CONSTRAINING STRONG C-WILF EQUIVALENCE USING CLUSTER POSET ASYMPTOTICS

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Abstract. Let $\pi \in \mathfrak{S}_m$ and $\sigma \in \mathfrak{S}_n$ be permutations. An *occurrence* of π in σ as a consecutive pattern is a subsequence $\sigma_i\sigma_{i+1}\cdots\sigma_{i+m-1}$ of σ with the same order relations as π. We say that patterns $\pi, \tau \in \mathfrak{S}_m$ are *strongly c-Wilf equivalent* if for all n and k, the number of permutations in $\mathfrak{S}_n$ with exactly k occurrences of π as a consecutive pattern is the same as for τ. In 2018, Dwyer and Elizalde [6] conjectured (generalizing a conjecture of Elizalde [8] from 2012) that if $\pi, \tau \in \mathfrak{S}_m$ are strongly c-Wilf equivalent, then (τ_1, τ_m) is equal to one of (π_1, π_m), (π_m, π_1), $(m+1-\pi_1, m+1-\pi_m)$, or $(m+1-\pi_m, m+1-\pi_1)$. We prove this conjecture using the cluster method introduced by Goulden and Jackson in 1979 [12], which Dwyer and Elizalde previously applied [6] to prove that $|\pi_1 - \pi_m| = |\tau_1 - \tau_m|$. A consequence of our result is the full classification of c-Wilf equivalence for a special class of permutations, the non-overlapping permutations. Our approach uses analytic methods to approximate the number of linear extensions of the "cluster posets" of Elizalde and Noy [11].

1. Introduction

Permutation patterns have held considerable interest ever since their introduction by Knuth in 1968 [13]. Of particular importance is the number of permutations in $\mathfrak{S}_n$ avoiding a fixed pattern π. Variations on this problem involve adding restrictions to the pattern: for instance, requiring that it appear in consecutive positions. This notion of *consecutive pattern avoidance* was first systematically studied by Elizalde and Noy in 2003 [10] and has been considered from multiple viewpoints [7, 8, 15]. For more information on the consecutive pattern avoidance literature, we refer the reader to the 2016 survey of Elizalde [9].

We now recall the definition of consecutive pattern avoidance. Given a sequence τ of m distinct positive integers, define the *standardization* of τ to be the sequence formed by replacing the ith smallest entry of τ by i for $1 \le i \le m$. Given permutations $\pi \in \mathfrak{S}_m$ and $\sigma \in \mathfrak{S}_n$, an *occurrence* of π in σ as a consecutive pattern is a subsequence $\sigma_i\sigma_{i+1}\cdots\sigma_{i+m-1}$ of σ whose standardization is π. We say that σ *avoids* π as a consecutive pattern if it has no occurrence of π as a consecutive pattern.

We say that $\pi, \tau \in \mathfrak{S}_m$ are *c-Wilf equivalent* and write $\pi \sim \tau$ if for all n, the number of permutations $\sigma \in \mathfrak{S}_n$ that avoid π (as a consecutive pattern) is the same as the number that avoid τ. We say that $\pi, \tau \in \mathfrak{S}_m$ are *strongly c-Wilf equivalent* and write $\pi \overset{s}{\sim} \tau$ if for all n and k, the number of permutations $\sigma \in \mathfrak{S}_n$ with k occurrences of π is the same as the number with k occurrences of τ. Clearly, if $\pi \overset{s}{\sim} \tau$ then $\pi \sim \tau$. Nakamura conjectured in 2011 that the reverse implication also holds [15, Conjecture 6].

A trivial example of strong c-Wilf equivalence is as follows. For any permutation $\pi = \pi_1 \cdots \pi_m$, define the *reverse* $\pi^R = \pi_m \cdots \pi_1$ and the *complement* $\pi^C = (m+1-\pi_1)\cdots(m+1-\pi_m)$. Then $\pi \overset{s}{\sim} \pi^R \overset{s}{\sim} \pi^C \overset{s}{\sim} \pi^{RC}$. However, there are other strong c-Wilf equivalences; for example, $1342 \overset{s}{\sim} 1432$. A comprehensive table of c-Wilf equivalence classes for permutations of length at most 5 can be found in [6, Table 1].

Following Dwyer and Elizalde [6], we say a permutation $\pi \in \mathfrak{S}_m$ is *standard* if $\pi_1 < \pi_m$ and $\pi_1 + \pi_m \le m + 1$. At least one of the permutations π, π^R, π^C, and π^{RC} is standard, so it suffices

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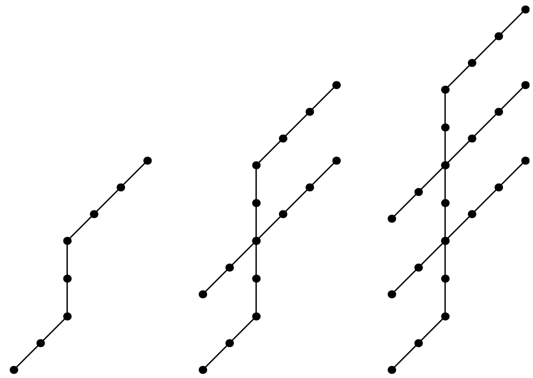

FIGURE 1. The Hasse diagrams of the cluster posets $P_1^{8,3,5}, P_2^{8,3,5}, P_3^{8,3,5}$.

to study (strong) c-Wilf equivalence of standard permutations. Still, the problem of determining whether two permutations are (strongly) c-Wilf equivalent seems difficult in general.

