Asymmetric Information and Security Design under Knightian Uncertainty*

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Abstract

We study optimal security design by an informed issuer when the investor faces Knightian uncertainty about the distribution of cash flows and demands robustness: she evaluates each security by the worst-case distribution at which she could justify it being offered by the issuer. First, we show that both standard outside equity and standard risky debt arise as equilibrium securities. Thus, the model provides a common foundation for two most widespread financial contracts based on one simple market imperfection, information asymmetry. Second, we show that the equilibrium security differs depending on the degree of uncertainty and on whether private information concerns assets in place or the new project. If private information concerns the new project and uncertainty is sufficiently high, standard equity arises as the unique equilibrium security. When uncertainty is sufficiently small, the equilibrium typically features risky debt. In the intermediate case, both risky debt and standard equity arise in equilibrium. In contrast, if private information concerns assets in place, standard equity is never issued in equilibrium, irrespective of the level of uncertainty, and the equilibrium security is (usually) risky debt.

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1 Introduction

Consider a firm with insufficient internal capital that needs to raise the extra from the investor to finance an investment project. The firm's owner knows the distribution of the project's cash flows, while the investor does not. The firm proposes a security to the investor, which the investor uses to infer information about the distribution of the project's cash flows. How will this security look like? This is the classic question of optimal security design under asymmetric information, which was first asked by Myers and Majluf (1984). Over the last decades, a large literature analyzed this classic question and its variations.[1]

With few exceptions, existing literature assumes that buyers of the security are confident about the nature of asymmetric information in the sense of holding a given prior about possible distributions of cash flows (typically, ranked by some stochastic order) that the project may have. An example is that it is common knowledge that the project's cash flows follow a log-normal distribution, whose mean is privately known by the issuer. While plausible in some settings, such as when a mature firm undertakes a project similar to a project this or other firms took in the past, this assumption can be unrealistic in other settings, such as when a young firm raises financing for a project that has few close comparables. A better description of these settings can be that the investor has some (possibly, vague) idea about possible cash flow distributions that the project may take, but lacks confidence to assign specific priors to them. Instead, she takes a robust approach to evaluating securities in the sense of having a strong preference for securities that are "robust" to the investor's misspecification of the project's cash flows.[2] The goal of this paper is to develop a theory of security design under asymmetric information in this setting.

Formally, we study the following model. The issuer has some resources and raises extra to finance his project. The issuer has private information about the distribution of cash flows. In contrast, the investor only knows that a certain set of distributions of the project's cash flows is possible, called the *uncertainty set*. It captures all distributions within the neighborhood (in the sense of total variation distance) of some base distribution. Importantly, we do not impose any further restrictions on the uncertainty set, and in particular, distributions in this

[1] An incomplete list of papers includes Brennan and Kraus (1987), Nachman and Noe (1994), DeMarzo and Duffie (1999), Fulghieri and Lukin (2001), DeMarzo (2005), Fulghieri et al. (2015), Yang (2015), Bond and Zhong (2016), Szydlowski (2017), Yang and Zeng (2017).

[2] Courtney et al. (1997) give the following recommendation to practitioners about the decision making under great uncertainty: In uncertain environments when "it is impossible ... to define a complete list of scenarios and related probabilities, it is impossible to calculate the expected value of different strategies. However, establishing the range of scenarios should allow managers to determine how robust their strategy is, identify likely winners and losers, and determine roughly the risk of following status quo strategies."

set are not ordered by some stochastic ordering. The uncertainty set can come from certain distributions being discarded by the investor (e.g., based on his analysis of the project) as not possible or sufficiently unlikely to neglect them in the decision. The size of the uncertainty set reflects the degree of the investor's uncertainty: When the uncertainty set is larger, the investor entertains more possible cash flow distributions and in this sense is more uncertain about the project.

While the investor believes that the distribution of the project's cash flows is some point in this uncertainty set, she lacks confidence about which one. Formally, she has infinitely many priors ("models of the world") where each prior puts probability one on a specific point in the uncertainty set. After the investor observes the security offered by the issuer, she re-evaluates ("tests") whether the security offer can be justified in each model and keeps only the models that can justify it, denoted the *set of justifiable models*. Given it, the investor demands robustness: she evaluates the security according to the *worst-case justifiable model*, i.e., the justifiable model that yields the lowest expected value of the security. Investor's preference for robustness can be viewed as the ambiguity aversion, as in Gilboa and Schmeidler (1989). Among other things, this preference for robustness reflects the Ellsberg paradox (Ellsberg (1961)), a finding that people prefer to take risk in situations when the odds are known than when the odds are unknown.

The "test" that the investor conducts to determine if the model can justify the security offer is similar in spirit to the Intuitive Criterion in Bayesian signaling games. Specifically, the investor evaluates whether the issuer is weakly better off, if the investor accepted the observed security offer, than in equilibrium. The set of justifiable models consists of all distributions of cash flows in the uncertainty set for which the answer is a "yes".

We characterize the equilibria of this non-Bayesian signaling game. Our first result is that the equilibrium is generically unique despite both types (distributions of cash flows) and signals (security offers) being multi-dimensional and weak restrictions on the structure of the uncertainty set. This contrasts sharply with Bayesian multi-dimensional signaling models, where severe multiplicity of equilibria is common place and general characterization is elusive. The key to our generic uniqueness is the robust approach of the investor to valuation of securities. Intuitively, because securities are evaluated according to the worst-case scenario, they are priced by the investor similarly on and out of the equilibrium path, which prevents the punishment of deviations by adverse beliefs that normally sustains a variety of equilibria in the Bayesian model.

The implications of our non-Bayesian model differ significantly from the standard Bayesian models of security design. In an important paper, Nachman and Noe (1994) deliver a stark result: signaling with securities is generally rather limited, and under certain conditions, risky debt is optimal and minimizes losses from mispricing. When investors demand robustness, signaling is richer, and generally, there is partial pooling and several securities are offered in equilibrium. The type of financing depends crucially on the degree of the investor's uncertainty, or specifically, on whether the investor entertains the possibility of negative net-present value (NPV) projects or not.

Our main results can be summarized as follows. If uncertainty is small in the sense that the project has a positive NPV for any distribution in the uncertainty set, then the investor evaluates any security according to the same model, which is the distribution of cash flows skewed maximally towards low realizations. In this case, risky debt arises as an equilibrium security for any issuer's type that dominates this "worst" point of the uncertainty set in the sense of monotone likelihood ratio property (MLRP). More interestingly, if uncertainty is large, i.e., there are points in the uncertainty set for which the project has a negative NPV, then standard equity arises as an equilibrium security for some issuer's types. Furthermore, when either the uncertainty set becomes sufficiently large (the investor considers more models possible) or the larger fraction of project types in the uncertainty set has negative NPV, the standard equity becomes a dominant source of financing: All types in the uncertainty set that choose to finance the project end up raising capital with the standard equity.

The key to the distinction between the high and low uncertainty environments is the interaction between investors' demand for robustness and learning from the security offer. When uncertainty is low, the signaling role of the security offer is limited, since the worst-case justifiable model is the same for any security offer. Moreover, when the investor's worst-case justifiable model and the actual distribution of cash flows are MLRP-ordered, it is cheaper for the issuer to pay in the low states, which he considers relatively less likely compared to investors, and debt emerges as the optimal security.

When the uncertainty is large, the signaling aspect of the security offer becomes important. When negative NPV types of projects are possible, the issuer chooses a security offer to signal that the project has a non-negative NPV. For example, under equity financing, the interests of the issuer and investors are partially aligned, and in particular, equity can be a credible way for the issuer to signal that his project has non-negative NPV and is worth financing in the first place. Similarly, a sufficiently high level of debt can be a credible signal of the non-negative NPV.

If the issuer's offer credibly signals that the project has non-negative NPV, then the worst-case scenario for investors is no longer the distribution of cash flows that is maximally skewed to the lowest realizations of cash flows, but rather it is one of many distributions in the uncertainty set with zero NPV. Since there are many such distributions, now the worst-case justifiable model varies with the type of security offered. For concave securities, such as debt, which are more valuable when cash flows are more concentrated around mean, the worst case scenario is the most dispersed distribution of cash flows among distributions with zero NPV. On the contrary, for convex securities, such as call option, the worst case scenario is the most concentrated distribution among distributions with zero NPV.

