Quantum ~~de Finetti theorems for local measurements~~ methods to analyze SDP hierarchies

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motivation/warmup

nonlinear optimization --> convex optimization

$$\max_{x \in S^n} \sum_{i,j=1}^{n} M_{i,j} x_i x_j$$

$$\langle A, B \rangle = \sum_{i,j=1}^{n} A_{i,j} B_{i,j}$$

$$S^n = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$$

$$= \max_{x \in S^n} \langle M, xx^T \rangle$$

$$= \max_{\rho \in D(n)} \langle M, \rho \rangle$$

D(n) = conv $\{xx^T : x \in S^n\}$ = density matrices

a harder problem

$$\max_{x\in S^n} \sum_{i_1,i_2,j_1,j_2} M_{i_1 i_2, j_1 j_2} x_{i_1} x_{i_2} x_{j_1} x_{j_2}$$

$(A \otimes B)_{i_1 i_2, j_1 j_2}$

$= A_{i_1, j_1} B_{i_2, j_2}$

$$= \max_{x\in S^n} \langle M, xx^T \otimes xx^T \rangle$$

$$= \max\{\langle M, \sigma\rangle : \sigma \in \mathrm{SepSym}(n,n)\}$$

SepSym(n,n) = conv $\{xx^T \otimes xx^T : x \in S^n\}$

$\subset$ Sep(n,n) = conv $\{xx^T \otimes yy^T : x,y \in S^n\} \subset D(n^2)$

polynomial optimization

$D(n)_n = \text{conv}\ \{xx^T : x \in S^n\}$ EASY

$\text{SepSym}(n,n) = \text{conv}\ \{xx^T \otimes xx^T : x \in S^n\}$ HARD

≈ tensor norms ≈ 2->4 norm ≈ small-set expansion

need to find relaxation!

k-extendable relaxation

want $\sigma \in \mathrm{SepSym}(n,n) = \mathrm{conv}\{xx^T \otimes xx^T : x \in S^n\}$

relax to $\tilde{\rho} \in D(n^k)$

$$\tilde{\rho}_{i_1 \ldots i_k, j_1 \ldots j_k} = \tilde{\rho}_{i_{\pi(1)} \cdots i_{\pi(k)}, j_{\pi'(1)} \cdots j_{\pi'(k)}} \quad \forall \pi, \pi' \in S_k$$

ideally $\tilde{\rho}_{i_1 \ldots i_k, j_1 \ldots j_k} = x_{i_1} \cdots x_{i_k} x_{j_1} \cdots x_{j_k}$

recover $\rho \in D(n^2)$

$$\rho_{i_1 i_2, j_1 j_2} = \sum_{i_3, \ldots, i_k} \tilde{\rho}_{i_1 i_2 i_3 \ldots i_k, j_1 j_2 i_3 \ldots i_k}$$

why?

1. partial trace = quantum analogue of marginal distribution
2. using $\Sigma_i x_i^2 = 1$ constraint

why should this work?

physics explanation:
"monogamy of entanglement"
only separable states are infinitely sharable

math explanation:

$$\tilde{\rho} = \sum_i (x_i x_i^T)^{\otimes k}$$

$$\rho = \sum_{i,j} x_i x_j^T \otimes x_i x_j^T \langle x_i, x_j \rangle^{2(k-2)}$$

