MODEL ORDER REDUCTION USING MAXIMAL REAL PART NORMS

Challenges of Model Reduction
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Discussion

MODEL REDUCTION

Efficiently find a simple model which matches well given complex system data.

For LTI systems:

“efficiently”: takes reasonable time on practical examples
“simple”: low order
“matches well”: optimal approximation
“data”: system equations or measurements

H_∞-optimal LTI model reduction is an open problem

HANKEL OPTIMAL MODEL REDUCTION

Hankel norm $\|G\|_H$ of transfer function $G \in H_\infty$ is the infimum of $\|G + \bar{Y}\|_\infty$, where Y ranges over H_∞.

Given a state space model of an n-th order transfer function G, there is an algorithm for finding a transfer function G_m of order less than m which minimizes $\|G - G_m\|_H$ in about n^4 operations.

What is the Hankel optimality good for?

HANKEL OPTIMAL MODEL REDUCTION: THE BENEFITS

- **the minimum $\sigma_m^H(G)$ is a lower bound for H_∞ optimal model reduction**

- **the Hankel optimal reduced model G_m^H has H_∞ approximation error not exceeding the sum of $\sigma_k^H(G)$ with $k \geq m$**

- **other used method of model reduction (e.g. moment mathing) guarantee even less**

Anything else?

HANKEL OPTIMAL MODEL REDUCTION: INCONVENIENT PROPERTIES

- **a finite dimensional state space model of G is required**
- **complexity of calculations grows quickly with n**
- **an analog for weighted norms is not available**

MAXIMAL REAL PART NORMS

For a function $W \in L_\infty$ which is non-zero on a set of positive measure, the corresponding maximal real part norm is defined on H_∞ by

$$\|G\|_R^W = \|W\,\mathrm{Re}(G)\|_\infty$$

The problem of minimizing $\|G - G_m\|_R^W$ over all stable transfer functions of order less than m is quasi-convex with respect to a re-parameterization with $2 * m - 1$ parameters.

THE QUASI-CONVEX PARAMETERIZATION

$$\text{Re } G_m(j\omega) = \frac{b(\omega^2)}{a(\omega^2)},$$

where b, a **are polynomials of degree less than** m**, and**

$$a(\omega^2) > 0 \quad \forall\ \omega \in \mathbf{R}$$

Then the level set $\|G - G_m\|_R^W < \gamma$ **is defined by the constraints**

$$a(\omega^2) > 0, \quad |b(\omega^2) - \text{Re } G(j\omega)a(\omega^2)| < \gamma a(\omega^2) \quad \forall\ \omega$$

which are convex with respect to a, b

Are the maximal real part norms good for anything?

MAX-REAL PART NORMS: STRONGER THAN HANKEL NORMS

Since $|G| \geq |\text{Re } G|$, **we have**

$$\|W\Delta\|_\infty \geq \|W\text{Re } \Delta\|_\infty$$

Since $\text{Re}(G) = 0.5(G + \bar{G})$, **we have**

$$\|W\text{Re } \Delta\|_\infty \geq 0.5 \inf_{Y \in H_\infty} \|W(\Delta + \bar{Y})\|_\infty$$

AN ASYMPTOTIC CONVERGENCE THEOREM

THEOREM:

Let

$$\rho_m(G) = \inf \|\mathrm{Re}(G - G_m)\|_\infty$$

where the infimum is taken over all stable transfer functions G_m of order less than m. Let G_m be stable transfer functions of order less than m such that

$$\rho_m^{-1}\|\mathrm{Re}(G - G_m)\|_\infty = O(1).$$

If $\sum \rho_m(G) < \infty$ then $\|G - G_m^R\|_\infty \to 0$.

