EXPONENTIAL IMPROVEMENTS FOR SUPERBALL PACKING UPPER BOUNDS

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ABSTRACT. We prove that for all fixed p > 2, the translative packing density of unit ℓ_p -balls in \mathbb{R}^n is at most $2^{(\gamma_p + o(1))n}$ with $\gamma_p < -1/p$. This is the first exponential improvement in high dimensions since van der Corput and Schaake (1936).

1. INTRODUCTION

The sphere packing problem asks for the densest packing of non-overlapping unit balls in \mathbb{R}^n . This is an old and difficult problem whose exact solution is only known in dimensions 1, 2, 3, 8, and 24. The problem is already non-trivial in two dimensions (see [8] for a short proof). The three-dimensional sphere packing problem is known as Kepler's conjecture, and it was solved by Hales [9] via a monumental computer-assisted proof. The problem in eight dimensions was recently resolved by Viazovska [23] in a stunning breakthrough, and the method was then quickly extended to solve the problem in twenty-four dimensions [3]. Dimensions 8 and 24 are special due to the existence of highly dense and symmetric lattices known as the E_8 lattice (dimension 8) and the Leech lattice (dimension 24). See the survey [2] and its references for background and recent developments.

In this paper, we study translative packings of ℓ_p -balls in high dimensions. Denote the ℓ_p -balls with radius R in \mathbb{R}^n by $\mathbf{B}_p^n(R) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_p \leq R\}$ and the unit ℓ_p -ball by $\mathbf{B}_p^n := \mathbf{B}_p^n(1)$. Here $\|(x_1, \ldots, x_n)\|_p := (|x_1|^p + \cdots + |x_n|^p)^{1/p}$ is the ℓ_p -norm. The name superball refers to ℓ_p -balls with p > 2 [19]. Superballs are more cube-like compared to the familiar ℓ_2 -balls. See [10, 11, 5] for studies of ℓ_p -ball packings in \mathbb{R}^3 . Although ℓ_p -balls do not possess rotational symmetry, in this paper we only consider translations of identical ℓ_p -balls, not allowing rotations. The best known lower bounds on high dimensional superball packing densities do not use rotations [6] (see Section 3.2).

Let $\Delta_p(n)$ denote the maximum translative packing density of copies of B_p^n in \mathbb{R}^n . Here *density* is the fraction of space occupied by these balls. For fixed $p \in [1, \infty)$, let

$$\gamma_p := \limsup_{n \to \infty} \frac{1}{n} \log_2 \Delta_p(n)$$

be the exponential rate of optimal packing densities in high dimensions. The precise value of γ_p is unknown for any $p \in [1, \infty)$, and the current best upper and lower bounds are quite far apart. For Euclidean balls, p = 2, the best high dimensional upper bound (apart from constant factors) is due to Kabatiansky and Levenshtein [12]:

$$\Delta_2(n) \leq 2^{(\kappa_{\rm KL}+o(1))n}$$
, where $\kappa_{\rm KL} := -0.5990...$

See Cohn and Zhao [4] and Sardari and Zargar [20] for constant factor improvements over Kabatiansky and Levenshtein [12]. For lower bounds, we have $\Delta_p(n) \geq 2^{-n}$ for all n and $p \geq 1$ since every maximal packing has density at least 2^{-n} . For p = 2, there have only been subexponential improvements, with the current best lower bound due to Venkatesh [22]. In summary, the best bounds on γ_2 are $-1 \leq \gamma_2 \leq \kappa_{\text{KL}} = -0.5990...$

YZ was supported by NSF Awards DMS-1362326 and DMS-1764176, and the MIT Solomon Buchsbaum Fund.



FIGURE 1. Upper bounds on the exponential rate γ_p of translative packing densities of identical ℓ_p -balls in high dimensions. For p > 2, the dashed blue curve is the previous upper bound -1/p and the solid red curve is our new upper bound. For $1 \le p < 2$, discussed in Section 3, the dashed blue curve is (3.1) due to Rankin [15] and the solid red curve is (3.3) derived from the Kabatiansky–Levenshtein [12] sphere packing bound.

For p > 2, the current best upper bound on the exponential rate of superball packing densities was first proved by van der Corput and Schaake [21] via Blichfeldt's method [1] (e.g., see [24, Section 6.3]), giving

$$\gamma_p \leq -1/p \quad \text{for } p > 2.$$

There have been subsequent subexponential upper bound improvements on $\Delta_p(n)$ for p > 2, e.g., Rankin [16, 17]. We defer to Section 3 for a discussion of known bounds on γ_p in other regimes.