This causes a discontinuous shift in the investor's worst-case justifiable model from most concentrated zero-NPV model to most dispersed when the issuer switches from convex to concave securities. This shift causes the drop in the valuation of concave securities, but does not affect equity. As a result, the equity is now optimal for a range of types. When the uncertainty increases or more projects become negative NPV, the change in the investor's worst-case justifiable model as one moves from convex to concave securities is larger, and hence, more types of issuer prefer equity, making it eventually the dominant source of financing.

Interestingly, the nature of the private information is important for the optimal security design. We consider the version of the model with the uncertainty about assets in place rather than cash flows from the new project. In this variation, when the worst-case justifiable model and the issuer's type are strictly ordered by the MLRP, the risky debt is optimal *irrespective* of the level of uncertainty, while equity is never an optimal security. When private information is about assets in place, there is an adverse selection problem: the issuer with low quality assets has a stronger preference to pledge them rather than the issuer with high quality assets. Thus, the worst case scenario for investors is always the distribution that is maximally skewed towards low realizations of cash flows, and by the same logic as in the baseline model with small uncertainty, risky debt emerges in equilibrium.

The paper has three implications. First, it shows that both standard risky debt and standard equity, i.e., two extremely popular financial contracts, arise as equilibrium outcomes from the same model of financing with only one friction – asymmetric information. Bayesian models of security design under asymmetric information often generate risky debt as the equilibrium security under some conditions, but when these conditions are violated, the equilibrium security is usually not standard equity — for example, it is common to have the opposite of risky debt, a call option, as the optimal security. For this reason, many papers that "operationalize" models of financing under asymmetric information restrict attention to debt and equity (e.g., Hennessy et al. (2010), Fulghieri et al. (2015)). In our model, the advantage of standard equity over any other security is that it, roughly speaking, minimizes the informational advantage of the issuer coming from knowing the exact form of distribution that achieves each level of the NPV.

Second, while this goes beyond the scope of the paper, our results suggest the following information-based theory of dynamic capital structure. Young firms have little assets in place, and investors face a lot of uncertainty about cash flows from their projects. As a consequence, young firms use outside equity as the source of external finance. As time goes by, they accumulate assets in place and the uncertainty about cash flows from their projects declines, as investors get enough data observations to discard some models as not plausible. For both reasons, risky debt becomes a better security to address information asymmetry problems, implying that the standard pecking order theory should be more applicable for mature firms, where there is little uncertainty and information asymmetry is primarily about assets in place.

This leads to a possible interpretation of some contradictory evidence on the validity of the pecking-order theory of financing. While pecking order works best for large mature firms (Shyam-Sunder and Myers (1999)), it does a poor job at describing financing decisions of small high-growth firms (Frank and Goyal (2003), Leary and Roberts (2010)), even though there is plausibly more information asymmetry about the latter. However, these findings can be consistent with security design implications in our paper.

Related Literature The paper is related to several strands of literature. First, as we already mentioned, we contribute to the literature on optimal security design under asymmetric information, started by Myers and Majluf (1984) (see footnote 1 for an incomplete list of papers). The formulation of our basic model is closest to Nachman and Noe (1994): like they, we consider the problem where all private information is about the investment project, and we impose the same restrictions on admissible securities. Our assets in place model is closer to DeMarzo and Duffie (1999), except that the issuer chooses the security *after* observing private information. The novelty of our setup is two-fold. First, instead of cash flow distributions all belonging to a certain class where issuer's "type" captures ranking in this class, the investor believes that "anything can happen" within some neighborhood of the base distribution. Thus, we put minimal structure on the set of distributions of cash flows that the investor considers possible.[3] The second novelty is the robust approach to

[3]At the same time, we want to stress that none of our results are driven by some exotic distributions that we allow for by relaxing the assumption about the structure of the uncertainty set. In fact, our primary

security pricing: investors evaluate securities by the worst-case justifiable model.[4] These novelties lead to results that are different from existing literature. Boot and Thakor (1993) show how riskless debt and equity can be rationalized, where the advantage of equity over risky debt is that it provides higher incentives for traders to get informed. In our model, we rule out safe debt by assuming that the lowest cash flow realization is zero.[5]

Several papers study security design problems with heterogeneous beliefs. Garmaise (2001) studies the problem when investors have diverse beliefs. Boot and Thakor (2011) studies how disagreement between the firm's initial owners and managers over project choice interacts with the firm's seurity issuance and allocation of control rights. Ortner and Schmalz (2016) study a problem of asset-backed security design when the issuer and investors disagree about probability distributions of different cash flow realizations. Kondor and Koszegi (2017) study a model in which competitive issuers design securities to sell to naive investors. In equilibrium of our model, the issuer and the investor also end up having heterogeneous beliefs, as they use different models to evaluate securities. The novelty of our model is that it results in the *endogenous* heterogeneity of beliefs: security design has a signaling role, like it does in standard Bayesian signaling models.[6]

Second, the paper is related to the growing literature on robust contracting. In this literature, the most related papers are models that study contracting in the presence of moral hazard under risk neutrality and limited liability. In Carroll (2015) and Antic (2015), the principal does not know what actions are available to the agent and demands robustness. Carroll (2015) shows that the optimal contract is linear in the setting when Knightian uncertainty of the principal is extreme. In Antic (2015), Knightian uncertainty is not extreme, so our setting is closer to his model. The conceptual difference is that these are moral hazard problems, while we study the adverse selection (signaling) problem. In other words, Carroll (2015) and Antic (2015) can be viewed as robust versions of Innes (1990), while ours can be viewed a robust version of Myers and Majluf (1984) and Nachman and Noe (1994). There are major differences in implications, but we postpone a detailed discussion until Section 7.

interest is in how the optimal financing changes with changes in the uncertainty set, i.e., as it becomes larger/smaller or includes more/less negative NPV projects.

[4]In Section 7, we discuss in more details the relationship between our approach and the classic Bayesian approach.

[5]If instead we assumed that the lowest realization is positive, then the issuer would issue as much safe debt as possible and then will issue a security prescribed by the equilibrium of our current model. However, because we have only one investor rather than many, our paper would not provide any insights about whether these claims should be combined into one or separate, which is one of the interesting insights of Boot and Thakor (1993).

[6]Less related, several papers study security design with investor's private information (Axelson (2007), DeMarzo et al. (2005), Gorbenko and Malenko (2011)).

Several other papers on robust contracting are also related. Lee and Rajan (2017) consider a robust version of Innes (1990) modeling ambiguity aversion of the principal by multiplier preferences towards model uncertainty, as in Hansen and Sargent (2001a).[7] Lee (2017) incorporates manager's ambiguity aversion into a "trade-off" theory of capital structure. Zhu (2015) uses an investor with a preference for robustness to provide a micro-foundation for refinanceable contracts, including refinanceable debt. Chassang (2013) and Miao and Rivera (2016) study dynamic agency problems when the principal demands robustness. Izhakian and Zender (2017) study contracting and disclosure by a risk-neutral principal with risk-averse agent when both parties are ambiguity averse. Bergemann and Schlag (2011) study monopoly pricing with the monopolist uncertain about the demand.

Third, we contribute to the literature on signaling with mulidimensional types and signals. In the Bayesian model, the characterization of the equilibrium set remains a hard, open question. Quinzii and Rochet (1985), Engers (1987) provide sufficient conditions for separating equilibria. We propose an alternative robust approach to the classic signaling model, which proves extremely tractable and allows for the complete characterization of the generically unique equilibrium. In this respect, we are close to Carroll (2016) who applies the robust approach to the multidimensional screening model, another open problem in the Bayesian formulation.

Finally, Dicks and Fulghieri (2015, forthcoming), Garlappi et al. (2017) study the role of ambiguity in other corporate finance decisions: allocation of control rights, bank runs, and a group decision about investment, respectively. An innovation of our approach is the introduction of a natural updating of models by a party with a preference for robustness. In this respect, our work is related to the literature on belief updating by ambiguity-averse agents (most closely to Epstein and Schneider (2007)).