We complete the solution to this problem for a certain restricted class of permutations. Following Bóna [2], we say that $\pi \in \mathfrak{S}_m$ is *non-overlapping* if $\pi_1 \cdots \pi_i$ and $\pi_{m-i+1} \cdots \pi_m$ have different standardizations for $2 \leq i \leq m-1$. Bóna showed that the fraction of non-overlapping permutations approaches $0.36409\ldots$. Furthermore, in 2011 Duane and Remmel [5] showed that if $\pi, \tau \in \mathfrak{S}_m$ are non-overlapping and $(\pi_1, \pi_m) = (\tau_1, \tau_m)$, then $\pi \overset{s}{\sim} \tau$.

In 2012, Elizalde conjectured a converse to this result, which we prove in this paper.

Theorem 1.1 (conjectured in [8, p.14])**.** *Let $\pi, \tau \in \mathfrak{S}_m$ be non-overlapping, standard permutations. If $\pi \sim \tau$, then $(\pi_1, \pi_m) = (\tau_1, \tau_m)$.*

This completes the classification of c-Wilf equivalence for non-overlapping standard permutations, and hence all non-overlapping permutations.

As a consequence of Theorem 1.1 and [6, Theorem 8], we have the following statement about strong c-Wilf equivalence of *all* permutations.

Corollary 1.2 ([6, Conjecture 7])**.** *Let $\pi, \tau \in \mathfrak{S}_m$ be standard permutations. If $\pi \overset{s}{\sim} \tau$, then $(\pi_1, \pi_m) = (\tau_1, \tau_m)$.*

Our proof of Theorem 1.1 uses the *cluster method* introduced by Goulden and Jackson in 1979 [12], which Dwyer and Elizalde previously applied to show that if $\pi, \tau \in \mathfrak{S}_m$ are standard and $\pi \overset{s}{\sim} \tau$, then $\pi_1 - \pi_m = \tau_1 - \tau_m$ [6, Theorem 9].

The cluster method applies to Theorem 1.1 as follows. In 2012, Elizalde and Noy [11] defined the *cluster posets* associated to a permutation $\pi \in \mathfrak{S}_m$. If π is non-overlapping, then it has only one cluster poset P_n^π. This poset depends only on m, n, $a = \pi_1$, and $b = \pi_m$, so we will often write it as $P_n^{m,a,b}$. It can be explicitly described as follows.

Definition 1.3 (cf. [6, Section 2.2])**.** Let a, b, and m be integers with $1 \leq a < b \leq m$. For any $n \geq 1$, define the cluster poset $P_n^{m,a,b}$ by gluing together n chains of length m as follows. The elements of $P_n^{m,a,b}$ are $A_{i,j}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, where $A_{i,b} = A_{i+1,a}$ for $1 \leq i \leq n-1$ and the $A_{i,j}$ are distinct otherwise (so $(m-1)n+1$ elements in total). The relations of $P_n^{m,a,b}$ are generated by the relations $A_{i,1} \leq \cdots \leq A_{i,m}$ for $1 \leq i \leq n$.

The bottommost chain corresponds to $A_{1,j}$. An example is shown for $m = 8, a = 3, b = 5$ in Figure 1.

Dwyer and Elizalde [6, Theorem 14] showed that if $\pi, \tau \in \mathfrak{S}_m$ are non-overlapping and $\pi \sim \tau$, then the posets P_n^π and P_n^τ have the same number of linear extensions for all n. Hence, to prove

Theorem 1.1, it suffices to approximate the number of linear extensions of $P_n^{m,a,b}$ accurately enough to determine the pair (a, b). Dwyer and Elizalde accomplished this for $b - a = 1$ [6, Proposition 20], and the case $b - a > 1$ is completed by the following theorem and its corollary. We use the notation $e(P)$ for the number of linear extensions of a finite poset P.

Theorem 1.4. *Fix integers a, b, and m with $1 \le a < b \le m$. Then*

$$\log e(P_n^{m,a,b}) = (m - b + a - 1)n \log n + c(m, a, b)n + O_{m,a,b}(\log n),$$

where

$$c(m, a, b) = (b - a) \log \mathrm{B}\left(\frac{a-1}{b-a} + 1, \frac{m-b}{b-a} + 1\right) - \log \mathrm{B}(a, m - b + 1) \\ - \log \Gamma(m - b + a + 1) + (m - 1) \log(m - 1) - (b - a) \log(b - a) - m + b - a + 1.$$

Here B *is the beta function, given by*

$$\mathrm{B}(\alpha, \beta) = \int_0^1 u^{\alpha-1}(1 - u)^{\beta-1} du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

Remark. The weaker statement that $\log e(P_n^{m,a,b}) = (m - b + a - 1)n \log n + O_{m,a,b}(n)$ is [6, Lemma 18].

