left(\mathbb{E}_{x\sim \mathrm{Bin}(k)}\left[\frac{x^2}{(k+1-x)^2}\right]\right)^{s/2} \\
&= \left(\sum_{\ell=0}^{k-1} \frac{\ell+1}{k-\ell}\binom{k}{\ell} 2^{-k}\right)^{s/2} \\
&= \left(\sum_{\ell=0}^{k-1} \left(\frac{k+2}{k} + \frac{4(k+1)(\ell-k/2)}{k^2} + \frac{(k+1)(k-2\ell)^2}{k^2(k-\ell)}\right)\binom{k}{\ell} 2^{-k}\right)^{s/2} \\
&\ge \left(\sum_{\ell=0}^{k-1} \left(\frac{k+2}{k} + \frac{4(k+1)(\ell-k/2)}{k^2}\right)\binom{k}{\ell} 2^{-k}\right)^{s/2} \\
&= \left(\frac{k+2}{k} - \frac{3k+4}{k} 2^{-k}\right)^{s/2} \\
&\ge 1
\end{aligned}
$$

if $k \ge 3$. Therefore, by Lemma 2.3, we have

$$
\begin{aligned}
\mathbb{P}_{\vec{x}}\left[\sup_{\vec{a}\in A} \frac{\mathbb{P}_{\vec{b}}[\vec{b} = \vec{x} - \vec{a}]}{\mathbb{P}_{\vec{b}}[\vec{b} = \vec{x}]} \ge e^{tn}\right] &\le |A| \inf_{s\ge 2} \exp(-stn) \left(\mathbb{E}_{x\sim \mathrm{Bin}(k)}\left(\frac{x}{k+1-x}\right)^s\right)^n \\
&\le |A| \inf_{2\le s\le k/(16\pi)} \exp(-stn) \left(\exp(10\pi s^2/k) + 2k^s(4/5)^k\right)^n \\
&\le |A| \inf_{2\le s\le k/(10\log k)} \exp(-stn) \left(\exp(12\pi s^2/k)\right)^n \\
&\le \begin{cases} |A| \exp\left(-\frac{kt^2 n}{48\pi}\right) & \text{if } \sqrt{\frac{96\pi\delta}{k}} \le t \le (\log k)^{-1} \\ |A| \exp\left(-\frac{kn}{48\pi(\log k)^2}\right) & \text{if } (\log k)^{-1} \le t \le \log k. \end{cases}
\end{aligned}
$$

Here, the second case follows by plugging in $s = k/(10\log k)$ and simplifying, and the first case follows from plugging in $s = kt/24\pi$ which satisfies $2 \le s \le k/10\log k$ by the restriction on t and δ. Finally, since

$$0 \le \sup_{\vec{a}\in A} \frac{\mathbb{P}_{\vec{b}}[\vec{b} = \vec{x} - \vec{a}]}{\mathbb{P}_{\vec{b}}[\vec{b} = \vec{x}]} \le \left(\max_{\ell} \frac{\binom{k}{\ell\pm 1}}{\binom{k}{\ell}}\right)^n \le k^n,$$

we have

$$\begin{aligned}
\mathbb{E}_{\vec{x}}\left[\sup_{\vec{a}\in A} \frac{\mathbb{P}_{\vec{b}}[\vec{b} = \vec{x} - \vec{a}]}{\mathbb{P}_{\vec{b}}[\vec{b} = \vec{x}]}\right] &= \int_{-\infty}^{\log k} \mathbb{P}\left[\sup_{\vec{a}\in A} \frac{\mathbb{P}_{\vec{b}}[\vec{b} = \vec{x} - \vec{a}]}{\mathbb{P}_{\vec{b}}[\vec{b} = \vec{x}]} \ge e^{tn}\right] ne^{tn} dt \\
&\le \int_{1/\log k}^{\log k} \cdot + \int_{\sqrt{96\pi\delta/k}}^{1/\log k} \cdot + \int_{-\infty}^{\sqrt{96\pi\delta/k}} ne^{tn} dt \\
&\le e^{\sqrt{96\pi\delta/k}n} + \int_{\sqrt{96\pi\delta/k}}^{1/\log k} |A| \exp\left(-\frac{kt^2 n}{48\pi}\right) ne^{tn} dt \\
&\quad + \int_{1/\log k}^{\log k} |A| \exp\left(-\frac{kn}{48\pi(\log k)^2}\right) ne^{tn} dt \\
&\le \exp\left(O(\sqrt{\delta/k})n\right) + \int_{\sqrt{96\pi\delta/k}}^{1/\log k} ne^{-tn} dt + 1 \\
&\le \exp\left(O(\sqrt{\delta/k})n\right). \qquad \square
\end{aligned}$$

References

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