$$\to \sum_i x_i x_i^T \otimes x_i x_i^T$$ as $k \to \infty$

convergence rate

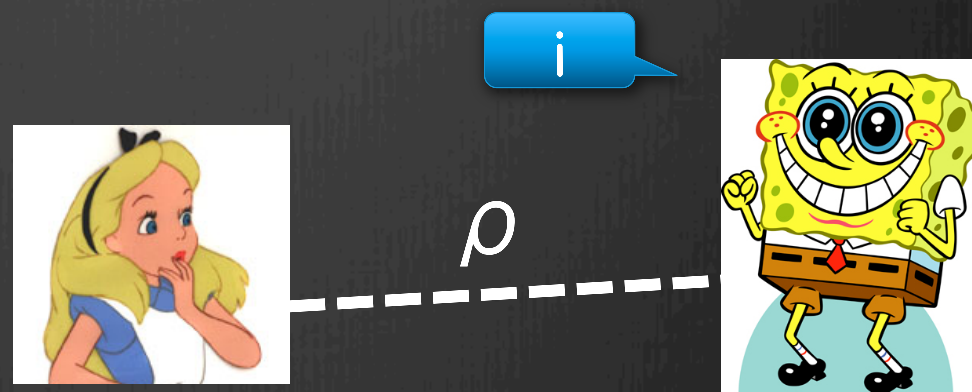
run-time = $n^{O(k)}$

dist(k-extendable, SepSym(n,n)) = f(k,n) = ??

trace dist(ρ , σ) = $\max_{0 \le M \le I} \langle M, \rho - \sigma \rangle$ ~ n/k → $n^{O(n)}$ time

[Brandão, Christandl, Yard; STOC '11]
distance ~ $(\log(n)/k)^{1/2}$
for M that are 1-LOCC
→ time $n^{O(\log(n))}$

<u>Def of 1-LOCC</u>
$M = \sum_i A_i \otimes B_i$ such that
$0 \le A_i \le I$
$0 \le B_i$
$\sum_i B_i = I$



Pr[accept | i] =
$\langle A_i \otimes B_i, \rho \rangle$ / Pr[i]

Pr[i] =
$\langle I \otimes B_i, \rho \rangle$

our results

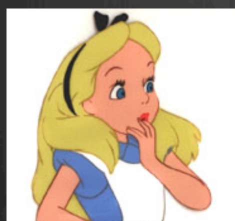
1. simpler proof of BCY 1-LOCC bound
2. extension to multipartite states
3. dimension-independent bounds if Alice is non-adaptive
4. extension to non-signaling distributions
5. explicit rounding scheme
6. (next talk) version without symmetry

applications

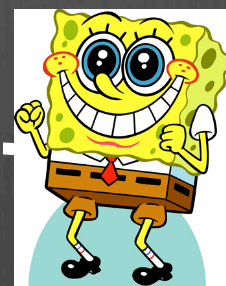
1. optimal algorithm for degree-$\sqrt{n}$ poly optimization (assuming ETH)
2. optimal algorithm for approximating value of free games
3. hardness of entangled games
4. QMA = QMA with poly(n) unentangled Merlins & 1-LOCC measurements
5. "pretty good tomography" without independence assumptions
6. convergence of Lasserre
7. multipartite separability testing

proof sketch

Further restrict to LO measurements



ρ



$\Pr[a,b] = \langle \rho, A_a \otimes B_b \rangle$

a

b

$M = \sum_{a,b} Y_{ab} A_a \otimes B_b$

$0 \le Y_{ab} \le 1$

$\sum_a A_a = I$

$\sum_b B_b = I$

Goal: $\max \sum_{a,b} \Pr[a,b]\ \gamma_{a,b}$

exact solutions ($\rho \in$ SepSym):

$\Pr[a,b] = \sum \lambda_i\ q_i(a)\ r_i(b)$

$\Pr[\text{accept} \mid a,b] = \gamma_{ab}$

rounding

$\Pr[a,b] = \langle \rho, A_a \otimes B_b \rangle$

Goal: max $\Sigma_{a,b} \Pr[a,b]\ \gamma_{a,b}$

$M = \Sigma_{a,b} Y_{ab} A_a \otimes B_b$

$0 \leq Y_{ab} \leq 1$

$\Sigma_a A_a = I$

$\Sigma_b B_b = I$

exact solutions ($\rho \in$ SepSym):