The structure of the paper is as follows. Section 2 describes the signalling game. Section 3 shows derives equilibrium pricing of securities and shows that the equilibrium is generically unique. Section 4 provides main resuts. Section 5 studies the version of the model with uncertainty about assets in place. Section 6 presents extensions. Section 7 discusses the existing empirical evidence, the relation to Bayesian signaling models and robust contracting models, and concludes. Key proofs are presented in the text, the rest are relegated to Appendix and Online Appendix.

[7] Hébert (forthcoming) considers a version of Innes (1990) in which the agent has a flexible moral hazard technology. However, there is no preference for robustness.

2 The Model

This section describes the model. In Section 2.1, we introduce the capital raising game between the informed issuer and the investor facing Knightian uncertainty about the distribution of the project's cash flows. We proceed by defining the equilibrium. In Section 2.2, we introduce the notion of a set of justifiable models, which specifies how the investor learns from observing the issuer's offer.

2.1 Model Setup

The issuer (male) has a project that requires investment K. The issuer has $W < K$ of his own resource and needs to complement it by raising financial capital $I \equiv K - W$ from the outside investor (female). Several interpretations of W are possible. In the case of a newly created firm, it is natural to think about the issuer as the entrepreneur and about W as the entrepreneur's outside option, such as the value from an alternative employment that he foregoes by undertaking the project. If the firm already operates, then it is natural to interpret W as the firm's assets in place and the issuer as the firm's management operating the firm in the interest of current shareholders (Myers and Majluf (1984)).

Information Structure The issuer knows the distribution f of the future cash flow from the project. We call f the *type of the issuer's project*, or simply, *the issuer's type*. The future cash flow z can take one of $N+1$ values in $Z = \{z_0, z_1, \ldots, z_N\}$, where $z_0 = 0 < z_1 < \cdots < z_N$. For any f, we denote probabilities of $z_0, z_1, \ldots, z_N$ by $f_0, f_1, \ldots, f_N$, respectively, and associate type f with point $(f_1, \ldots, f_N)$ in the probability simplex $\Delta(Z) \equiv \{f \in \mathbb{R}_+^N : \sum_{n=1}^N f_n \leq 1\}$. Denote by F the c.d.f. of distribution f.

The investor faces Knightian uncertainty about distribution f. We capture it by a set of distributions $B \subset \Delta(Z)$, referred to as the *uncertainty set*. It includes all distributions in the neighborhood of some base distribution $g = (g_0, ..., g_N)$, satisfying $g \in \mathbb{R}_+^N$ and $\sum_{n=1}^N g_n \leq 1$. To define set B, we need to choose a specific metric to quantify the distance between two probability distributions. We focus on neighborhoods induced by the total variation distance, because it is one of the most widely used probability metrics and because it has a natural interpretation. Specifically, set B is the set of all distributions whose total variation distance from base distribution g does not exceed ν:

$$B = \left\{ f \in \Delta(Z) : \sup_{A \subset Z} \left| \mathbb{P}_f(A) - \mathbb{P}_g(A) \right| \leq \nu \right\}, \tag{1}$$

where A is any measurable event (any subset of Z) and $\mathbb{P}_f(A)$ and $\mathbb{P}_g(A)$ are the probabilities of it occuring under distribution f and base distribution g, respectively. Total variation distance is a natural notion of "closeness" of distributions capturing that two distributions are close if they assign sufficiently similar probabilities (different at most by ν) to any event. Because the state space is countable, definition (1) is equivalent to (e.g., Huber (2011)):

$$B = \left\{ f \in \Delta(Z) : \sum_{n=0}^{N} |f_n - g_n| \leq 2\nu \right\}. \tag{2}$$

We will refer to ν as the *degree of (investor's) uncertainty.* The larger ν, the more uncertain the investor in the sense that she considers more distributions of cash flows as possible.[8]

Remark 1. *One possible interpretation of B is as follows. The investor after observing data comes up with an estimate g of the distribution of cash flows and forms a certain confidence region B around this estimate. If g is a maximum-likelihood estimator and B consists of all distributions that pass the likelihood-ratio test, then this procedure coincides with the model of learning under ambiguity in Epstein and Schneider (2007). It is not important for our results how the investor comes up with g, because only the set B itself plays role. Further, it is expositionally more convenient to analyze sets of the form* (2)*, however, as we argue in Section* 6.3*, a particular shape of set B is not crucial for our main results.*

To have a non-trivial problem, we assume that the project has positive NPV for at least one point in set B: $\{f \in B : \mathbb{E}_f[z] \geq K\} \neq \emptyset$. If this condition is violated, then the investor believes that the project has a negative NPV, so, as it will be clear from what follows, the project does not get financed for any $f \in B$.

Figure 1 illustrates set B in the case of $N = 2$. In this case, B can be parametrized by f_1 and f_2 both in $[0, 1]$ that satisfy the following constraints $f_0 \in [g_0 - \nu, g_0 + \nu]$, $f_1 \in [g_1 - \nu, g_1 + \nu]$, $f_2 \in [g_2 - \nu, g_2 + \nu]$. Set B has a natural interpretation: the investor has a reference distribution of cash flows, g, but allows for a possibility that she knows the probability of each realization $z \in \{0, z_1, z_2\}$ not exactly, but with some error ν.[9]

Timing and Actions The timing of the game is as follows:

[8] While our model assumes that set B is induced by the total variation distance, our main results do not rely on the specific form of B. In Section 6.3, we show that many of our results are not sensitive to the specific form of B, as long as B is convex with a non-empty interior. Furthermore, when $N = 2$, many probability metrics result in the same set B: Kolmogorov, Levy, and Prokhorov metrics result in exactly the same set B as the total variation distance, provided that possible realizations of z are sufficiently far apart.

[9] Note that $f_0 \in [g_0 - \nu, g_0 + \nu]$ is equivalent to $f_1 + f_2 \in [g_1 + g_2 - \nu, g_1 + g_2 + \nu]$.

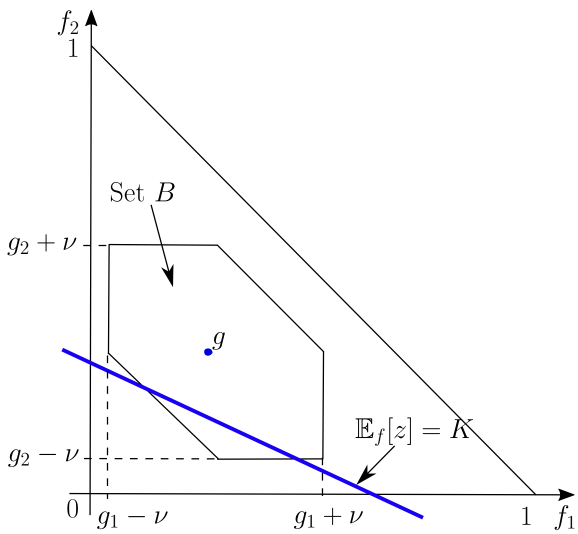


Figure 1: ***Illustration of the uncertainty set*** B

The uncertainty set B is the neighborhood of radius ν around the base distribution $\mathbf{f}$. *The blue, bold line represents types f satisfying $\mathbb{E}_f[z]=K$.*

1. The issuer makes an offer to the investor of security $s=(s_0,s_1,...,s_N)$ that pays s_n if the cash flow realization is z_n. The issuer can also choose not to pursue the investment, which we refer to as offering security $s=\mathbf{0}$, which pays zero for any cash flow realization. In this case, the issuer's payoff is W, and the investor's payoff is zero.

2. Having observed the security offer s, the investor chooses $\sigma\in\{0,1\}$ whether to invest I in exchange for security s ($\sigma=1$) or not ($\sigma=0$).

3. If the investor accepts the offer ($\sigma=1$), the investment is made, and the cash flow from the project $z\in\{z_0,...,z_N\}$ is realized. The issuer gets $z-s$ and the investor gets s. If the investor rejects the offer ($\sigma=0$), then the investor keeps her investment I and the issuer keeps his resource W.

