The princess and the EPR pair

or

Entanglement spread, communication complexity and information theory.

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# Quantum Information Basics

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entanglement

An old mystery of quantum theory:
“[not] one, but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought.”

--- Schrödinger, 1935

Spooky action at a distance
“This makes the reality of [quantities] P and Q depend upon the process of measurement carried out on the first system, which does not disturb the second system in any way. No reasonable definition of reality could be expected to permit this.”

--- Einstein, Podolsky and Rosen [EPR], 1935

canonical form:
EPR pair

\[ |\Phi_2\rangle = \frac{|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]
entanglement as resource
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- Bell’s theorem [1964] describes a set of distributed measurements on $|\Phi_2\rangle$ that produce outcomes inconsistent with any correlated classical probability distribution.
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- **Quantum key distribution** achieves information-theoretic security using entanglement either implicitly [BB84] or explicitly [E91].

- **Quantum computing** exploits the exponential scaling to perform calculations that are hard to simulate classically.
Two-party entanglement:
Alice and Bob share $|\psi\rangle = \sum_{ij} c_{ij} |i\rangle \otimes |j\rangle$. 
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But what if classical communication isn’t free?
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Measurements and unitary evolutions are constrained to respect this partition.
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Idea:
If Alice and Bob are allowed only local unitaries (LU) then the Schmidt coefficients of their state remain exactly the same.

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1. $|\Phi_2\rangle^\otimes$ and $|\Phi_2\rangle^\otimes$ are only approximately orthogonal.
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**Really?**
1. $|\Phi_2\rangle^{\otimes k}$ and $|\Phi_2\rangle^{\otimes l}$ are only approximately orthogonal.
2. Technically we can only approximately decompose $|\psi\rangle$ into

$$\sum_{k \geq 0} \sqrt{p_k} |k\rangle_A |k\rangle_B |\Phi_{2^{\epsilon_k}}\rangle_{AB}$$
implications

1. Any transformation using local unitaries and Q qubits of communication has off-diagonal blocks decaying as \( \leq 2^Q \frac{|k-\ell|}{2} \).

\[
\begin{pmatrix}
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & *
\end{pmatrix}
\]

2. ‘Exotic’ states, such as \( |01\rangle^\otimes n \pm |\Phi_2\rangle^\otimes n / \sqrt{2} \), should be difficult to create, and are potentially valuable.
Traditional version: A mysterious woman appears at the castle claiming to be a princess. That night, a single pea placed under twenty mattresses keeps her from sleeping. The prince realises that she is genuine and immediately asks her to marry him.
A bipartite fairy tale

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**Quantum version:** Our heroine is so delicate that she can distinguish from any orthogonal state. In particular, she can distinguish it from

\[
\frac{\left|\psi_1\right> + \left|\psi_2\right>}{\sqrt{2}}
\]

and

\[
\frac{\left|\psi_1\right> - \left|\psi_2\right>}{\sqrt{2}}
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**Quantum version:** Our heroine is so delicate that she can distinguish from any orthogonal state. In particular, she can distinguish it from $\left\{\begin{array}{c} + \\ \sqrt{2} \end{array}\right\}$ however! Adding or removing lots of mattresses is difficult.

$\left|\begin{array}{c} \rule{30pt}{2pt} \\ \rule{30pt}{2pt} \end{array}\right\rangle + \left|\begin{array}{c} \rule{30pt}{2pt} \\ \rule{30pt}{2pt} \end{array}\right\rangle$ requires $\left|\begin{array}{c} \rule{30pt}{2pt} \\ \rule{30pt}{2pt} \end{array}\right\rangle$
Distinguishing \( \frac{1}{\sqrt{2}} \left| \text{True} \right> + \frac{1}{\sqrt{2}} \left| \text{False} \right> \) from \( \frac{1}{\sqrt{2}} \left| \text{True} \right> - \frac{1}{\sqrt{2}} \left| \text{False} \right> \) with a reversible quantum circuit allows us to apply a phase (-1) to one of the states.

Should he marry her?
Distinguishing \( \frac{\sqrt{2}}{\sqrt{2}} \left| \begin{array}{c} \uparrow \\ \downarrow \end{array} \right\rangle + \left| \begin{array}{c} \uparrow \\ \downarrow \end{array} \right\rangle \) from \( \frac{\sqrt{2}}{\sqrt{2}} \left| \begin{array}{c} \uparrow \\ \downarrow \end{array} \right\rangle - \left| \begin{array}{c} \downarrow \\ \uparrow \end{array} \right\rangle \) with a reversible quantum circuit allows us to apply a phase \((-1)\) to one of the states.

