A COUNTEREXAMPLE TO THE BOLLOBÁS–RIORDAN CONJECTURES ON SPARSE GRAPH LIMITS

ASHWIN SAH, MEHTAAB SAWHNEY, JONATHAN TIDOR, AND YUFEI ZHAO

Abstract. Bollobás and Riordan, in their paper "Metrics for sparse graphs," proposed a number of provocative conjectures extending central results of quasirandom graphs and graph limits to sparse graphs. We refute these conjectures by exhibiting a sequence of graphs with convergent normalized subgraph densities (and pseudorandom C_4-counts), but with no limit expressible as a kernel.

Pseudorandom and quasirandom graphs, whose studies were initiated by Thomason [13, 14] and Chung, Graham, and Wilson [5], play central roles in graph theory. A particularly nice consequence is that that many notions of quasirandomness are in fact equivalent for dense graph sequences. The theory of graph limits [11], developed by Lovász and collaborators, further generalizes these concepts. Some of the central results in these developments are summarized below. Here we are considering a sequence of graphs G_n. We write $|G|$ and e_G respectively for the number of vertices and edges of G, and $t(F,G) = \hom(F,G)|G|^{-|F|}$ for the homomorphism density of F in G.

(1) *C_4 counts control quasirandomness* [5]. If $t(K_2,G_n) \to p$ and $t(C_4,G_n) \to p^4$ for some constant p, then $t(F,G_n) \to p^{|F|}$ for all graphs F, and furthermore G_n converges to p in the cut norm (i.e., satisfies the discrepancy condition).
(2) *Existence of graph limits* [5]. If $t(F,G_n)$ converges as $n \to \infty$ for every F, then there exists a graphon $W\colon [0,1]^2 \to [0,1]$ such that $t(F,G_n) \to t(F,W)$.
(3) *Equivalence of convergence* [2]. $t(F,G_n)$ converges as $n \to \infty$ for every F if and only if G_n is a Cauchy sequence with respect to the cut metric.

Implications concerning subgraph densities often fail for naive generalizations to sparse graphs. Here we call a sequence of graphs G_n *sparse* if $e_{G_n}/|G_n|^2 \to 0$ as $n \to \infty$. We normalize all the quantities considered according to the decaying edge-density.

There is much interest in extending the above ideas to sparse graphs. The first such systematic studies was undertaken by Bollobás and Riordan [1]. They considered natural notions of convergence and metrics for sparse graphs, and gave many interesting results, examples, as well as a long list of provocative conjectures. A recurring theme in their paper, as well as in other works in this area, is that one quickly runs into difficulties as soon as subgraph counts are involved. The lack of a general purpose "counting lemma" in sparse graphs appears to be a fundamental difficulty. This issue lies at the heart of the sparse regularity method of Conlon–Fox–Zhao [6, 7, 8], which developed novel counting lemmas in sparse graphs and hypergraphs under additional pseudorandomnesses hypotheses, which built on and simplified the Green–Tao theorem on arithmetic progressions in the primes [9]. Some of the subsequent extensions of the Bollobás–Riordan sparse graph limit theory, in particular the L^p theory of sparse graph limits [3, 4], largely avoids the issues of subgraph counts in favor of other metrics.

Given real $p > 0$ and graphs F and G, we define the normalized F-density in G to be

$$t_p(F,G) = \frac{\hom(F,G)}{p^{e_F}|G|^{|F|}}.$$

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Here we will primarily be concerned with N-vertex graphs with edge density $p = N^{-o(1)}$, so that there is only a lower order difference between homomorphism counts and subgraph counts (after accounting for automorphisms of H). The normalization in $t_p(F, G)$ is chosen so that for a sequence of random graphs $G_n = G(n, p)$, one has $t_p(F, G_n) \to 1$ for all F almost surely.

A *kernel* is a symmetric measurable function $W\colon [0,1]^2 \to [0, \infty)$, where symmetric means that $W(x, y) = W(y, x)$. (The word *graphon* is often used in the literature for kernels with $[0, 1]$-values.) We say that a kernel is *bounded* if there is some real C so that $0 \le W \le C$ holds pointwise. Given a graph H, we define the H*-density* of a kernel W to be

$$t(H, W) = \int_{[0,1]^{V(H)}} \prod_{uv \in E(H)} W(x_u, x_v) \prod_{v \in V(H)} dx_v.$$

Bollobás and Riordan [1] proposed the following conjectures. Throughout, let G_n be a sequence of graphs with edge-density $p_n = 2e_{G_n}/|G_n|^2$ satisfying $p_n = |G_n|^{-o(1)}$. For a graph F, write

$$c_F = \lim_{n\to\infty} t_{p_n}(F, G_n).$$

- [1, Conjecture 3.4] If c_F exists and is finite for all graphs F, then there is some kernel W such that $t(F, W) = c_F$ for all graphs F.
- [1, Conjecture 3.3] If c_F exists for all graphs F and $\sup_F c_F^{1/e_F} < \infty$, then there is a bounded kernel W such that $t(F, W) = c_F$ for all graphs F.
- [1, Conjecture 3.21] If c_F exists and is finite for all graphs F and $c_{C_4} = 1$, then $c_{K_3} = 1$.
- [1, Conjecture 3.9] If c_F exists and is finite for all graphs F and $c_{C_4} = 1$, then $c_F = 1$ for all graphs F.