We make the standard assumption in the security design literature (e.g., Nachman and Noe (1994), DeMarzo and Duffie (1999)) that the set of feasible securities, denoted $\mathcal{S}$, consists of all monotone securities that satisfy the limited liability condition:

Definition 1. *Security $s=(s_0,...,s_N)$ is* feasible *if it satisfies: (1) $0\leq s_n\leq z_n$ for all $n=0,1,\ldots,N$ (limited liability); and (2) s_n and z_n-s_n are weakly increasing in n (monotonicity). Set $\mathcal{S}$ is defined as the set of all feasible securities.*

The first condition states that repayment to either party must be non-negative. In particular, since $z_0 = 0$, $s_0 = 0$ for any $s \in \mathcal{S}$. The second condition states that the payoff of each party must be weakly increasing in the realized cash flow z.[10]

Equilibrium The *issuer's pure strategy* $s^*(\cdot)$ maps any $f \in B$ into a feasible security $s^*(f) \in \mathcal{S}$ that the issuer's type f offers to the investor. The *investor's pure strategy* $\sigma^*(s)$ is a mapping from any security $s \in \mathcal{S}$, $s \neq \mathbf{0}$ into the decision whether to accept ($\sigma^*(s) = 1$) or reject ($\sigma^*(s) = 0$) it.

The key feature of our model is that the investor faces Knightian uncertainty over the distribution f. In the case of Bayesian uncertainty, the investor has a single "model of the world," represented by a prior belief $\mu \in \Delta(B)$. In contrast, here the investor lacks confidence in assigning prior beliefs to distributions in B. Formally, she has infinitely many "models of the world". Each model is a degenerate distribution that puts probability one on the particular distribution $f \in B$.[11] Thus, we can identify the set of models (i.e., the set of degenerate distributions with support in B) with the uncertainty set B, and use terms "model f" and "distribution f" interchangeably. After receiving a security offer s, the investor reevaluates the set of models B into a subset $B(s) \subseteq B$ of *justifiable models*. Specifically, for each model $f \in B$, the investor runs a test, which we specify in the next subsection, determining whether offer s can be justified in model $f \in B$, and keeps only models that pass the test. We refer to $B(\cdot) : \mathcal{S} \to 2^B$ as the *model updating mapping* that maps securities in $\mathcal{S}$ into subsets of B.

Having reevaluated the set of models, the investor is averse to uncertainty and demands robustness in the sense that given security offer s and the set of justifiable models $B(s)$, the investor evaluates the security by the justifiable model that yields the lowest value of the security. Formally, the investor values security $s \in \mathcal{S}$ at[12]

$$P(s) \equiv \min_{f \in B(s)} \mathbb{E}_f[s]. \tag{3}$$

The investor's uncertainty aversion can be interpreted via a game played between the investor and adversarial nature: The investor believes that after she accepts security offer s, the

[10] Monotonicity can be justified by a "sabotage" argument: if a security were non-monotone, one of the parties would be better off destroying some output for some realizations of z. See, e.g., Hart and Moore (1995).

[11] One can define the model more generally as any distribution over the issuer's types in B. Denote the set of such models by $\hat{B}$, and by $\hat{B}(s)$ the set of all distributions over $B(s)$. This more general definition would not affect our results, as it can be easily verified that the minimum in (4) when $f \in \hat{B}(s)$ is always attained by some distribution that puts probability one on some distribution in B (i.e., $f(s) \in B(s)$).

[12] Given the definition of $B(s)$ in the next subsection, $B(s)$ is a compact subset of B, so the minimum in (3) is attained.

adversarial nature will pick $f \in B(s)$ with the objective of minimizing the investor's payoff. This game has two conceptual differences from the standard moral hazard problem: first, the adversarial nature has no incentive compatibility constraint to be satisfied; second, the set of actions that the nature can take is affected by the issuer's security choice. In what follows, we call the minimization problem in (3) the nature's problem.

We refer to the distribution that solves program (3) as the *worst-case justifiable model* and denote it by $f^*(s)$.[13] We call $P(s)$ the *valuation* of security s by the investor. Then, the investor's utility from investing I in exchange for s equals

$$V(s) \equiv P(s) - I = \min_{f \in B(s)} \mathbb{E}_f [s - I]. \tag{4}$$

Utility in (4) coincides with Gilboa and Schmeidler's (1989) maxmin expected utility representation of ambiguity aversion. However and importantly, the issuer's security choice puts a restriction on the set of distributions over which the expected value of the security is minimized.

We next define the equilibrium:

Definition 2. *A pair of strategies $(s^*(\cdot), \sigma^*(\cdot))$ and a model updating mapping $B(\cdot)$ constitute* an equilibrium *if*

1. *For any $f \in B$,*
$$s^*(f) \in \arg\max_{s \in \mathcal{S}} \{\sigma^*(s)(\mathbb{E}_f[z - s] - W)\},$$
with $s^*(f) = \mathbf{0}$ for any f such that $\max_{s \in \mathcal{S}:\sigma^*(s)=1} \{\mathbb{E}_f[z - s]\} < W$.

2. *For any $s \in \mathcal{S}$, $\sigma^*(s) = 1$ if and only if $P(s) \geq I$, where $P(\cdot)$ is given by* (3)*.*

3. *For any $s \in \mathcal{S}$, $B(s)$ is a set of justifiable models, defined in subsection* 2.2 *below.*

The first condition is rationality of the issuer: Given his known distribution of cash flows f and the equilibrium action of the investor $\sigma^*(s)$ for each security $s \in \mathcal{S}$, the issuer chooses the security to maximize his expected payoff.[14] Note that by definition of the equilibrium, any type of the issuer that in equilibrium does not get financing offers zero security $s = \mathbf{0}$,

[13] If there are multiple f that solve (3), all of them yield the same investor's expected utility, and we specify $f^*(s)$ to be any arbitrary selection from the solution to (3).

[14] As we show below, in equilibrium, only a Lebesgue measure zero of types is indifferent between issuing some security and not making an offer ($s = \mathbf{0}$), and so it is essentially without loss of generality to suppose that the issuer prefers to issue some security to not issuing when indifferent. We break ties in the investor's decision in favor of acceptance of the offer to guarantee that the maximum in the issuer's problem is indeed attained.

and thus any non-zero security offered by some type in equilibrium gets accepted. We denote the set of these securities by $\mathcal{S}^* \equiv \bigcup_{f \in B : \sigma^*(s^*(f))=1} \{s^*(f)\}$. The second condition states that the investor accepts any security that she values weakly above her investment I and rejects any security she values below that. Finally, the last condition of Definition 2 requires the investor's learning from observing the issuer's offer s to be "reasonable." Next, we define what we mean by "reasonable."

2.2 Learning under Knightian Uncertainty

The offer of the issuer potentially conveys information to the investor about the distribution of the project's cash flows. When the investor observes security s, she rules out some models $f \in B$ as implausible and keeps only subset $B(s)$ of models she could justify. We define $B(s)$ as:

Definition 3. *Fix $s^*(f)$ and let $U^*(f)$ be the issuer's expected utility from offering $s^*(f)$:*

$$U^*(f) = \begin{cases} \mathbb{E}_f \left[z - s^*(f) \right], & \text{if } s^*(f) \in \mathcal{S}^*, \\ W, & \text{otherwise.} \end{cases}$$

The model updating mapping $B(\cdot)$ is justifiable *if*

$$B(s) = \{ f \in B : \mathbb{E}_f \left[z - s \right] \geq U^*(f) \}, \tag{5}$$

whenever this set is non-empty. If it is empty, then $B(s) = B$.

This definition is critical for the results of the paper, so it is worth describing it in detail. For each model $f \in B$, the investor runs a test whether she can justify the issuer offering security s given $f \in B$. In this test, the investor asks the following question for any $f \in B$:

> "If I accepted offer s, would the issuer be weakly better off than if he followed his equilibrium strategy (i.e., if he instead issued an equilibrium security $s^*(f)$ or chose not to invest in the project entirely)?"