PROOF (A SKETCH):

$$
\begin{aligned}
&\|G - G_m\|_\infty \\
&\qquad \le \|G - G_m^H\|_\infty + \|G_m^H - G_m\|_\infty \\
&\qquad \le \|G - G_m^H\|_\infty + 2m\|G_m^H - G_m\|_H \\
&\qquad \le \|G - G_m^H\|_\infty + 2m\|G_m^H - G\|_H + 2m\|G_m - G\|_H \\
&\qquad \le \|G - G_m^H\|_\infty + 2m\|G_m^H - G\|_H + 4m\|\text{Re }(G_m - G)\|_\infty
\end{aligned}
$$

MAXIMAL REAL PART NORM MODEL REDUCTION OVER FINITE FREQUENCY SETS

The problem of minimizing

$$\max_k |W_k \mathrm{Re}(G(j\omega_k) - G_m(j\omega_k))|$$

over all stable transfer functions G_m of order less than m can be re-parameterized as a quasi-convex optimization in a similar way.

With a finite number of frequencies, this is an LMI optimization.

This way, a complete model of G is not needed!

USING VECTOR MAX-REAL NORMS

$$\left\| \begin{bmatrix} \text{Re } G(j\omega) - b(\omega^2)/a(\omega^2) \\ \text{Im } G(j\omega) - \omega c(\omega^2)/a(\omega^2) \end{bmatrix} \right\|_\infty \to \text{min}$$

This yields a better lower bound

EXAMPLE 1

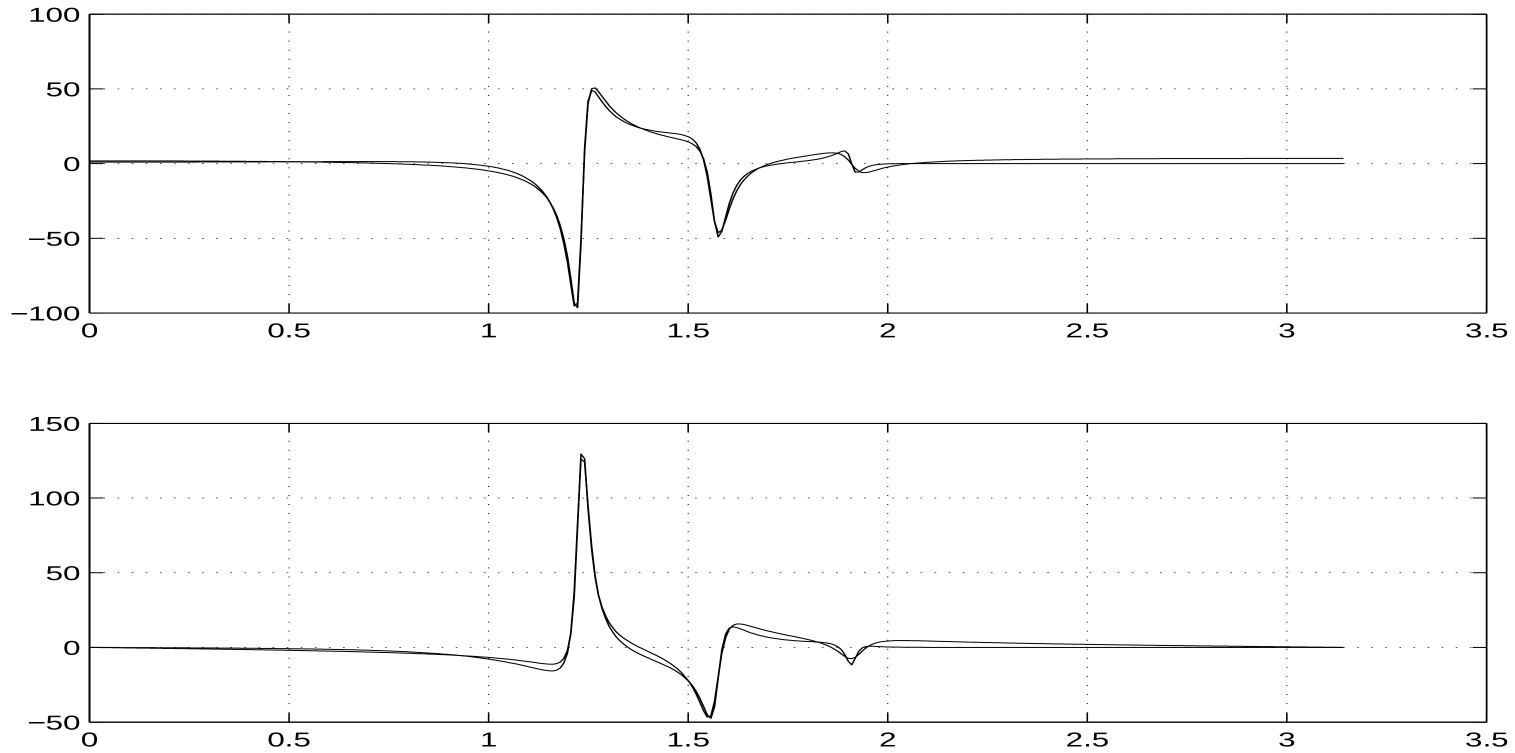
A 6-th order reduced model for a 12-th order transfer function with three resonant peaks was constructed using Hankel and maximal real part optimality.