Corollary 1.5 ([6, Conjecture 21])**.** *Let a, b, a', b', and m be integers with $1 \le a < b \le m$ and $1 \le a' < b' \le m$. Suppose that $b - a = b' - a' > 1$ and that $a + b < a' + b' \le m + 1$. Then for sufficiently large n we have*

$$e(P_n^{m,a,b}) < e(P_n^{m,a',b'}).$$

In Section 2, we will begin the proof of Theorem 1.4 by defining a poset $Q_n^{m,a,b}$ such that $e(P_n^{m,a,b})$ and $e(Q_n^{m,a,b})$ are within a polynomial factor in n. In Section 3, we will write $e(Q_n^{m,a,b})$ as a $(n+1)$-dimensional integral using a probabilistic interpretation of linear extensions. In Section 4, we will compute sharp asymptotics for this integral, proving Theorem 1.4. In Section 5, we will use Theorem 1.4 to prove Corollary 1.5 and conclude Theorem 1.1. In Section 6, we will discuss possible extensions of this technique and directions for further research.

2. Modifying the Cluster Poset

In this section, we will define a poset $Q_n^{m,a,b}$ with the property that

$$\log e(P_n^{m,a,b}) = \log e(Q_n^{m,a,b}) + O_{m,a,b}(\log n). \tag{1}$$

Hence, to prove Theorem 1.4, it will suffice to show that

$$\log e(Q_n^{m,a,b}) = (m - b + a - 1)n \log n + c(m, a, b)n + O_{m,a,b}(\log n). \tag{2}$$

Definition 2.1. Let a, b, and m be integers with $1 \le a < b \le m$. For any $n \ge 1$, define $Q_n^{m,a,b}$ to be $P_n^{m,a,b} \cup \{A_{0,b+1}, A_{0,b+2}, \ldots, A_{0,m}, A_{n+1,1}, A_{n+1,2}, \ldots, A_{n+1,a-1}\}$ with the added relations $A_{1,a} \le A_{0,b+1} \le A_{0,b+2} \le \cdots \le A_{0,m}$ and $A_{n+1,1} \le A_{n+1,2} \le \cdots \le A_{n+1,a-1} \le A_{n,b}$.

This is a poset with cardinality $(m-1)n + m - b + a$. An example is shown for $m = 8, a = 3, b = 5$ in Figure 2. Since $P_n^{m,a,b}$ is an induced sub-poset of $Q_n^{m,a,b}$ and $|Q_n^{m,a,b}| - |P_n^{m,a,b}| = m - b + a - 1$, we have

$$e(P_n^{m,a,b}) \le e(Q_n^{m,a,b}) \le |Q_n^{m,a,b}|^{m-b+a-1} e(P_n^{m,a,b}),$$

which implies (1).

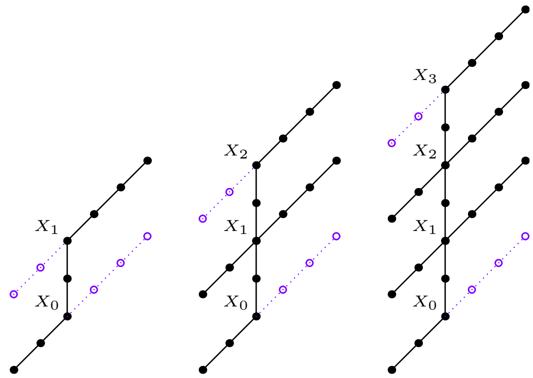

Figure 2. The Hasse diagrams of the modified cluster posets $Q_1^{8,3,5}, Q_2^{8,3,5}, Q_3^{8,3,5}$. The added elements and relations are colored purple and dashed.

3. Probabilistic Interpretation of Linear Extensions

In this section, we will outline the first steps toward counting the linear extensions of $Q_n^{m,a,b}$. The key ingredient is the following probabilistic interpretation of the number of linear extensions of a poset.

Let $(P, \leq)$ be a finite poset and let $\varphi : P \to (0,1)$ be a uniformly random function. Assuming that φ is injective, which occurs with probability 1, it induces a uniformly random linear order $\prec$ on P given by $x \prec y$ if and only if $\varphi(x) < \varphi(y)$. The linear order $\prec$ is a linear extension of P if and only if φ is (strictly) *order-preserving*; that is, if $\varphi(x) < \varphi(y)$ whenever $x < y$. Hence, we may interpret $e(P)$ probabilistically, viz.

$$\frac{e(P)}{|P|!} = \Pr(\varphi \text{ is order-preserving}).$$

Applying this to the case $P = Q_n^{m,a,b}$, we obtain

$$\frac{e(Q_n^{m,a,b})}{((m-1)n+m-b+a)!} = \Pr(\varphi \text{ is order-preserving}).$$