There are additional conjectures in [1] that we do not state here precisely since they require additional definitions. In particular, Conjecture 3.22 concerns graphs of sparser densities and would imply Conjecture 3.21. Conjecture 5.5 would imply Conjecture 3.3. Conjectures 5.6 and 5.7 propose equivalences between convergence of subgraph densities and convergence in cut metric, and they would imply Conjecture 5.5.

We provide a counterexample that refutes all conjectures in [1]. This counterexample illustrates a fundamental difficulty with counting in sparse graphs, and suggest additional hypotheses, such as those in [6, 8], may indeed be necessary. It remains interesting to propose and explore further weakenings of the Bollobás–Riordan conjectures.

Theorem 1. *There exists a sequence of graphs G_n with $|G_n| \to \infty$ and edge density $p_n = |G_n|^{-o(1)}$ such that for every graph F, writing $\triangle_F$ for the number of triangles in F,*

$$t_{p_n}(F, G_n) \to e^{-\triangle_F} \qquad \text{as } n \to \infty.$$

Furthermore, there is no kernel W satisfying $t(F, W) = e^{-\triangle_F}$ for all graphs F.

Proof. Let $G = G_n = K_n^{\otimes n^2}$, the n^2-th tensor power of K_n. Its edge density is $p = p_n = (1 - n^{-1})^{n^2} = (1 + o(1))e^{-n}$. Note that $\hom(F, K_n)$ counts proper n-colorings of F. It is a standard result in graph theory (easily proved using inclusion-exclusion) that

$$\hom(F, K_n) = n^{|F|} - e_F n^{|F|-1} + \left(\binom{e_F}{2} - \triangle_F\right) n^{|F|-2} + O_F(n^{|F|-3}).$$

Since $\hom(F, K_n^{\otimes n^2}) = \hom(F, K_n)^{n^2}$,

$$\begin{aligned} t_p(F,G) &= p^{-e_F}|G|^{-|F|}\hom(F,K_n)^{n^2} \\ &= (1-n^{-1})^{-e_F n^2}\left(1 - e_F n^{-1} + \binom{e_F}{2}n^{-2} - \triangle_F n^{-2} + O_F(n^{-3})\right)^{n^2} \\ &= \left(1 - \triangle_F n^{-2} + O_F(n^{-3})\right)^{n^2} \\ &\to e^{-\triangle_F} \qquad \text{as } n \to \infty. \end{aligned}$$

It follows from [1, Lemma 3.5] that there does not exist a kernel W satisfying $t(F,W) = e^{-\triangle_F}$ for all F. We include the short argument here for the convenience of the reader. Since $t(K_2, W) = 1$ and $t(K_3, W) = e^{-1}$, the kernel W averages to 1 but is not constant. So there exist subsets $A, B \subseteq [0,1]$ of positive measure such that W averages to some $c > 1$ on $A \times B$. We find that, for every positive integer m,

$$1 = t(K_{m,m}, W) \geq t(K_{m,m}, W1_{A\times B}) \geq \mu(A)^m \mu(B)^m c^{m^2},$$

where the second inequality follows from two applications of Hölder's inequality (i.e., Sidorenko's conjecture [12] for $K_{m,m}$). Taking m sufficiently large gives a contradiction. $\square$

Remark. The above sequence in fact converges to the constant kernel in normalized cut norm. This is a result of the following lemma applied with W_n being the associated graphon of G_n divided by p_n. As a consequence (see [1, Lemma 4.2]), the graph sequence satisfies the bounded density assumption [1, Assumption 4.1] (also known under the names "no dense spots" [10] and "L^∞ upper regular" [3, 4]).

One can obtain a sequence of graphs with similar properties and $|G_n| = n$ by slowly blowing-up the above construction (see [1, Remark 3.14]).

Recall the cut norm of $U\colon [0,1]^2 \to \mathbb{R}$ is defined by $\|U\|_\square = \sup_{A,B\subset[0,1]} \left|\int_{A\times B} U\right|$.

Lemma 2. *If a sequence W_n of kernels satisfies $t(F, W_n) \to 1$ whenever F is a subgraph of C_4, then $\|W_n - 1\|_\square \to 0$.*

Proof. Applying Cauchy–Schwarz twice (e.g., [11, Lemma 8.12]) and expanding,

$$\begin{aligned} \|W_n - 1\|_\square^4 &\leq t(C_4, W_n - 1) \\ &= t(C_4, W_n) - 4t(P_3, W_n) + 4t(K_{2,1}, W_n) + 2t(K_2, W_n)^2 - 4t(K_2, W_n) + 1 \\ &\to 0. \end{aligned}$$ $\square$

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Massachusetts Institute of Technology, Cambridge, MA 02139, USA
E-mail address: {asah,msawhney,jtidor,yufeiz}@mit.edu