Quantum Information Theory in Quantum Hamiltonian Complexity

Aram Harrow (MIT)
Simons Institute
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Entanglement

Original motivation for quantum computing [Feynman ‘82]

Nature isn’t classical, dammit, and if you want to make a simulation of Nature, you’d better make it quantum mechanical, and by golly it’s a wonderful problem, because it doesn't look so easy.

\[ N \text{ systems in product state} \rightarrow O(N) \text{ degrees of freedom} \]

\[ N \text{ entangled systems} \rightarrow \exp(N) \text{ degrees of freedom} \]

Describes cost of simulating dynamics or even describing a state.

This talk: can we do better when a system is only lightly entangled?
success story: quantum circuits

Classical simulation possible in time $O(T \cdot \exp(k))$, where
- $k =$ treewidth [Markov-Shi ‘05]
- $k =$ max # of gates crossing any single qubit
  [Yoran-Short ‘06, Jozsa ‘06]

+ Complexity interpolates between linear and exponential.
- Treating all gates as “potentially entangling” is too pessimistic.
success story: 1-D systems

\[ H = H_{12} + H_{23} + \ldots + H_{n-1,n} \]

Classically easy to minimize energy, calculate \( \text{tr } e^{-H/T} \), etc.

Quantumly QMA-complete to estimate ground-state energy (to precision \( 1/\text{poly}(n) \) for \( H \) with gap \( 1/\text{poly}(n) \)).

Extension to trees:

- [Landau-Vazirani-Vidick, '13]
- \( n \) qudits with gap \( \lambda \) and precision \( \varepsilon \) \( \rightarrow \) runtime \( \exp(\exp(d/\lambda)\log(n)) \text{ poly}(1/\varepsilon) \)

intuition:
- spectral gap of \( H \)
- exponential decay of correlations
- entanglement area law
- efficient MPS description

Hastings '07, etc.
meta-strategy

1. solve trivial special case (e.g. non-interacting theory)
2. treat corrections to theory as perturbations
partial success: stabilizer circuits

exact version:
Clifford gates on n qubits = \{U \text{ s.t. } UPU^\dagger \text{ is a Pauli for all Paulis } P\}
Generated by various single-qubit gates and CNOTs.

[Gottesman-Knill ’98] Clifford circuits simulable in time $\tilde{O}(nT)$.
intuition: Paulis $\cong \mathbb{F}_2^{2n}$, Cliffords $\cong \text{Sp}_{2n}(\mathbb{F}_2)$

interpolation theorem [Aaronson-Gottesman ‘04]
Circuits with $k$ non-Clifford gates simulable in time $\tilde{O}(nT \exp(k))$.

+ Can simulate some highly entangled computations including most quantum error-correction schemes.
- Almost all single-qubit gates are non-Clifford gates.
Theorem [Brandão-Harrow, 1310.0017]
If $H$ is a 2-local Hamiltonian on a $D$-regular graph of $n$ qudits with $H = \sum_{i<j} H_{i,j}$ and each $\|H_{i,j}\| \leq 1$, then there exists a product state $|\psi\rangle = |\psi_1\rangle \otimes \ldots \otimes |\psi_n\rangle$ such that

$$\lambda_{\min} \leq \langle \psi | H | \psi \rangle \leq \lambda_{\min} + O(d^{2/3} / D^{1/3})$$

Corollary
The ground-state energy can be approximated to accuracy $O(d^{2/3} / D^{1/3})$ in $\text{NP}$.

interpretation: quantum PCP [tomorrow] impossible unless $D = O(d^2)$
intuition from physics: mean-field approximation

used in limit of high degree, e.g.