To approximate this probability, we will first write it as a $(n+1)$-dimensional integral. Consider the elements $X_0, \ldots, X_n \in Q_n^{m,a,b}$ defined by $X_i = A_{i+1,a}$ for $0 \leq i \leq n-1$ and $X_i = A_{i,b}$ for $1 \leq i \leq n$. (These definitions coincide when $1 \leq i \leq n-1$.) Let $x_0, \ldots, x_n \in (0,1)$. Conditioned on $\varphi(X_i) = x_i$ for $0 \leq i \leq n$, the probability that φ is order-preserving can be computed as follows. If it is not the case that $x_0 < \cdots < x_n$, then φ is not order-preserving. Otherwise, it is order-preserving with probability

$$\prod_{i=0}^{n} \frac{x_i^{a-1}}{(a-1)!}\frac{(1-x_i)^{m-b}}{(m-b)!}\prod_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^{b-a-1}}{(b-a-1)!}.$$

Each factor $\frac{x_i^{a-1}}{(a-1)!}$ in this product is the probability that $\varphi(A_{i+1,1}) < \cdots < \varphi(A_{i+1,a-1}) < x_i$, each factor $\frac{(1-x_i)^{m-b}}{(m-b)!}$ is the probability that $x_i < \varphi(A_{i,b+1}) < \cdots < \varphi(A_{i,m})$, and each factor $\frac{(x_{i+1}-x_i)^{b-a-1}}{(b-a-1)!}$ is the probability that $x_i < \varphi(A_{i+1,a+1}) < \cdots < \varphi(A_{i+1,b-1}) < x_{i+1}$.

Hence

$$\frac{e(Q_n^{m,a,b})}{((m-1)n+m-b+a)!} = \int\limits_{0<x_0<\cdots<x_n<1} \prod_{i=0}^{n} \frac{x_i^{a-1}}{(a-1)!}\frac{(1-x_i)^{m-b}}{(m-b)!}\prod_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^{b-a-1}}{(b-a-1)!}d\mathbf{x}. \tag{3}$$

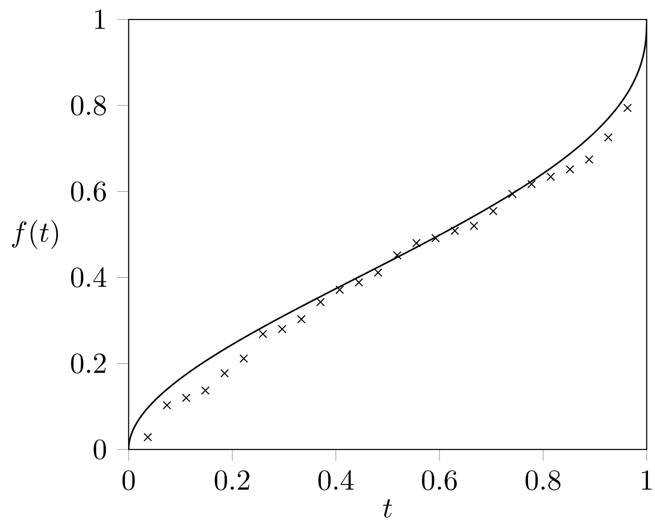

FIGURE 3. A plot of f for $m = 8$, $a = 3$, $b = 5$. The marked points are

$$\left(\frac{i+1}{27}, \widetilde{\mathrm{ht}}_{\prec}(X_i)\right)$$

for $i = 0, \ldots, 25$, where $\prec$ is a random linear extension of $P_{25}^{8,3,5}$ generated by the Markov chain Monte Carlo algorithm of [3]. Here $\widetilde{\mathrm{ht}}_{\prec}(X_i)$ denotes the fraction of elements of $P_{25}^{8,3,5} \setminus \{X_i\}$ that precede X_i in the order $\prec$. Since the marked points are close to the plot of f, this figure agrees with the prediction of Conjecture 6.1.

4. Bounding $e(Q_n^{m,a,b})$

Let I_n be the integral on the right-hand side of (3). In this section we prove Theorem 1.4 by applying a change of variables to the integral I_n.

Define the function $g : [0,1] \to [0,1]$ by

$$g(t) = \frac{1}{\mathrm{B}\left(\frac{a-1}{b-a}+1, \frac{m-b}{b-a}+1\right)} \int_0^t u^{\frac{a-1}{b-a}} (1-u)^{\frac{m-b}{b-a}} \, du.$$

This is a strictly increasing function, independent of n, with $g(0) = 0$ and $g(1) = 1$. Hence we may define its inverse $f = g^{-1} : [0,1] \to [0,1]$, which is also strictly increasing and independent of n. An example is shown for $m = 8, a = 3, b = 5$ in Figure 3. Observe that

$$g'(t) = \frac{1}{\mathrm{B}\left(\frac{a-1}{b-a}+1, \frac{m-b}{b-a}+1\right)} t^{\frac{a-1}{b-a}} (1-t)^{\frac{m-b}{b-a}}.$$