If $f \in B$ satisfies $\mathbb{E}_f \left[z - s \right] \geq \max \{ \mathbb{E}_f \left[z - s^*(f) \right], W \}$, i.e., if condition (5) holds, then the answer to this question is a "yes", meaning that the investor can justify the observation of s when $f \in B$. Otherwise, the issuer cannot justify it. The set of justifiable models, $B(s)$, consists of all models $f \in B$ that pass this test. If security s is such that no model $f \in B$

passes this test, then the investor does not learn anything and believes that all models from set B are plausible.

The idea behind our specification of the set of justifiable models is that the investor tries to learn about the distribution of cash flows from the fact that the issuer wants to undertake the project keeping the residual security. One can see the parallel between (5) and the Intuitive Criterion of Cho and Kreps (1987) for Bayesian signaling games. According to the Intuitive Criterion, the receiver cannot rationalize a sender type to send a certain signal if this type can only do worse with this signal than in equilibrium. If some types can be rationalized while others cannot, the receiver's belief must place positive probability only on the former set of types. Condition (5) is essentially the same test as the Intuitive Criterion: Similar to how the Intuitive Criterion would require the investor to put zero belief on types that violate (5), our definition of the set of justifiable models requires the investor to "discard" models that violate (5). Therefore, our updating rule can be viewed as the Intuitive Criterion applied to the game where the investor faces multiple degenerate priors instead of holding one non-degenerate prior.

In the definition of $B(\cdot)$, we do not distinguish between securities offered on the equilibrium path ($s \in \mathcal{S}^*$) and out of the equilibrium path ($s \notin \mathcal{S}^*$). One could argue that it is reasonable to additionally require that for on-path offers $s \in \mathcal{S}^*$, $B(s)$ is equal to $\{f \in B : s^*(f) = s\}$ so that $P(s)$ given by (3) is the infimum of the expected value of security s over all types that issue s in equilibrium. As we will show below, in equilibrium, for any $s \in \mathcal{S}^*$ that is offered by a positive Lebesgue measure of types, $B(s)$ given by (5) coincides with the closure of $\{f \in B : s^*(f) = s\}$, and so, $P(s) = \inf_{f \in B : s^*(f) = s} \mathbb{E}_f[s]$.

Remark 2. *There is no standard equilibrium concept for signaling games with non-Bayesian receivers. However, our equilibrium concept constitutes perhaps the smallest (and least controversial) departure from standard solution concepts. To see this, recall that the standard solution concept in Bayesian signaling model is sequential equilibrium, which involves two conditions: sequential rationality and Bayesian updating. Because of multiplicity of equilibria, researcher often apply the Intuitive Criterion (or stronger refinements). In our analysis, we essentially keep the sequential rationality and the Intuitive Criterion, but dispense of Bayesian updating, which is the most controversial point when we talk about non-Bayesian receivers.*

As we shall see after we solve the model, for any security $s \in \mathcal{S}^*$, i.e., for any security that some issuer's type $f \in B$ offers in equilibrium, our model updating rule becomes very similar to the model of learning under ambiguity introduced by Epstein and Schneider (2007). In

Section 7, we discuss alternative models of learning from security offers.

3 Equilibrium Valuation of Securities

The game may seem potentially intractable because the valuation of each security $P(s)$ depends on the as-yet-unknown equilibrium via set $B(s)$. Furthermore, because there are usually multiple equilibria in Bayesian signaling games, it can be natural to expect our game to have multiple equilibria. In this section, we give two technical results that simplify the valuation problem considerably. Then, we use these results to show that the equilibrium is generically unique.

The first result shows that it is without loss of generality for the equilibrium analysis to restrict attention to securities that make the investor indifferent, i.e., securities that satisfy $P(s) = I$. Intuitively, there is no value for the issuer to offer the investor more than she requires. If the investor more than breaks even at security s, the issuer can also get the investor's acceptance by offering a security with the same shape but lower than s. Formally:

Lemma 1. *For any $s \in \mathcal{S}$ such that $P(s) > I$, there exists $\gamma \in (0,1)$ such that $P(\gamma s) = I$. In particular, $P(s) = I$ for any $s \in \mathcal{S}^*$.*

Lemma 1 has two implications. First, to verify that a certain strategy profile $\{s^*(f), f \in B\}$ is an equilibrium issuer's strategy, it is sufficient to consider deviations to securities satisfying $P(s) = I$. Second, the issuer's problem can be restated in terms of the minimization of mispricing. To see this, for any s such that $P(s) = I$, the issuer's payoff from the project, net of W, is:

$$\mathbb{E}_f[z - s] - W = \underbrace{\mathbb{E}_f[z] - K}_{\text{NPV}} - \underbrace{(\mathbb{E}_f[s] - I)}_{\text{Mispricing}}.$$

The issuer's expected payoff consists of the project's NPV and the underpricing term arising because the investor prices the security according to the worst-case justifiable model rather than the true distribution f. Putting together these two implications, we can find any equilibrium of our game by solving the following program. For any $f \in B$, we first solve

$$\min_{s \in \mathcal{S}} \{\mathbb{E}_f[s] \text{ s.t. } P(s) = I\}. \tag{6}$$

That is, each issuer type f determines the set cheapest securities (the ones that mimize $\mathbb{E}_f[s]$) from the set of securities at which the investor breaks even ($P(s) = I$). Denote this set by $S^*(f)$. Equivalently, the issuer minimizes underpricing subject to the investor accepting the security. If this lowest underpricing, $\mathbb{E}_f[s] - I$, is greater than the NPV of the project $\mathbb{E}_f[z] - K$, then the issuer type f does not raise financing. Otherwise, the equilibrium security $s^*(f)$ is contained in $S^*(f)$.

However, the value of each security depends on the as-yet-uknown set $B(s)$. The next lemma is the central result of the section. It shows that the value of any security, which can be potentially relevant for the analysis, can be calculated without the knowledge of $B(s)$. Let us introduce the following subsets of B:

$$B_+ \equiv \{f \in B : \mathbb{E}_f[z] \geq K\},\ B_0 \equiv \{f \in B : \mathbb{E}_f[z] = K\},\ B_- \equiv \{f \in B : \mathbb{E}_f[z] < K\}.$$

Thus, B_+, B_0, and B_- are the sets of, respectively, non-negative, zero and negative NPV projects in B. If $B_+ = B$, the investor is confident that the issuer's project is positive-NPV, while when $B_+ \subset B$, the investor entertains the possibility that the NPV of the project is negative. Our main technical result is:

Lemma 2 (**Pricing Lemma**). *For any security $s \in \mathcal{S}$ such that $P(s) = I$ and set $\{f \in B : \mathbb{E}_f[z - s] \geq U^*(f)\}$ is not empty, it holds $f^*(s) \in \arg\min_{f \in B_+} \mathbb{E}_f[s]$ and*

$$P(s) = \min_{f \in B_+} \mathbb{E}_f[s]. \tag{7}$$

For any security $s \in \mathcal{S}$ such that $\min_{f \in B_+} \mathbb{E}_f[s] = I$, it holds $P(s) = I$.

Pricing Lemma has two implications. First, it shows that for all relevant securities, the equilibrium pricing can be determined without the knowledge of $B(s)$: we can simply minimize $\mathbb{E}_f[s]$ over the set B_+. (However, note that this does not mean that $B(s) = B_+$.) This implies that we can rewrite the program (6) as:

$$\min_{s \in \mathcal{S}} \{\mathbb{E}_f[s] \text{ s.t. } \min_{h \in B_+} \mathbb{E}_h[s] = I\}. \tag{8}$$

Second, Pricing Lemma implies that the issuer is able to signal that the project has a non-negative NPV, and this way the equilibrium investment is always efficient. Equity financing is an example of a security that can be a credible signal that the NPV is non-negative. Indeed, if the issuer offers an equity stake I/K, then the issuer credibly signals that the project has non-negative NPV, because $\mathbb{E}_f[z - (I/K)z] \geq W$ if and only if $\mathbb{E}_f[z] \geq K$. Another example is a sufficiently high level of debt: debt level d such that $\max_{f \in B_-} \mathbb{E}_f[\max\{0, z - d\}] < W$.