In this paper, we prove a new upper bound on γ_p for all p > 2, giving the first exponential improvement since 1936 on the upper bound of superball packing densities in high dimensions.

Theorem 1.1. For all $p \geq 2$,

$$\gamma_p \le \inf_{0 < \theta < \pi/2} \left(\frac{1 + \sin \theta}{2 \sin \theta} \log_2 \frac{1 + \sin \theta}{2 \sin \theta} - \frac{1 - \sin \theta}{2 \sin \theta} \log_2 \frac{1 - \sin \theta}{2 \sin \theta} + \frac{2}{p} \log_2 \sin \frac{\theta}{2} \right).$$

In particular, $\gamma_p < -1/p$ for all $p \ge 2$.

See Figure 1 for a plot of the bounds.

Remark. Theorem 1.1 with p = 2 recovers $\gamma_p \leq \kappa_{\text{KL}}$. Our upper bound on γ_p is continuous with p, whereas the previous best bounds were not continuous¹ at p = 2. The fact that our bound at p = 2 recovers the Kabatiansky–Levenshtein bound is not a coincidence, as our proof relies on the Kabatiansky–Levenshtein bound for spherical codes.

2. Proof of main theorem

2.1. Kabatiansky–Levenshtein spherical code bound. Denote the ℓ_p -sphere in \mathbb{R}^n of radius R by $S_p^{n-1}(R) := \{ \boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_p = R \}$ and the unit ℓ_p -sphere by $S_p^{n-1} := S_p^{n-1}(1)$. Let $A_p(n,d)$ to be the maximum number of points on S_p^{n-1} with pairwise ℓ_p -distance at least 2d, i.e., an ℓ_p -spherical code. Note that $A_p(n,d) = 1$ unless $d \in [0,1]$. Note that $A_2(n, \sin(\theta/2))$ is the maximum size of a

¹It is unknown whether $p \mapsto \gamma_p$ is continuous. Lemma 3.1 implies that γ_p is continuous at all but at most countably many points.

spherical code in \mathbb{R}^n with pairwise angle at least θ . Kabatiansky and Levenshtein [12] proved that for all² $0 < \theta < \pi/2$,

$$\limsup_{n \to \infty} \frac{1}{n} \log_2 A_2(n, \sin(\theta/2)) \le a(\theta)$$
(2.1)

where

$$a(\theta) := \frac{1 + \sin \theta}{2\sin \theta} \log_2 \frac{1 + \sin \theta}{2\sin \theta} - \frac{1 - \sin \theta}{2\sin \theta} \log_2 \frac{1 - \sin \theta}{2\sin \theta}$$

A projection argument (see [4, Section 2]) shows that

$$\Delta_2(n) \le \sin^n(\theta/2) A_2(n+1, \sin(\theta/2))$$

so (2.1) gives

$$\lim_{n \to \infty} \frac{1}{n} \log_2 \Delta_2(n) \le a(\theta) + \log_2 \sin \frac{\theta}{2}.$$

The bound $\gamma_2 \leq \kappa_{\text{KL}} = -0.5990...$ is obtained by choosing $\theta = \theta_{\text{KL}} = 1.0995...$ to minimize the upper bound above.

2.2. ℓ_p -twist. Fix $p \ge 2$. Define

$$x^* := \operatorname{sgn}(x)|x|^{p/2}, \qquad x \in \mathbb{R}$$

For $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, write $\boldsymbol{x}^* := (x_1^*, \ldots, x_n^*)$, and for $X \subseteq \mathbb{R}^n$, write $X^* := \{\boldsymbol{x}^* : \boldsymbol{x} \in X\}$. Observe that for all $x, y \in \mathbb{R}$,

$$|x^* - y^*| \ge 2^{1 - p/2} |x - y|^{p/2}.$$
(2.2)

Indeed, without loss of generality it suffices to consider two cases: $x \ge 0 \ge y$ and $x \ge y \ge 0$. The former case is an immediate consequence of Hölder's inequality (or the convexity of $x \mapsto x^{p/2}$). In the latter case, we have

$$x^{p/2} - y^{p/2} \ge (x - y)^{p/2} \ge 2^{1 - p/2} (x - y)^{p/2}.$$

Here we use $(w+z)^{p/2} \ge w^{p/2} + z^{p/2}$ for w, z > 0, which can be proved by first normalizing to w+z = 1 and noting that $w^{p/2} + z^{p/2} \le w+z = 1$.

Lemma 2.1. For all $p \ge 2$ and $d \in (0, 1]$, we have $A_p(n, d) \le A_2(n, d^{p/2})$.