But \( \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \) in the \( \frac{\sqrt{2}}{\sqrt{2}} \left| \begin{array}{c} \uparrow \\ \downarrow \end{array} \right\rangle \pm \left| \begin{array}{c} \downarrow \\ \uparrow \end{array} \right\rangle \) basis is equivalent to

\( \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \) in the \( \frac{\sqrt{2}}{\sqrt{2}} \left| \begin{array}{c} \uparrow \\ \downarrow \end{array} \right\rangle , \left| \begin{array}{c} \downarrow \\ \uparrow \end{array} \right\rangle \) basis.
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Distinguishing \[\frac{1}{\sqrt{2}}\left|\alpha\right\rangle + \frac{1}{\sqrt{2}}\left|\beta\right\rangle\] from \[\frac{1}{\sqrt{2}}\left|\alpha\right\rangle - \frac{1}{\sqrt{2}}\left|\beta\right\rangle\] with a reversible quantum circuit allows us to apply a phase (-1) to one of the states.

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This performs

Conclusion: The “princess” is stronger than she looks!
The relevance to entanglement
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- Performing $|01\rangle^\otimes n \leftrightarrow |\Phi_2\rangle^\otimes n$ requires transmitting $n$ qubits.

$|01\rangle^\otimes n = \left|\begin{array}{c}
\text{state 1}
\end{array}\right\rangle$

$|\Phi_2\rangle^\otimes n = \left|\begin{array}{c}
\text{entangled state}
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|1\rangle
\end{array}$ $|\Phi_2\rangle^\otimes n = \begin{array}{c}
|\alpha\rangle \\
|\beta\rangle
\end{array}$

• Therefore, distinguishing $|01\rangle^\otimes n \pm |\Phi_2\rangle^\otimes n / \sqrt{2}$ requires transmitting $n/2$ qubits.

Why? Because any measurement in the $\{|\alpha\rangle , |\beta\rangle\}$ basis using $Q$ qubits of communication implies that the operation $|\alpha\rangle\langle \alpha| - |\beta\rangle\langle \beta|$ can be performed using $2Q$ qubits of communication.
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$$|01\rangle^\otimes n = \begin{array}{c} 0 \\
1
\end{array} \quad \begin{array}{c} 0 \\
1
\end{array} \quad |\Phi_2\rangle^\otimes n = \begin{array}{c} \Phi_2 \\
\Phi_2
\end{array}$$

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- This bound holds even given unlimited EPR pairs.

*Why? Because for any $m$, the same argument applies to the states $|\Phi_2\rangle^\otimes m \otimes (|01\rangle^\otimes n \pm |\Phi_2\rangle^\otimes n / \sqrt{2})$*
The general rule:
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- If $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$, then preparing $|\psi\rangle = \sum_i \sqrt{\lambda_i} |i\rangle \otimes |i\rangle$ from EPR pairs requires $\log(r\lambda_1)/2$ qubits of communication (i.e. the “entanglement spread” of $|\psi\rangle$).

**Why?** $r$ and $\lambda_1$ each change by at most 2 for each qubit sent. For EPR pairs $r\lambda_1=1$.

[Hayden, A. Winter. quant-ph/0204092]
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- If $|\psi\rangle = \sum_k \sqrt{p_k} |k\rangle |k\rangle |\Phi_2\rangle \otimes^k$, then
  $\log(r) \approx \max\{k : p_k>0\}$ and $\log(\lambda_1) \approx -\min\{k : p_k>0\}$.
So the spread of $|\psi\rangle \approx$ the diameter of the support of $p$. 
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  So the spread of $|\psi\rangle \approx$ the diameter of the support of $p$.

- **Corollary:** For $|01\rangle \otimes^n + |\Phi_2\rangle \otimes^n / \sqrt{2}$, $r\lambda_1 \approx 2^n$. Therefore creating the state requires $\approx n/2$ qubits of communication.
Application to information theory
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- **Example:** If $|\psi\rangle$ is an entangled state, then $|\psi\rangle^{\otimes n}$ is very close to a state with spread $O(\sqrt{n})$. Therefore, $O(\sqrt{n})$ bits of communication are necessary and sufficient to prepare $|\psi\rangle^{\otimes n}$ from EPR pairs. (a.k.a. entanglement dilution.) [Harrow and Lo; quant-ph/0204096]
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- However, even in i.i.d. settings, entanglement spread can be size $O(n)$. 
Example: Channel simulation
Shannon’s (noisy coding) theorem:
Any noisy channel $N$ using input distribution $p^A$ can code at rate

$$C_{N,p} = H(A)_p + H(B)_p - H(AB)_p.$$
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(Classical) Reverse Shannon Theorem: $\textbf{N}$ can be simulated on $p^{\otimes n}$ using communication $C_{N,p}$ and shared randomness $R_{N,p} = H(AB)_p - H(A)_p$.