Here

$$\|G - G_7^H\|_H \approx 3.56, \quad \|G - G_7^H\|_\infty \approx 5.76,$$

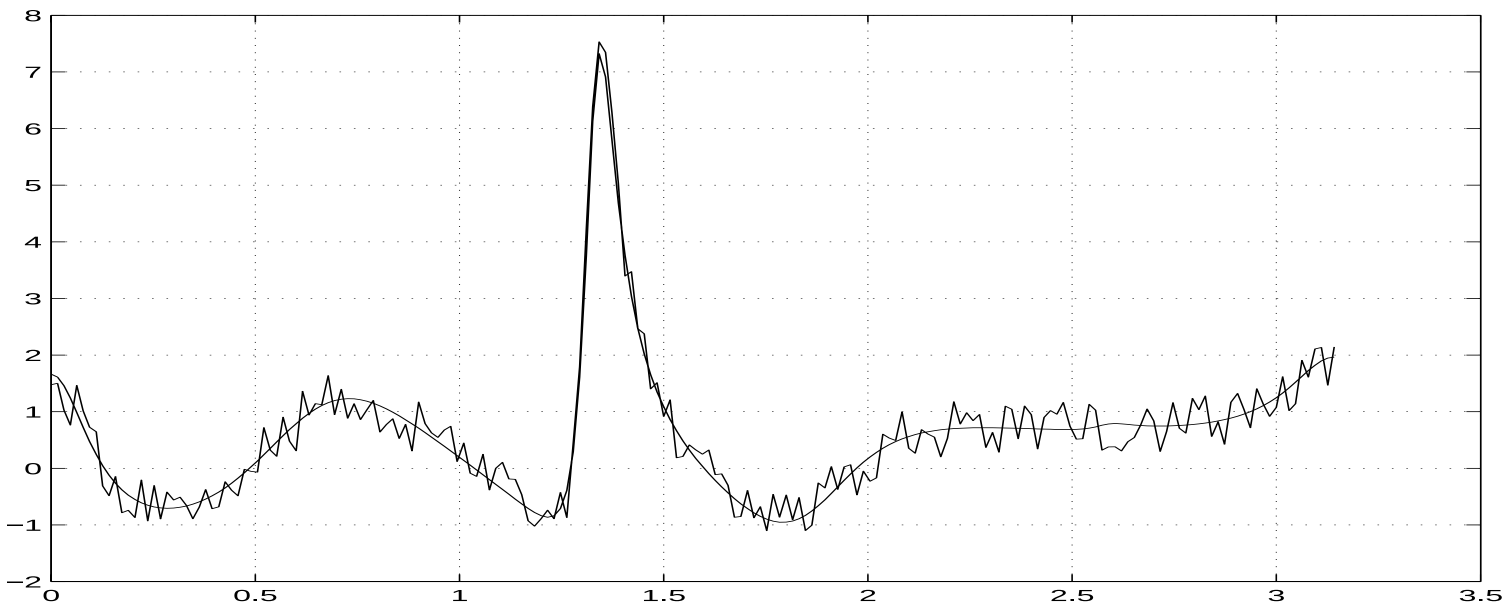
$$\|\mathrm{Re}(G - G_7^R)\|_\infty \approx 3.37[4.0], \quad \|G - \bar{G}_7^R\|_\infty \approx 4.57.$$