Using the formula $f'(t) = \frac{1}{g'(f(t))}$ yields

$$f'(t)^{b-a} f(t)^{a-1} (1-f(t))^{m-b} = \mathrm{B}\left(\frac{a-1}{b-a}+1, \frac{m-b}{b-a}+1\right)^{b-a}. \tag{4}$$

It follows that f' is unimodal: there exists $\lambda \in [0,1]$ such that $f'(t)$ is decreasing for $t \in [0, \lambda]$ and increasing for $t \in [\lambda, 1]$. (The constant λ is given explicitly by $\lambda = g\left(\frac{a-1}{m-b+a-1}\right)$.) It also follows that there exists an integer $N = N(m,a,b) > 0$ such that for all $t \in (0,1)$, we have

$$\frac{1}{(f'(t))^{b-a-1}} \geq t^N (1-t)^N. \tag{5}$$

Now perform the substitution $y_i = g(x_i)$, so that $x_i = f(y_i)$. This yields

$$\begin{aligned}
I_n &= \int\limits_{0<x_0<\cdots<x_n<1} \prod_{i=0}^{n} \frac{x_i^{a-1}}{(a-1)!}\frac{(1-x_i)^{m-b}}{(m-b)!}\prod_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^{b-a-1}}{(b-a-1)!}d\mathbf{x} \\
&= \int\limits_{0<y_0<\cdots<y_n<1} \prod_{i=0}^{n} \frac{f(y_i)^{a-1}}{(a-1)!}\frac{(1-f(y_i))^{m-b}}{(m-b)!}\prod_{i=0}^{n-1}\frac{(f(y_{i+1})-f(y_i))^{b-a-1}}{(b-a-1)!}\prod_{i=0}^{n} f'(y_i)d\mathbf{y}, \\
&= (b-a-1)!\left(\frac{\mathrm{B}\left(\frac{a-1}{b-a}+1,\frac{m-b}{b-a}+1\right)}{(a-1)!(m-b)!(b-a-1)!}\right)^{n+1} \int\limits_{0<y_0<\cdots<y_n<1}\left(\frac{\prod_{i=0}^{n-1}(f(y_{i+1})-f(y_i))}{\prod_{i=0}^{n} f'(y_i)}\right)^{b-a-1} d\mathbf{y},
\end{aligned} \tag{6}$$

using (4) to pass from the second to third line.

We will now bound the integrand of this last integral.

Lemma 4.1. *Let $0 < y_0 < \cdots < y_n < 1$. Then, we have*

$$\frac{f'(\lambda)}{f'(y_0)f'(y_n)}\prod_{i=1}^{n-1}(y_{i+1}-y_i) \le \frac{\prod_{i=0}^{n-1}(f(y_{i+1})-f(y_i))}{\prod_{i=0}^{n} f'(y_i)} \le \frac{1}{f'(\lambda)}\prod_{i=1}^{n-1}(y_{i+1}-y_i).$$

Proof. By the mean value theorem, for each i with $0 \le i \le n-1$, there exists a real number u_i with $y_i < u_i < y_{i+1}$ such that $f'(u_i) = \frac{f(y_{i+1})-f(y_i)}{y_{i+1}-y_i}$. It remains to show that

$$\frac{f'(\lambda)}{f'(y_0)f'(y_n)} \le \frac{\prod_{i=0}^{n-1} f'(u_i)}{\prod_{i=0}^{n} f'(y_i)} \le \frac{1}{f'(\lambda)}. \tag{7}$$

Recall that $f'(t)$ is unimodal for $t \in (0,1)$ and minimized at $t = \lambda$. Also recall that $0 < y_0 < u_0 < y_1 < u_1 < \cdots < y_{n-1} < u_{n-1} < y_n < 1$. We will prove (7) using only these facts. We consider four cases, depending on where λ falls between the elements of the sequence $0, y_0, u_0, \ldots, u_{n-1}, y_n, 1$.

Case 1 ($\lambda \le y_0$)**.** Then for $0 \le i \le n-1$ we have $\lambda \le y_i < u_i < y_{i+1}$, so $f'(y_i) \le f'(u_i) \le f'(y_{i+1})$. Hence

$$\frac{\prod_{i=0}^{n-1} f'(u_i)}{\prod_{i=0}^{n} f'(y_i)} = \frac{1}{f'(y_n)}\prod_{i=0}^{n-1}\frac{f'(u_i)}{f'(y_i)} \ge \frac{1}{f'(y_n)} \ge \frac{f'(\lambda)}{f'(y_1)f'(y_n)}$$

and

$$\frac{\prod_{i=0}^{n-1} f'(u_i)}{\prod_{i=0}^{n} f'(y_i)} = \frac{1}{f'(y_1)}\prod_{i=0}^{n-1}\frac{f'(u_i)}{f'(y_{i+1})} \le \frac{1}{f'(y_1)} \le \frac{1}{f'(\lambda)}.$$

The inequality (7) is proved.