1-D

2-D

3-D

∞-D

Bethe lattice := Cayley graph
clustered approximation

Given a Hamiltonian $H$ on a graph $G$ with vertices partitioned into $m$-qudit clusters $(X_1, ..., X_{n/m})$, can approximate $\lambda_{\text{min}}$ to error with a state that has no entanglement between clusters.

\[ \Phi(X_i) = \Pr_{(u,v) \in E} \left( v \notin X_i \mid u \in X_i \right) \]

good approximation if

1. expansion is $o(1)$
2. degree is high
3. entanglement satisfies subvolume law
proof sketch

mostly following [Raghavendra-Tan, SODA ’12]

Chain rule Lemma:
\[ I(X:Y_1...Y_k) = I(X:Y_1) + I(X:Y_2|Y_1) + ... + I(X:Y_k|Y_1...Y_{k-1}) \]
\[ \rightarrow I(X:Y_t|Y_1...Y_{t-1}) \leq \log(d)/k \text{ for some } t \leq k. \]

Decouple most pairs by conditioning:
Choose \( i, j_1, ... , j_k \) at random from \{1, ..., n\}
Then there exists \( t < k \) such that
\[ \mathbb{E}_{i,j,j_1,...,j_t} I(X_i : X_j | X_{j_1} \ldots X_{j_t}) \leq \frac{\log(d)}{k} \]

Discarding systems \( j_1, ..., j_t \) causes error \( \leq k/n \) and leaves a distribution \( q \) for which
\[ \mathbb{E}_{i,j} I(X_i : X_j)_q \leq \frac{\log(d)}{k} \]
\[ \mathbb{E}_{i \sim j} I(X_i : X_j)_q \leq \frac{n \log(d)}{D} \frac{1}{k} \]
Does this work quantumly?

What changes?

😊 Chain rule, Pinsker, etc, still work.

😢 Can’t condition on quantum information.

😭 $I(A:B|C)_\rho \approx 0$ doesn’t imply $\rho$ is approximately separable

[Ibinson, Linden, Winter ‘08]

Key technique: informationally complete measurement maps quantum states into probability distributions with poly(d) distortion.

$$d^{-3} \| \rho - \sigma \|_1 \leq \| M(\rho) - M(\sigma) \|_1 \leq \| \rho - \sigma \|_1$$

quantum trace distance

classical variational distance

quantum trace distance
Proof of qPCP no-go

1. Measure $\varepsilon n$ qudits and condition on outcomes. Incur error $\varepsilon$.

2. Most pairs of other qudits would have mutual information
   $\leq \log(d) / \varepsilon D$ if measured.

3. Thus their state is within distance $d^2(\log(d) / \varepsilon D)^{1/2}$ of product.

4. Witness is a global product state. Total error is $\varepsilon + d^2(\log(d) / \varepsilon D)^{1/2}$.
   Choose $\varepsilon$ to balance these terms.
Here is the QCD Hamiltonian. Can you describe the wavefunction of the proton in a way that will let me compute its mass?

Greetings! The proton is the ground state of the u, d, and u quarks.

Can you give me some description I can use to get a 0.1% accurate estimate using fewer than $10^{50}$ steps?

No. I can, however, give you many stars protons, whose mass you can measure.
better approximation?

- There is no guaranteed way to improve the approximation with a larger witness.

Approximation quality depends on:
- degree (fixed)
- average expansion (can change, but might always be high)
- average entropy (can change, but might always be high)

SDP hierarchy:
variables = \{density matrices for all sets of \leq k qubits\}
constraints = overlap compatibility + global PSD constraint (tomorrow)

Can prove this finds a good product state when \( k \gg \text{poly}(\text{threshold rank}) \).
Clearly converges to the true ground state energy as \( k \rightarrow n \).

SDP relaxation \( \leq \) true ground state energy \( \leq \) variational bounds
improves with \( k \) need better ansatz, eg MPS
quantifying entanglement

bipartite pure states – the nice case

\[ |\psi\rangle = \sum_{i=1}^{d} \sum_{j=1}^{d} c_{i,j} |i\rangle \otimes |j\rangle \]

\[ = \sum_{i=1}^{d} \sqrt{\lambda_i} |a_i\rangle \otimes |b_i\rangle \]