Proof. Let $X \subseteq \mathbf{S}_p^{n-1}$ with $|X| = A_p(n, d)$ and $\|\mathbf{x} - \mathbf{y}\|_p \ge 2d$ for all distinct $\mathbf{x}, \mathbf{y} \in X$. We have $\|\mathbf{x}^*\|_2 = \|\mathbf{x}\|_p = 1$ for all $\mathbf{x} \in X$, so $X^* \subseteq \mathbf{S}_2^{n-1}$. For distinct $\mathbf{x}, \mathbf{y} \in X$, we have

$$\|\boldsymbol{x}^* - \boldsymbol{y}^*\|_2^2 = \sum_{i=1}^n |x_i^* - y_i^*|^2 \ge 2^{2-p} \sum_{i=1}^n |x_i - y_i|^p \ge 2^2 d^p,$$

by (2.2). Thus X^* is a subset of S_2^{n-1} whose points have pairwise ℓ_2 -distance at least $2d^{p/2}$. Hence $|X| = |X^*| \le A_2(n, d^{p/2})$.

Remark. The same argument shows that $A_p(n,d) \leq A_q(n,d^{p/q})$ for all $1 \leq q \leq p$ and $d \in (0,1]$.

Lemma 2.2. For every $p \ge 1$, $d \in (0,1]$, and $n \in \mathbb{N}$, we have $\Delta_p(n) \le d^n A_p(n+1,d)$.

Proof. Let $\rho < \Delta_p(n)$ be arbitrary. Consider a translative packing $\{\boldsymbol{x} + \boldsymbol{B}_p^n(d) : \boldsymbol{x} \in X\}$ in \mathbb{R}^n with density greater than ρ , where $X \subseteq \mathbb{R}^n$ is the set of centers of the ℓ_p -balls. By an averaging argument³, there exists some translate of a unit ℓ_p -ball that contains at least $d^{-n}\rho$ points of X. Translating X if necessary, we may assume that $|X \cap \boldsymbol{B}_p^n| \ge d^{-n}\rho$. Add an (n + 1)-st coordinate to each point in $X \cap \boldsymbol{B}_p^n$ to obtain a set X' of points on the unit ℓ_p -sphere in \mathbb{R}^{n+1} . In other words, X'

²A simple geometric argument (see [13, (17)]) shows that the upper bound (2.1) can be improved for $\theta < \theta_{\text{KL}} :=$ 1.0995... to $a(\theta_{\text{KL}}) + \log_2 \sin(\theta_{\text{KL}}/2) - \log_2 \sin(\theta/2)$, but this improvement does not benefit our bounds.

³A uniform random translation of a unit ℓ_p -ball inside $[-R, R]^n$ contains more than $d^{-n}\rho + o_{R\to\infty}(1)$ points of X.

is obtained by projecting the points of X contained in the unit ball "upward" to the hemisphere one dimension higher. Since the points in X are pairwise at least 2d apart in ℓ_p -distance, the same holds for X'. So X' is an ℓ_p -spherical code whose points are pairwise separated by ℓ_p -distance at least 2d, and hence $d^{-n}\rho \leq |X \cap \mathbf{B}_p^n| = |X'| \leq A_p(n+1,d)$. Since ρ can be arbitrarily close to $\Delta_p(n)$, we obtain the claimed inequality.

Remark. As in [4], the above argument can be modified so that we do not need to add a new dimension when $d \in [1/2, 1]$, resulting in a slightly better bound $\Delta_p(n) \leq d^{-n}A_p(n, d)$. We omit the details of this modification since this improvement does not affect the exponential asymptotics.

Proof of Theorem 1.1. Applying Lemmas 2.1 and 2.2, we have, for every $0 < \theta < \pi/2$,

$$\Delta_p(n) \le \sin(\theta/2)^{2n/p} A_p(n+1, \sin(\theta/2)^{2/p}) \le \sin(\theta/2)^{2n/p} A_2(n+1, \sin(\theta/2)).$$

Applying (2.1), we obtain

$$\gamma_p = \limsup_{n \to \infty} \frac{1}{n} \log_2 \Delta_p(n) \le a(\theta) + \frac{2}{p} \log_2 \sin(\theta/2).$$

The main result follows by taking the infimum of the bound over $\theta \in (0, \pi/2)$.

Setting $\theta = \pi/2 - \eta$, we have, with $p \ge 2$ fixed and $\eta \to 0^+$,

$$\gamma_p \le -\frac{1}{p} - \frac{\eta}{p\ln 2} + o(\eta).$$

So choosing $\eta > 0$ sufficiently small gives $\gamma_p < -1/p$ for all $p \ge 2$.