Case 2 ($y_\ell \le \lambda \le u_\ell$ for some ℓ with $0 \le \ell \le n-1$)**.** Then for $0 \le i \le \ell - 1$, we have $u_i < y_{i+1} \le \lambda$, so $f'(u_i) \ge f'(y_{i+1})$. Similarly, for $\ell + 1 \le i \le n-1$, we have $\lambda \le y_i < u_i$, so $f'(u_i) \ge f'(y_i)$. Hence

$$\frac{\prod_{i=0}^{n-1} f'(u_i)}{\prod_{i=0}^{n} f'(y_i)} = \frac{f'(u_\ell)}{f'(y_0)f'(y_n)}\prod_{i=0}^{\ell-1}\frac{f'(u_i)}{f'(y_{i+1})}\prod_{i=\ell+1}^{n-1}\frac{f'(u_i)}{f'(y_i)} \ge \frac{f'(u_\ell)}{f'(y_0)f'(y_n)} \ge \frac{f'(\lambda)}{f'(y_0)f'(y_n)}.$$

Additionally for $0 \le i \le \ell - 1$ we have $y_i < u_i \le \lambda$, so $f'(u_i) \le f'(y_i)$. Similarly, for $\ell \le i \le n-1$ we have $\lambda \le u_i < y_{i+1}$, so $f'(u_i) \le f'(y_{i+1})$. Hence

$$\frac{\prod_{i=0}^{n-1} f'(u_i)}{\prod_{i=0}^{n} f'(y_i)} = \frac{1}{f'(y_\ell)}\prod_{i=0}^{\ell-1}\frac{f'(u_i)}{f'(y_i)}\prod_{i=\ell}^{n-1}\frac{f'(u_i)}{f'(y_{i+1})} \le \frac{1}{f'(y_\ell)} \le \frac{1}{f'(\lambda)}.$$

The inequality (7) is proved.

Case 3 ($u_\ell \le \lambda \le y_{\ell+1}$ for some ℓ with $0 \le \ell \le n-1$)**.** Then (7) follows from an argument similar to the one used in Case 2.

Case 4 ($\lambda \ge y_n$)**.** Then (7) follows from an argument similar to the one used in Case 1. ☐

From Lemma 4.1 and (5) we obtain

$$\begin{aligned}
&(f'(\lambda))^{b-a-1} \int\limits_{0<y_0<\cdots<y_n<1} y_0^N(1-y_0)^N y_n^N(1-y_n)^N \prod_{i=1}^{n-1}(y_{i+1}-y_i)^{b-a-1} d\mathbf{y} \\
&\le \int\limits_{0<y_0<\cdots<y_n<1} \left(\frac{\prod_{i=0}^{n-1}(f(y_{i+1})-f(y_i))}{\prod_{i=0}^{n} f'(y_i)}\right)^{b-a-1} d\mathbf{y} \\
&\le \frac{1}{(f'(\lambda))^{b-a-1}} \int\limits_{0<y_0<\cdots<y_n<1} \prod_{i=1}^{n-1}(y_{i+1}-y_i)^{b-a-1} d\mathbf{y}. \qquad (8)
\end{aligned}$$

We now evaluate the the left- and right-hand sides of (8). The integral on the right-hand side of (8) is a multivariate beta (or Dirichlet) integral [4] and evaluates to

$$\frac{1}{(f'(\lambda))^{b-a-1}} \frac{(\Gamma(b-a))^{n-1}(\Gamma(1))^2}{\Gamma((b-a)(n-1)+2)} = \frac{(\Gamma(b-a))^n}{((b-a)n)!} n^{O_{m,a,b}(1)}. \qquad (9)$$

The left-hand side of (8) can be evaluated by expanding $(1-y_0)^N$ and $y_n^N = (1-(1-y_n))^N$ using the binomial theorem and evaluating the result as a multivariate beta integral. This yields

$$\begin{aligned}
&(f'(\lambda))^{b-a-1} \int\limits_{0<y_0<\cdots<y_n<1} y_0^N(1-y_0)^N y_n^N(1-y_n)^N \prod_{i=1}^{n-1}(y_{i+1}-y_i)^{b-a-1} d\mathbf{y} \\
&= (f'(\lambda))^{b-a-1} \sum_{j=0}^{N}\sum_{k=0}^{N}(-1)^{j+k}\binom{N}{j}\binom{N}{k} \int\limits_{0<y_0<\cdots<y_n<1} y_0^{N+j}(1-y_n)^{N+k} \prod_{i=1}^{n-1}(y_{i+1}-y_i)^{b-a-1} d\mathbf{y} \\
&= (f'(\lambda))^{b-a-1} \sum_{j=0}^{N}\sum_{k=0}^{N}(-1)^{j+k}\binom{N}{j}\binom{N}{k} \frac{\Gamma(N+j+1)\Gamma(N+k+1)(\Gamma(b-a))^{n-1}}{\Gamma((b-a)(n-1)+2N+j+k+2)}.
\end{aligned}$$