EXAMPLE 1: REAL/IMAGINARY VALUES



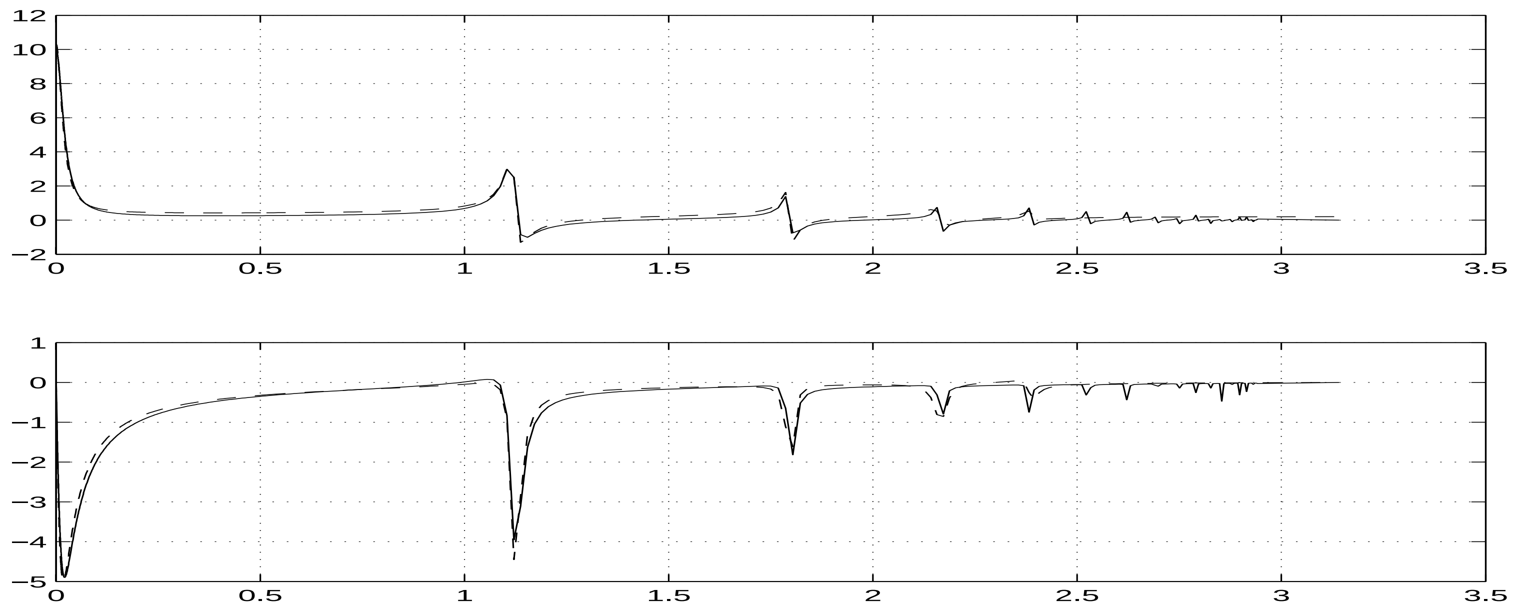
EXAMPLE 2: NOISY DATA

200 noisy samples of rational trigonometric function of degree 10.



EXAMPLE 3: SYSTEM WITH DELAY

10-th order approximation of $(1 - .9e^{-s})^{-1}(1 + .3s)^{-1}$
Lower bound 0.35, actual 1.4[0.54]



EXAMPLE 4: FOCUS SERVO OF A DVD PLAYER

59 frequency samples

10th order fit: lower bound ≈ 1.57**, actual** $\approx 6.2[4.9]$