- \( \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_d \geq 0 \) determine equivalence under local unitaries
- LOCC can modify \( \lambda \) according to majorization partial order
- entanglement can be quantified by [Rényi] entropies of \( \lambda \)
- asymptotic entanglement determined by \( H(\lambda) = S(\psi^A) = S(\psi^B) \)
  “entropy of entanglement” \( \rightarrow \) entanglement as resource
  [Bennett, Bernstein, Popescu, Schumacher '95]
mixed / multipartite

mixed-state and/or multipartite entanglement measures form a zoo
• relating to pure bipartite entanglement (formation/distillation)
• distance to separable states (relative entropy of entanglement, squashed ent.)
• easy to compute but not operational (log negativity, concurrence)
• operational but hard to compute (distillable key, geometric measure, tensor rank)
• not really measuring entanglement (ent. of purification, ent. of assistance)
• regularized versions of most of the above

Generally “entropic” i.e. match on pure states. Hopefully convex, continuous, monotonic, etc.

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Brandão-Christandl-Yard ’10

Christandl ’06
conditional mutual information and Markov states

\[ I(A:B|C) = H(A|C) + H(B|C) - H(AB|C) \]
\[ = H(AC) + H(BC) - H(ABC) - H(C) \]
\[ = \sum_c p(C=c) I(A:B|C=c) \]
\[ \geq 0 \]

Classical

TFAE:
- \( I(A:B|C)=0 \)
- \( p(a,b,c) = p_1(c) p_2(a|c) p_3(b|c) \)
- \( p = \exp(H_{AC} + H_{BC}) \) for some \( H_{AC}, H_{BC} \)
  [Hammersley-Clifford]
- A & B can be reconstructed from C

Quantum

\[ I(A:B|C)=0 \]

\[ C \cong \bigoplus_i C_{A,i} \otimes C_{B,i} \]

\[ \rho^{ABC} = \sum_i p_i \alpha^{AC}_{A,i} \otimes \beta^{BC}_{B,i} \]

\( \rho^{AB} \) is separable

[Hayden, Jozsa, Petz, Winter ‘04]
conditional mutual information

\[ I(A:B|C) = 0 \iff \rho \text{ is a Markov state} \]
\[ I(A:B|C) = \varepsilon \iff \rho \text{ is an approximate Markov state?} \]

**Classical**

\[ I(A:B|C)_\rho = \min_q \text{Markov } D(p \parallel q) \]

\( I(A:B|C) \) small \( \Rightarrow \) can approximately reconstruct \( A,B \) from \( C \).

**Quantum**

\[ I(A:B|C)_\rho \leq \min_\sigma \text{Markov } D(\rho \parallel \sigma) \]

\( I(A:B|C) \) can be \( \ll \) RHS

[Ibinson, Linden, Winter ‘06]

\( \rho^{AB} \) can be far from separable in trace distance but not 1-LOCC distance. [Brandão, Christandl, Yard ‘10]

approximate reconstruction? [Winter]

application to Hamiltonians?

[Poulin, Hastings ‘10] [Brown, Poulin ‘12]
approximate quantum Markov state

three possible definitions

1. $I(A:B|C)_\rho \leq \text{small}$

2. $\min_{\sigma \text{ Markov}} D(\rho \parallel \sigma) \leq \text{small}$

3. reconstruction: There exists a map $T:C \rightarrow BC$ such that $T(\rho^{AC}) \approx \rho^{ABC}$

$\rho^{AB}$ is
$\approx \text{k-extendable}$
Can we simulate lightly entangled dynamics? i.e. given the promise that entanglement is always \( \leq k \) is there a simulation that runs with overhead \( \exp(k) \)?

Time evolution of quantum systems

\[
\frac{d\rho}{dt} = -i(H\rho - \rho H) + \text{noise terms that are linear in } \rho
\]

Can we simulate lightly entangled dynamics? i.e. given the promise that entanglement is always “\( \leq k \)” is there a simulation that runs with overhead \( \exp(k) \)?

- **Ideal QC**: possible
- **FTQC possible**: \( 10^{-2} \)-ish
- **Classical simulation possible**: \( \approx 0.3 \)
- **Classical simulation possible**: 1
If exponential quantum speedup/hardness is due to entanglement, then can we make this quantitative?

Answer may include:
• saving the theory of entanglement measures from itself
• new classical ways to describe quantum states (e.g. MPS)
• conditional mutual information
• the right definition of “approximate quantum Markov states”