Consider the expression $\Gamma((b-a)(n-1)+2N+j+k+2)$ appearing in this sum. By applying the identity $\Gamma(z) = \frac{1}{z}\Gamma(z+1)$ to this expression $2N-j-k$ times, we may write the sum as

$$q_{m,a,b}(n) \frac{(\Gamma(b-a))^n}{\Gamma((b-a)(n-1)+4N+2)} = \frac{(\Gamma(b-a))^n}{((b-a)n)!} n^{O_{m,a,b}(1)} \qquad (10)$$

where $q_{m,a,b}$ is a polynomial independent of n. By (8), (9), and (10), we have

$$\int\limits_{0<y_0<\cdots<y_n<1} \left(\frac{\prod_{i=0}^{n-1}(f(y_{i+1})-f(y_i))}{\prod_{i=0}^{n} f'(y_i)}\right)^{b-a-1} d\mathbf{y} = \frac{(\Gamma(b-a))^n}{((b-a)n)!} n^{O_{m,a,b}(1)}. \qquad (11)$$

Combining (1), (3), (6), and (11) (and taking logarithms) yields

$$\begin{aligned}\log e(P_n^{m,a,b}) &= O_{m,a,b}(\log(n)) + \log(((m-1)n+m-b+a)!) \\ &\quad + \log\left((b-a-1)!\left(\frac{\mathrm{B}\left(\frac{a-1}{b-a}+1, \frac{m-b}{b-a}+1\right)}{(a-1)!(m-b)!(b-a-1)!}\right)^{n+1}\right) \\ &\quad + \log\left(\frac{(\Gamma(b-a))^n}{((b-a)n)!} n^{O_{m,a,b}(1)}\right).\end{aligned}$$

Theorem 1.4 now follows from applying Stirling's approximation to the expressions $((m-1)n+m-b+a)!$ and $((b-a)n)!$. □

5. Applications to strong c-Wilf equivalence

We now turn to the proofs of Corollary 1.5, Theorem 1.1, and Corollary 1.2, which we now restate.

Corollary 1.5 ([6, Conjecture 21])**.** *Let a, b, a', b', and m be integers with $1 \le a < b \le m$ and $1 \le a' < b' \le m$. Suppose that $b-a = b'-a' > 1$ and that $a+b < a'+b' \le m+1$. Then for sufficiently large n we have*

$$e(P_n^{m,a,b}) < e(P_n^{m,a',b'}).$$

Theorem 1.1 (conjectured in [8, p.14])**.** *Let $\pi, \tau \in \mathfrak{S}_m$ be non-overlapping, standard permutations. If $\pi \sim \tau$, then $(\pi_1, \pi_m) = (\tau_1, \tau_m)$.*

Corollary 1.2 ([6, Conjecture 7])**.** *Let $\pi, \tau \in \mathfrak{S}_m$ be standard permutations. If $\pi \overset{s}{\sim} \tau$, then $(\pi_1, \pi_m) = (\tau_1, \tau_m)$.*

Proof of Corollary 1.5. By Theorem 1.4, it suffices to show that $c(m,a,b) < c(m,a',b')$. Let $d = b-a = b'-a' > 1$ and define the function $A : \mathbb{R}_{\ge 0} \to \mathbb{R}$ by

$$A(t) = d \log \Gamma\left(\frac{t}{d}+1\right) - \log \Gamma(t+1).$$

We may compute

$$A''(t) = \frac{1}{d}\psi^{(1)}\left(\frac{t}{d}+1\right) - \psi^{(1)}(t+1)$$

where $\psi^{(1)}$ is the trigamma function [1, 6.4.1]. By the series expansion for the trigamma function [1, 6.4.10], we may write this as

$$\begin{aligned}A''(t) &= \frac{1}{d}\sum_{k=1}^{\infty}\frac{1}{\left(\frac{t}{d}+k\right)^2} - \sum_{j=1}^{\infty}\frac{1}{(t+j)^2} \\ &= \frac{1}{d}\sum_{k=1}^{\infty}\frac{1}{\left(\frac{t}{d}+k\right)^2} - \sum_{k=1}^{\infty}\sum_{j=d(k-1)+1}^{dk}\frac{1}{(t+j)^2} \\ &< \frac{1}{d}\sum_{k=1}^{\infty}\frac{1}{\left(\frac{t}{d}+k\right)^2} - d\sum_{k=1}^{\infty}\frac{1}{(t+dk)^2} \\ &= 0.\end{aligned}$$

Hence A is strictly concave.

For any t, we may write

$$\begin{aligned} c(m,t,t+d) &= A(t-1) + A(m-t-d) \\ &\quad - d\log\Gamma\left(\frac{m-d-1}{d}+2\right) + (m-1)\log(m-1) - d\log d - m + d + 1, \end{aligned}$$

which is a strictly concave function of t that is symmetric about $t = \frac{m-d+1}{2}$. Hence it is unimodal and maximized at $t = \frac{m-d+1}{2}$. Since $a < a'$ and $a' = \frac{a'+b'-d}{2} \leq \frac{m-d+1}{2}$, we have

$$c(m,a,b) = c(m,a,a+d) < c(m,a',a'+d) = c(m,a',b'),$$

as desired. ☐

Proof of Theorem 1.1. Let $(a,b) = (\pi_1, \pi_m)$ and $(a',b') = (\tau_1, \tau_m)$. Since π and τ are standard, we have the inequalities $a < b$ and $a + b \leq m + 1$ and $a' < b'$ and $a' + b' \leq m + 1$.