[BSST01,Cuff08]
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A \[\begin{array}{c|c|c} N & \text{(asymptotically)} & B \end{array}\]

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On general inputs:
The capacity and simulation cost are replaced by $C(N) = \max_p C_{N,p}$.
Randomness cost for simulation is $\max_p H(B)_p - C(N)$.

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\[ A \longrightarrow N \longrightarrow B \]

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\[ __________ \]

\[ \text{(asymptotically)} \]

\[ __________ \]

\[ \text{(assuming free shared randomness)} \]

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Simulating quantum channels
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Coding with quantum channels: Using shared EPR pairs, a quantum channel $\mathcal{N}$ can send noiseless qubits at rate

$$\max_\rho Q_{\mathcal{N},\rho} = \max_\rho \left( H(A)_\rho + H(B)_\rho - H(AB)_\rho \right) / 2.$$
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  \[
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- This requires either extra communication (forward or back) or embezzling states.
The general goal: LOSE
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- **Definition:** LOSE (local operations and shared entanglement) operations can be performed with local operations and arbitrary shared entangled states, but no communication.
The general goal: LOSE

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- **Communication complexity**: Special case in which Alice holds $x \in \{0,1\}^n$, Bob holds $y \in \{0,1\}^n$ and they want to compute the bit $f(x,y)$. 
Uses of non-standard entanglement:
I. Embezzling states
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1. Embezzling states

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- On the other hand, there is a family of $k \times k$-qubit “embezzling states” [van Dam and Hayden. quant-ph/0201041]

\[ |\zeta_k\rangle \propto \sum_{i=1}^{2^k} \frac{1}{\sqrt{i}} |i\rangle \otimes |i\rangle \]

such that for any $n \times n$-qubit entangled state $|\psi\rangle$, Alice and Bob can map $|\zeta_k\rangle$ to $|\zeta_k\rangle \otimes |\psi\rangle$ with no communication, up to error $O(n/k)$. 
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- The proper definition of “free entanglement” is thus closer to “an embezzling state of arbitrary finite size” than “unlimited EPR pairs.” In particular, the entangled state in LOSE operations can be taken to be an embezzling state w.l.o.g.
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$m-1$ copies

$|\alpha\rangle$  $|\alpha\rangle$  $|\alpha\rangle$  $|\alpha\rangle$  $|\alpha\rangle$  $|\beta\rangle$
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- $\langle\alpha|\beta\rangle = 0$

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  Problem reduces to projecting onto symmetric subspace.
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- **Corollary:** $U$ can asymptotically create $O(n)$ EPR pairs/use, but can only send $O(\log(n))$ bits/use.
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• Can non-standard entanglement (e.g. embezzling states) save even more communication?
Claim: General entanglement is not much better than EPR pairs in reducing communication complexity.
Communication complexity

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**Proof**: Let $|\psi\rangle = \sum_k \sqrt{p_k} |k\rangle|k\rangle|\Phi_2\rangle^\otimes k$ be our starting state for a protocol that uses $Q$ qubits of communication. Then $\Pr[\text{accept}]$ is of the form
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$$
\text{tr} \left( 
\begin{pmatrix}
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
\end{pmatrix}
\right)
\begin{pmatrix}
p_1 & \sqrt{p_1 p_2} & \sqrt{p_1 p_3} & \cdots \\
\sqrt{p_1 p_2} & p_2 & \sqrt{p_2 p_3} & \cdots \\
\sqrt{p_1 p_3} & \sqrt{p_2 p_3} & p_3 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
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* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
\end{array} \right) \left( \begin{array}{cccc}
p_1 & \sqrt{p_1p_2} & \sqrt{p_1p_3} & \cdots \\
\sqrt{p_1p_2} & p_2 & \sqrt{p_2p_3} & \cdots \\
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\vdots & \vdots & \vdots & \ddots \\
\end{array} \right)$$

Thus we can replace $|\psi\rangle$ with a mixture of states with spread $O(Q/\varepsilon)$ and incur error $\leq \varepsilon$. 
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• Can spread be quantified and described as a resource, like EPR pairs?
  (First step: \(\log(r\lambda_1)/2 + O(\log 1/\epsilon)\) qubits suffice to produce a state with Schmidt coefficients \(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r\) up to accuracy \(\epsilon\) [Harrow & Hayden].)
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• In communication complexity, how useful even are EPR pairs? Can spread be used to argue that $n$ EPR pairs are not useful for a $Q$-qubit protocol when $n \gg Q$?
And they all lived happily ever after.
The end.