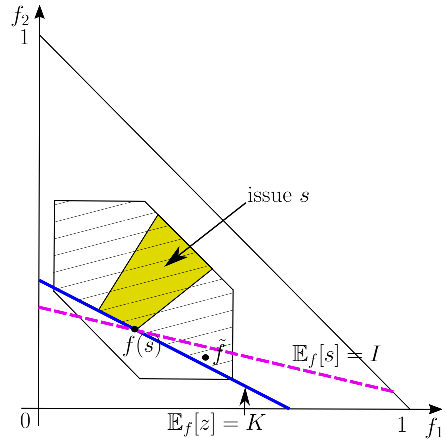


Figure 2: ***Illustration for Pricing Lemma***
The hatched region is B_+, the shaded region is $B(s)$, the solid line is $\mathbb{E}_f[z] = K$, the dashed line is $\mathbb{E}_f[s] = I$. In the figure, security s issued in equilibrium by types in the shaded region is priced at $P(s) = \mathbb{E}_{f(s)}[s] = I$ above $\min_{f \in B_+} \mathbb{E}_f[s]$. (This follows from the fact that iso-line $\mathbb{E}_f[s] = I$ passing through $f(s)$ intersects the interior of set B_+). Type $\tilde{f}$ prefers to issue security s to the security $\tilde{s} = s^*(\tilde{f})$ that he issues in equilibrium, as for type $\tilde{f}$ the mispricing from security s is negative, while it is non-negative from $\tilde{s}$. Thus, this is impossible in equilibrium.

Corollary 1. *Any equilibrium is efficient: all types in B_+ issue some security in equilibrium, and all types in B_- do not raise financing.*

Cne may expect that some type of the issuer could reduce mispricing even further by signaling information through the security choice, i.e., $\min_{f \in B_+} \mathbb{E}_f[s] < P(s)$ for some $s \in \mathcal{S}^*$. Pricing Lemma shows that this is not possible. To illustrate why, consider the case of $N = 2$ depicted in Figure 2. Suppose that types in the shaded region issue security s that is priced at $I = \mathbb{E}_{f(s)}[s] > \min_{f \in B_+} \mathbb{E}_f[s]$. Consider type $\tilde{f} \in B_+$ such that $I > \mathbb{E}_{\tilde{f}}[s]$. In equilibrium, such type issues some security $\tilde{s} = s^*(\tilde{f})$, and by Lemma 1, this security is not overvalued by the investor. However, if type $\tilde{f}$ issues security s, it will be overvalued, and thus s would constitute a profitable deviation for type $\tilde{f}$, which is a contradiction.

We next proposition shows that the equilibrium in our model is generically unique:

Proposition 1. *The equilibrium is generically unique, i.e., the set of issuer types that offer different securities in different equilibria has Lebesgue measure zero.*

Since the proof is somewhat technical, we relegate it to the appendix. It follows from the linearity of the objective function in s and sets B_+ and $\mathcal{S}$ being convex polyhedra.

The generic uniqueness of equilibrium is somewhat surprising in a signaling model with multidimensional types and signals. To provide some intuition, it is useful to recall the reason for multiplicity of equilibria in Bayesian signaling games. There, multiple equilibria can be sustained by the adverse inference in the case of deviation: The receiver believes that all deviations come from the "worst" type. Since, in contrast, on-path actions are evaluated according to the posterior beliefs about the sender's type, it is easy to deter deviations and sustain multiple equilibria. In contrast, in our model, securities on and off the equilibrium path are evaluated similarly, because all securities are evaluated by their worst-case justifiable models.

4 Main Results

In this section, we present the main results for the model with private information about the new project. As we will see, the equilibrium is quite different depending on whether the investor contemplates the possibility that the investment project has a negative NPV ($B_+ \subset B$) or the investor is confident that the project's NPV is non-negative ($B_+ = B$).

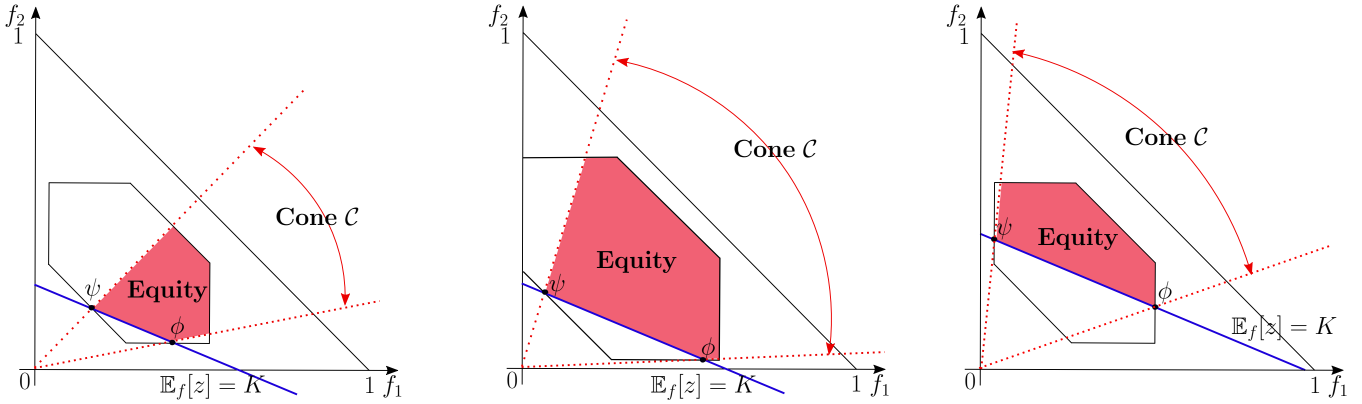
Note that for any base distribution $g : \mathbb{E}_g[z] \geq K$, there exists a cut-off degree of investor's uncertainty ν, such that $B_+ \subset B$, if the degree of investor's uncertainty exceeds this cut-off, and $B_+ = B$, otherwise. Thus, we will refer to these two cases as the case of large uncertainty and the case of small uncertainty, respectively.

4.1 Large Uncertainty

Suppose that the degree of investor's uncertainty $\nu$is large in the sense that she is not confident that the project has a positive NPV: $B_+ \subset B$. We show that in this case standard outside equity, i.e., a security that promises the investor a fraction of the realized project's payoff z, is a credible and relatively cheap signal that the issuer has a positive-NPV project. This is due to the fact that under unlevered equity the issuer and the outside investor will hold securities with the same shape. As a consequence, there is a subset of types in B_+ that issues standard unlevered equity and it expands as uncertainty increases or the quality of the project worsens.

Equity Region Let $\mathcal{C}$ denote the convex cone generated by vectors in the relative interior of B_0, and let $\mathcal{E} \equiv \mathcal{C} \cap B_+$ be the intersection of this cone and the set of non-negative NPV projects.[15] Note that sets $\mathcal{C}$, $\mathcal{E}$, and B_+ depend on ν and K, which we suppress in

[15]Formally, $\mathcal{C}$ consists of all vectors of the form $\sum_{i=1}^{k} \alpha^i f^i$ where $k \geq 1$, and for all $i = 1, \ldots, k$, $f^i \in \mathrm{relint}(B_0)$ and $\alpha_i^i \geq 0$. Cones $\mathcal{C}$ and $\mathcal{E}$ are well-defined, because the set of zero-NPV projects B_0 is non-empty



(a) **Illustration of Equilibrium** (b) **Higher Uncertainty** (c) **Worse Project Quality**

Figure 3: ***Equity Region***

the notation for brevity. Let $cl(\mathcal{E})$ denote the closure of $\mathcal{E}$. The next theorem shows that, loosely speaking, standard equity is issued in equilibrium if and only if the issuer's type is inside region $\mathcal{E}$, and that this region expands with $\nu$and K:[16]

Theorem 1. *Suppose $B_+ \subset B$. Then,*

1. *For all $f \in \mathcal{E}$, $s^*(f) = (I/K)z$. For all $f \notin cl(\mathcal{E})$, $s^*(f) \neq (I/K)z$.*