$\Pr[a,b] = \Sigma \lambda_i\, q_i(a)\, r_i(b)$

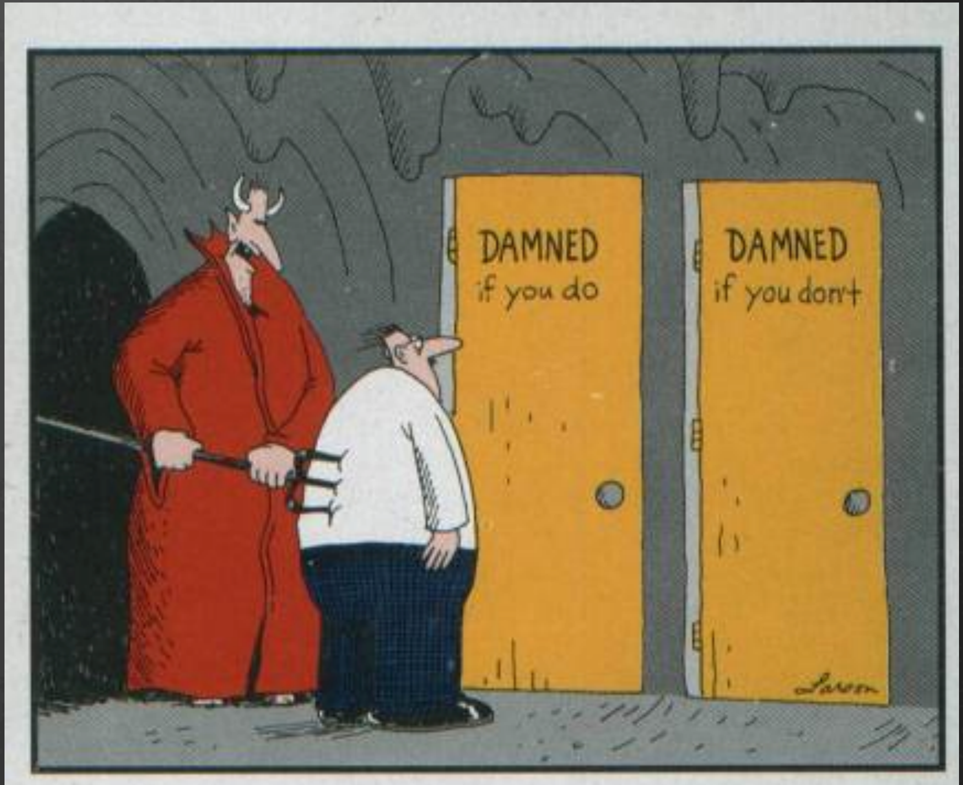
relaxation

$$\Pr[a,b] = \sum_{b_2,\ldots,b_k} p_{a,b,b_2,\ldots,b_k}$$

$$p_{a,b_1,\ldots,b_k} = \langle \tilde{\rho}, A_a \otimes B_{b_1} \otimes \cdots \otimes B_{b_k} \rangle$$

proof idea

- good approximation if $\Pr[a,b_1] \approx_\varepsilon \Pr[a] \cdot \Pr[b_1]$
- otherwise $H(a|b_1) < H(a) - \varepsilon^2$



"C'mon, c'mon — it's either one or the other."

information theory

[Raghavendra-Tan, SODA '12]

$\log(n) \geq I(a:b_1 \dots b_k)$
$= I(a:b_1) + I(a:b_2|b_1) + \dots + I(a:b_k|b_1 \dots b_{k-1})$

$\therefore I(a:b_j|b_1 \dots b_{j-1}) \leq \log(n)/k$ for some j

$\therefore \rho \approx \mathrm{Sep}$ for this particular measurement

Note: Brandão-Christandl-Yard based on quantum version of $I(a:b|c)$.

open questions

1. Improve 1-LOCC to SEP
   would imply QMA = QMA with poly(n) Merlins
   and quasipolynomial-time algorithms for tensor problems

2. Better algorithms for small-set expansion / unique games

3. Make use of "partial transpose" symmetry

4. Understand quantum conditional mutual information

5. extension to entangled games that would yield
   NEXP $\subseteq$ MIP*. (see paper)

6. More counter-examples / integrality gaps.