As noted in Section 1, Dwyer and Elizalde [6, Theorem 14] showed that if $\pi, \tau \in \mathfrak{S}_m$ are non-overlapping and $\pi \sim \tau$, then the posets $P_n^\pi = P_n^{m,a,b}$ and $P_n^\tau = P_n^{m,a',b'}$ have the same number of linear extensions for all n.

By Theorem 1.4, we have $\log e(P_n^\pi) = (m-b+a-1)n\log n + O_{m,a,b}(n)$ and $\log e(P_n^\tau) = (m-b'+a'-1)n\log n + O_{m,a',b'}(n)$, so $b - a = b' - a'$.

Assume for the sake of contradiction that $(a,b) \neq (a',b')$. Without loss of generality $a < a'$. If $b - a = b' - a' > 1$ then by Corollary 1.5 we have that $e(P_n^\pi) < e(P_n^\tau)$ for sufficiently large n, which is a contradiction.

Finally, if $b - a = b' - a' = 1$ we have $e(P_2^\pi) > e(P_2^\tau)$, as noted by Dwyer and Elizalde [6, Proposition 20], which is again a contradiction. The theorem is proved. ☐

Proof of Corollary 1.2. By [6, Theorem 8], Theorem 1.1 implies the result. ☐

6. Further remarks

6.1. **Other Variational Problems.** It is clear that our method of approximating the $(n+1)$-dimensional integral I_n can be extended to approximate integrals of the form

$$\int\limits_{0<x_0<\cdots<x_n<1} \prod_{i=0}^{n} h(x_i) \prod_{i=0}^{n-1} (x_{i+1}-x_i)^\beta d\mathbf{x} \tag{12}$$

for any sufficiently well-behaved function $h : [0,1] \to \mathbb{R}_{\geq 0}$ and any $\beta \geq 0$. The method is to perform a substitution $x_i = j(y_i)$, where $j : [0,1] \to [0,1]$ plays the role of f from Section 4. The function j can be found by solving the differential equation that $h(j(t)) \cdot j'(t)^{\beta+1}$ is constant, with the initial conditions $j(0) = 0, j(1) = 1$.

Using the calculus of variations, the function j can also be described as the function that minimizes

$$\int_0^1 (\log h(j(t)) + (\beta+1)\log j'(t))dt, \tag{13}$$

subject to $j(0) = 0, j(1) = 1$. Observe that if $x_i = j\left(\frac{i+1}{n+2}\right)$ for $0 \leq i \leq n$ for n large, we have

$$\log\left(\prod_{i=0}^{n} h(x_i) \prod_{i=0}^{n-1}(x_{i+1}-x_i)^\beta\right) \approx n\int_0^1 (\log h(j(t)) + \beta\log j'(t))dt - \beta n\log n,$$

and the integral appearing here is formally similar to the one appearing in (13). It would be interesting to explore why this is the case and provide a satisfying explanation for the extra weight of $\log j'(t)$ in (13).

6.2. **Conjectures and Further Questions.** Several outstanding questions about random linear extensions of the posets $P_n^{m,a,b}$ and $Q_n^{m,a,b}$ remain. Namely, we wish to demonstrate a concentration result for "typical" linear extensions. The following conjecture is motivated by the observation that the volume of the integral I_n from (3) appears to be tightly concentrated near the point

$$(x_0, \dots, x_n) = \left(f\left(\frac{1}{n+2}\right), \dots, f\left(\frac{n+1}{n+2}\right)\right),$$

where $(y_0, \dots, y_n) = \left(\frac{1}{n+2}, \dots, \frac{n+1}{n+2}\right)$ are equally spaced.

Conjecture 6.1. *Fix integers m, a, b with $1 \le a < b \le m$. Then, with probability tending towards 1 as $n \to \infty$, the height of X_i in a uniformly random linear extension of $P_n^{m,a,b}$ is*

$$|P_n^{m,a,b}| \left(f\left(\frac{i+1}{n+2}\right) + o_{m,a,b}(1)\right)$$

for $0 \le i \le n$.

The expected height of an element in a uniformly random linear extension of a poset is an object of independent interest [16].

Nakamura also conjectured the following, which appears to defy the techniques we have outlined above. We remark that the natural analogue of this conjecture for ordinary Wilf-equivalence is already false for patterns of length 3 [6].

Conjecture 6.2 ([15, Conjecture 6])**.** *Let $\pi, \tau \in \mathfrak{S}_m$. Then $\pi \sim \tau$ if and only if $\pi \overset{s}{\sim} \tau$.*

This conjecture has already been resolved in the case that π, τ are non-overlapping [8, 14].

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