2. *Holding K fixed, as ν increases, set $\mathcal{E}$ continuously expands and $\mathcal{E} = B_+$ whenever ν is sufficiently high so that $B = \Delta(Z)$.*

3. *Suppose ν is such that B is contained in the interior of $\Delta(Z)$, and let $\overline{K} \equiv \max_{f \in B} \mathbb{E}_f[z]$. Then, as K increases, set $B_+ \backslash \mathcal{E}$ continuously shrinks, and there is $\underline{K} < \overline{K}$ such that $cl(\mathcal{E}) = B_+$ for all $K \in [\underline{K}, \overline{K}]$.*

To see the intuition, consider the model with three states, illustrated in Figure 3*a*.[17] In this case, set B_0 is a segment connecting distributions $\psi \equiv \arg\min_{f \in B}\{f_1 \text{ s.t. } f_1 z_1 + f_2 z_2 = K\}$ and $\phi \equiv \arg\max_{f \in B}\{f_1 \text{ s.t. } f_1 z_1 + f_2 z_2 = K\}$, and cone $\mathcal{C}$ consists of all distributions with $f_2/f_1 > \phi_2/\phi_1$ and $f_2/f_1 < \psi_2/\psi_1$. The equity region $\mathcal{E}$ is the intersection of this cone and set B.

Consider any issuer type $f \in \mathcal{E}$. When there are three states, any non-equity security is either concave or convex. Consider why any type in $\mathcal{E}$ prefers to issue equity over any

whenever $B_+ \subset B$.

[16]This statement is "loose" because of possible indeterminacy at the boundary.

[17]The formal analysis of the model with three states is provided in Appendix.

concave security s at which the investor breaks even ($P(s) = I$). By Pricing Lemma, the worst-case justifiable model for such security is the most dispersed zero-NPV distribution ψ. Intuitively, when facing a concave security, the investor is worried that

In words, the investor is concerned that the project has zero NPV, but also that there is a relatively high probability of upside (z_2), but she would not gain from the upside much by holding a concave security. Suppose that the issuer decreases s_1 by ε, and increases s_2 by $\varepsilon\psi_1/\psi_2$. If ε is sufficiently small, then the new security is still concave, and so, the worst-case justifiable model is still ψ and the investor will accept it. Because $f_2/f_1 < \psi_2/\psi_1$, this modification increases the expected payoff of type f by $(f_1 - f_2\psi_1/\psi_2)\varepsilon$. Therefore, equity $s = (I/K)z$ dominates any concave security.

Now, let us show that equity dominates any convex security s such that $P(s) = I$. By Pricing Lemma, the worst-case justifiable model changes and is now the most concentrated zero-NPV distribution ϕ. In words, the investor is concerned that the project has zero NPV, but also that there is not much upside in the project, and hence, convex securities are less valuable. If the issuer increases s_1 by ε, and decreases s_2 by $\varepsilon\phi_1/\phi_2$, then for small enough ε, the new security is still convex and the worst-case justifiable model is still ϕ, hence, it is accepted by the investor. Because $f_2/f_1 > \phi_2/\phi_1$, this modification increases the expected payoff of type f by $(-f_1 + f_2\phi_1/\phi_2)\varepsilon$. Therefore, equity $s = (I/K)z$ dominates any convex security. Importantly, when the issuer type f offers equity, he does not have incentives to slightly increase s_1 and decrease s_2 (or vice versa), because this would make the security concave (respectively, convex) and by the argument above such a security is dominated by equity for type f.

The special feature of equity is that both the investor and the issuer hold the security with the same shape. Since the issuer invests his own W into the project, the fact that the issuer keeps stake (W/K) in the company signals to the investor that $\mathbb{E}_f[(W/K)z] \geq W$. This implies that $\mathbb{E}_f[(I/K)z] \geq I$, and so, the investor is willing to finance the project. In contrast, any non-linear security s only signals that the security that the issuer keeps, $z - s$, is good enough. However, this could occur, because security s pays most in states that are relatively less likely, and pays little in states that are relatively more likely. For example, when s is concave, the investor is concerned that the distribution of cash flows is very dispersed so that the issuer can break even, because he is exposed to the upside by holding $z - s$, but the investor would not benefit from the upside by holding concave s.

The second implication of Theorem 1 is that equity becomes more prevalent as the uncertainty set expands (when ν increases) or the investment project becomes uniformly worse (when investment cost K increases). To illustrate this, we again turn to the model with three states. In Figure 3b compared to Figure 3a, when the uncertainty ν gets larger, the gap between the two extreme zero-NPV models, ψ and ϕ, increases. This way there is a bigger change in the investor's worst-case justifiable model, when the issuer switches from convex to concave securities, and so, more types find it optimal to offer equity. Geometrically, as ν increases, cone $\mathcal{C}$ expands, and so, region $\mathcal{E}$ expands. For ν sufficiently large, region $\mathcal{E}$ takes over the whole B_+ area.

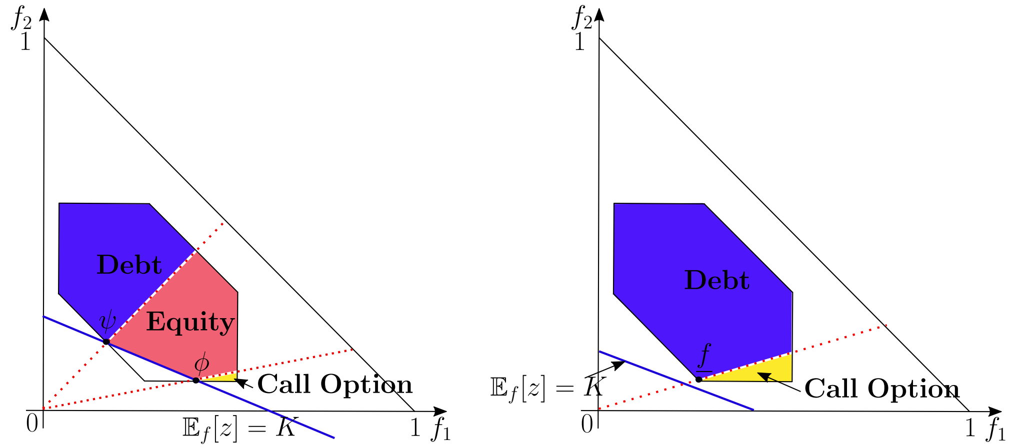
Similarly, in Figure 3c compared to Figure 3a, as K increases and fewer projects in set B are profitable, there are fewer types in B_+ with $f_2/f_1 > \psi_2/\psi_1$ and with $f_2/f_1 < \phi_2/\phi_1$, and so, fewer types issue securities different from equity. Again, as K becomes sufficiently large, all issuer types in B_+ issue equity.

Third, the two implications described above hold for any N. In fact, the geometric argument is exactly the same: equity is issued by types in B_+ inside cone $\mathcal{C}$, and as ν or K increase this cone expands, which leads to more prevalence of equity. Thus, our results are not driven by the investor's uncertainty about a particular moment of the distribution of cash flows, such as mean or variance. Rather, what is important for our results is that the worst-case justifiable model changes with the shape of the security. In particular, that set B has full dimensionality. In words, equity becomes special, because whenever there are non-linear features of the security, the investor becomes concerned that the issuer would gain from such features at the expense of the investor.

Finally, it is interesting to point out that a variety of project types pool on the same equity contract, $s = (I/K)z$, which has a natural interpretation: the investor gets the share of cash flows that is proportional to her contribution to the investment cost K. This is a common profit sharing rule in practice. However, this security does not arise in the complete information case, where the optimal contract is determined by the outside option of the investor rather than contributions of both parties.

Debt Region We say that f *likelihood ratio dominates* f', denoted $f \succ_{LRD} f'$, if f_n/f'_n is strictly increasing in n. Let $F^* \equiv \{f : f = f^*(s) \text{ for some } s \text{ s.t. } P(s) = I\}$ be the set of worst-case justifiable models for all securities that make the investor break even (and can potentially be issued in equilibrium). Denote by $\mathcal{D} \equiv \{f : f \succ_{LRD} f' \text{ for all } f' \in F^*\}$ the set of issuer types that likelihood ratio dominate any model in F^*. We say that a security is *risky debt* if it takes the form $s = \min\{z, d\}$ for some $d \in (0, z_N)$.