3. Remarks

3.1. Asymptotics. Setting $\theta = \pi/2 - (p \ln p)^{-1}$, we obtain

$$\gamma_p \leq -\frac{1}{p} - \frac{1}{\ln 4} \cdot \frac{1}{p^2 \ln p} + O\left(\frac{1}{p^2 \ln^2 p}\right), \quad \text{as } p \to \infty.$$

Taking $\theta = \theta_{\rm KL}$ gives

$$\gamma_p \le \kappa_{\text{KL}} + \frac{2-p}{p} \log_2 \sin \frac{\theta_{KL}}{2}, \text{ for all } p \ge 2.$$

Thus, as $\epsilon \to 0^+$,

$$\gamma_{2+\epsilon} \le \gamma_{\mathrm{KL}} - \left(\frac{1}{2}\log_2 \sin\frac{\theta_{\mathrm{KL}}}{2}\right)\epsilon + O(\epsilon^2) = (-0.5990\dots) + (0.4650\dots)\epsilon + O(\epsilon^2).$$

3.2. Review of other bounds on γ_p . Here we survey other existing bounds on γ_p .

For p = 2, the best known bounds are $-1 \le \gamma_2 \le \kappa_{\rm KL} = -0.5990...$ as discussed earlier.

For p > 2, the best known upper bounds are the ones given in this paper. For lower bounds, extending on methods developed by Rush [18] and Rush–Sloane [19], Elkies, Odlyzko, and Rush [6] proved $\gamma_p > -1$ for all p > 2, thereby exponentially beating the Minkowski–Hlawka lower bound. See [6] for the precise bound. Their bounds have the following asymptotics:

$$\gamma_p \ge -(1+o(1))\frac{\ln \ln p}{p\ln 2}, \quad \text{as } p \to \infty,$$

and

$$\gamma_{2+\epsilon} \ge -1 + \left(\frac{\sqrt{\pi}\zeta(3)}{2\ln 2} + o(1)\right) \frac{\epsilon}{\ln^{3/2}(1/\epsilon)}, \quad \text{as } \epsilon \to 0^+.$$

Here ζ denotes the Riemann zeta function. See [14] for some later improvements using algebraicgeometric codes for some specific integers p.

For $1 \le p < 2$, no improvement over the Minkowski–Hlawka lower bound $\gamma_p \ge -1$ is known. The best upper bound on γ_p is due to Rankin [15], based on Blichfeldt's method [1]:

$$\gamma_p \le \inf_{\frac{1}{2} \le \frac{1}{q} \le \frac{1}{3} \left(1 + \frac{1}{p}\right)} \left(b(p) - b(q) - 1 + 1/p + (1/q - 1/p) \log_2 \left(\frac{2 - 1/q}{1 - 1/q}\right) \right)$$
(3.1)

where

$$b(p) := \lim_{n \to \infty} \frac{1}{n} \log_2 \operatorname{vol} \mathbf{B}_p^n(n^{1/p}) = 1 + \log_2 \Gamma\left(1 + \frac{1}{p}\right) + \frac{1}{p} \log_2(pe).$$
(3.2)

Recall that vol $B_p^n = 2^n \Gamma(1 + 1/p)^n / \Gamma(1 + n/p)$. For packings of congruent cross-polytopes (i.e., unit ℓ_1 -balls) allowing rotations, Fejes Tóth, Fodor, and Vígh [7] proved an exponentially decaying upper bound in high dimensions. For translative packing of unit ℓ_1 -balls, the upper bound (3.1) remains best known in high dimensions.

We note that the above bound (3.1) can be improved on the region $p \in [1.494..., 2)$ using the Kabatiansky-Levenshtein bound via the following folklore observation.

Lemma 3.1. For $1 \le p \le q \le \infty$, $\gamma_p - b(p) \le \gamma_q - b(q)$.

Proof. By monotonicity of norms, we have $n^{-1/p} \| \boldsymbol{x} \|_p \le n^{-1/q} \| \boldsymbol{x} \|_q$, so $\boldsymbol{B}_p^n(n^{1/p}) \supseteq \boldsymbol{B}_q^n(n^{1/q})$. Any packing of $\boldsymbol{B}_p^n(n^{1/p})$ can be shrunk into a packing of $\boldsymbol{B}_q^n(n^{1/q})$. Hence

$$\frac{\Delta_p(n)}{\operatorname{vol} \boldsymbol{B}_p^n(n^{1/p})} \le \frac{\Delta_q(n)}{\operatorname{vol} \boldsymbol{B}_p^n(n^{1/q})}$$

Taking log, dividing by n, and letting $n \to \infty$ yields the lemma.

Using $\gamma_2 \leq \kappa_{\rm KL}$, we find that

$$\gamma_p \le \kappa_{\rm KL} - b(2) + b(p) = (-0.5990...) - b(2) + b(p) \text{ for } 1 \le p < 2.$$
 (3.3)

Thus

 $\gamma_p \leq \min\{\text{RHS of } (3.1), \text{RHS of } (3.3)\}$

See Figure 1 for an illustration of the above bounds.

Acknowledgments

This work began during Y.Z.'s internship at Microsoft Research New England, and he would like to thank Henry Cohn for discussions and mentorship and Microsoft Research for its hospitality.

References

- [1] H. F. Blichfeldt, The minimum value of quadratic forms, and the closest packing of spheres, Math. Ann. 101 (1929), 605-608.
- [2] Henry Cohn, A conceptual breakthrough in sphere packing, Notices Amer. Math. Soc. 64 (2017), 102–115.
- [3] Henry Cohn, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko, and Maryna Viazovska, The sphere packing problem in dimension 24, Ann. of Math. (2) 185 (2017), 1017–1033.
- [4] Henry Cohn and Yufei Zhao, Sphere packing bounds via spherical codes, Duke Math. J. 163 (2014), 1965–2002.
- [5] M. Dostert and F. Vallentin, New dense superball packings in three dimensions, arXiv:1806.10878.
- [6] N. D. Elkies, A. M. Odlyzko, and J. A. Rush, On the packing densities of superballs and other bodies, Invent. Math. **105** (1991), 613–639.
- [7] G. Fejes Tóth, F. Fodor, and V. Vígh, The packing density of the n-dimensional cross-polytope, Discrete Comput. Geom. 54 (2015), 182–194.
- [8] Thomas C. Hales, Cannonballs and honeycombs, Notices Amer. Math. Soc. 47 (2000), 440–449.
- [9] Thomas C. Hales, A proof of the Kepler conjecture, Ann. of Math. (2) 162 (2005), 1065–1185.
- [10] Y. Jiao, F. H. Stillinger, and S. Torquato, Optimal packings of superballs, Phys. Rev. E (3) 79 (2009), 041309, 12.
- [11] Y. Jiao, F. H. Stillinger, and S. Torquato, Distinctive features arising in maximally random jammed packings of superballs, Phys. Rev. E (3) 81 (2010), 041304, 8.

- [12] G. A. Kabatiansky and V. I. Levenshtein, Bounds for packings on a sphere and in space, Problemy Peredači Informacii 14 (1978), 3–25.
- [13] V. I. Levenshtein, Maximal packing density of n-dimensional euclidean space with equal balls, Mat. Zametki 18 (1975), 301–311.
- [14] Li Liu and Chao Ping Xing, Packing superballs from codes and algebraic curves, Acta Math. Sin. (Engl. Ser.) 24 (2008), 1–6.
- [15] R. A. Rankin, On sums of powers of linear forms. III, Nederl. Akad. Wetensch., Proc. 51 (1948), 846–853 = Indagationes Math. 10, 274–281 (1948).
- [16] R. A. Rankin, On sums of powers of linear forms. I, Ann. of Math. (2) 50 (1949), 691–698.
- [17] R. A. Rankin, On sums of powers of linear forms. II, Ann. of Math. (2) 50 (1949), 699-704.
- [18] J. A. Rush, A lower bound on packing density, Invent. Math. 98 (1989), 499-509.
- [19] Jason A. Rush and N. J. A. Sloane, An improvement to the Minkowski-Hlawka bound for packing superballs, Mathematika 34 (1987), 8–18.
- [20] Naser T. Sardari and Masoud Zargar, A new upper bound for spherical codes, arXiv:2001.00185.
- [21] J. G. van der Corput and G. Schaake, Anwendung einer Blichfeldtschen Beweismethode in der Geometrie der Zahlen, Acta Arith. 2 (1936), 152–160.
- [22] Akshay Venkatesh, A note on sphere packings in high dimension, Int. Math. Res. Not. IMRN (2013), 1628–1642.
- [23] Maryna S. Viazovska, The sphere packing problem in dimension 8, Ann. of Math. (2) 185 (2017), 991–1015.
- [24] Chuanming Zong, Sphere packings, Universitext, Springer-Verlag, New York, 1999.

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