Proposition 2. *Suppose $B_+ \subset B$. Issuer types $f \in \mathcal{D}$ pool on the risky debt security.*



(a) **Large Uncertainty** (b) **Small Uncertainty**

Figure 4: ***Equilibrium Characterization in the Model with Three States***

The argument for the optimality of debt in region $\mathcal{D}$ is most clearly seen in the model with three states. In this case, as we argued above, worst-case justifiable models for securities s such that $P(s) = I$ span the segment B_0 that connects distributions ψ and ϕ, respectively, most dispersed and most concentrated zero-NPV distributions in set B. Thus, region $\mathcal{D}$ consists of all distributions that likelihood ratio dominate any distribution in B_0, which are distributions with $f_2/f_1 > \psi_2/\psi_1$. (See Figure 4a for an illustration.) Intuitively, all securities are underpriced in equilibrium. In region $\mathcal{D}$, the likelihood of high cash flow z_2 relative to low cash flow z_1 is higher compared to that in any worst-case justifiable model for securities that can potentially be offered in equilibrium. In words, the investor is more concerned about the downside compared to the issuer. Hence, it is optimal for the issuer to provide maximal downside protection and offer debt.

Characterization with Three States We have shown that both risky debt and standard equity may appear in equilibrium when the uncertainty in large. In general, one can obtain the complete characterization of equilibrium from program (8). This characterization is particularly simple in the model with three states.

Proposition 3. *Suppose that $B_+ \subset B$ and $N = 2$. Then, in equilibrium, only types in B_+ issue securities and i) types in B_+ with $f_2/f_1 > \psi_2/\psi_1$ offer the risky debt with face value d given by $\mathbb{E}_\psi[\min\{z, d\}] = I$; ii) types in B_+ with $\psi_2/\psi_1 > f_2/f_1 > \phi_2/\phi_1$ offer the standard equity with stake I/K; iii) types in B_+ with $\phi_2/\phi_1 > f_2/f_1$ offer the call option with strike* *price k given by* $\mathbb{E}_\phi[\max\{z-k,0\}] = I$.[18]

Figure 4*a* illustrates the equilibrium. We have argued above why debt and equity arise in regions $\mathcal{D}$ and $\mathcal{E}$, and it is only left to show that call-option is optimal for types in B_+ with $\phi_2/\phi_1 > f_2/f_1$. The argument is symmetric to the optimality of debt in region $\mathcal{D}$. When type f is such that $\phi_2/\phi_1 > f_2/f_1$, it is cheaper for issuer type f to pay the investor in state z_2 (rather than state z_1), which he considers relatively less likely compared to the investor, who uses one of models in B_0 to value equilibrium securities. Thus, call-option is the optimal security for such types.

4.2 Small Uncertainty

We now turn to the case of small uncertainty in the sense that all projects in B have positive NPV, i.e., $B_+ = B$. Let $\underline{f} \equiv \arg\min_{f\in B} \mathbb{E}_f[z]$ be the distribution with the lowest NPV in B. In Lemma 3 in Appendix, we show that such a distribution is unique and it is the distribution that maximally shifts the probability mass from high states into low states.

Theorem 2. *Suppose $B_+ = B$. Then,*

1. *The investor's worst-case justifiable model is $\underline{f}$ for any security.*

2. *For any s that is offered by a positive Lebesgue measure of types, it holds that for all $n = 1, \ldots, N$ either $s_n = s_{n-1}$ or $z_n - s_n = z_{n-1} - s_{n-1}$.*

3. *If in equilibrium true cash flow distribution f likelihood ratio dominates the investor's worst-case justifiable model $\underline{f}$, then the issuer with type f finances the project with risky debt with nominal d such that $\mathbb{E}_{\underline{f}}[\min\{d,z\}] = I$.*

To provide intuition for Theorem 2, let us again first consider the case of three states. The first implication of Theorem 2 is that all relevant securities are priced using the most pessimistic model in B, model $\underline{f}$. Model $\underline{f}$ is the distribution in B that is maximally skewed towards low realizations: it puts probability $\min\{g_0+\nu,1\}$ on $z=0$ and probability $\max\{g_2-\nu,0\}$ on $z=z_2$. Intuitively, because the investor is certain that the project has a non-negative NPV, she knows that the issuer is weakly better off investing for any distribution of the project's cash flows and any security at which the investor just breaks even. Thus, distribution $\underline{f}$ is in the set of justifiable models for any security at which the

[18] Recall that call option is the security that pays $\max\{z-k,0\}$ for some strike price k.

investor breaks even. Since it is also the distribution at which the value of any security is minimized, it must be the worst-case justifiable model.

Given that any relevant security is priced by the investor at $\mathbb{E}_{\underline{f}}[s]$, it is straightforward to solve the issuer's problem: the issuer of type $f \in B$ simply chooses the cheapest security (i.e., with the lowest $\mathbb{E}_f[s]$) from the set of securities that satisfy $\mathbb{E}_{\underline{f}}[s] = I$. The following proposition characterizes the equilibrium:

Proposition 4. *Suppose that $B_+ = B$ and $N = 2$. Then, in equilibrium, (i) types f with $f_2/f_1 > \underline{f}_2/\underline{f}_1$ offer a standard risky debt security with face value d such that $\mathbb{E}_{\underline{f}}[\min\{z, d\}] = I$; (ii) types f with $f_2/f_1 < \underline{f}_2/\underline{f}_1$ offer a call option with strike price k such that $\mathbb{E}_{\underline{f}}[\max\{z - k, 0\}] = I$.*

Figure 4*b* illustrates the equilibrium. The dashed line depicts types that are indifferent between all securities satisfying $\mathbb{E}_{\underline{f}}[s] = I$. Types above the dashed line (with $f_2/f_1 > \underline{f}_2/\underline{f}_1$) issue risky debt. Intuitively, the issuer is endogenously more optimistic about the project than the investor. Furthermore, for any distribution above the dashed line, the issuer is uniformly more optimistic about higher realizations of the cash flows (in the sense of monotone likelihood ratio property (MLRP)) than the investor. Thus, the issuer is better off keeping the security that gives him the maximum upside and giving the investor the security that gives her the highest payoff in the low cash flow realizations. The security that achieves this is the standard debt security. Types below the dashed line (with $f_2/f_1 < \underline{f}_2/\underline{f}_1$) issue a call option. For these distributions, even though the issuer is more optimistic about the project than the investor, his optimism is not monotone in the states: he knows that the probability of the highest cash flow realization is rather low, while the probability of the medium cash flow realization is rather high. Thus, she wants to keep the security that gives her the highest payoff for medium realizations of the cash flow while selling the security that gives the investor paid in the highest cash flow realizations.

Theorem 2 shows that the intuition from the case of three states carries out to the general case. When the uncertainty is small, the worst-case justifiable model for securities that make the investor break even is the same for all such securities. Because of this, equity that is "immune" to changes in the worst-case justifiable model does not play a special role. Instead, any security s that is issued by a positive Lebesgue measure of types is some extreme point of the set $\mathcal{S}$: in any states either constraint $s_n \geq s_{n-1}$ or $z_n - s_n \geq z_{n-1} - s_{n-1}$ binds. Such securities can be represented as a collection of tranches, i.e., $s = \min\{d_1, z\} + \min\{d_2 - d_1, z - d_1\} + \cdots + \min\{d_J - \sum_{j=1}^{J-1} d_j, z - \sum_{j=1}^{J-1} d_j\}$ for some $d_1 < d_2 < \cdots < d_J$. For example, in the case of three states, there are two such securities – one representing the senior tranche (the risky debt), and another the junior tranche (the call option). As in the case of large uncertainty in Proposition 2, types f that likelihood ratio dominate the worst-case justifiable model $\underline{f}$ issue risky debt in equilibrium.

5 Private Information about Assets in Place

In this section, we show that the implications of the model change drastically if the private information of the issuer concerns assets in place rather than the new project.

Consider the following modification of the baseline model.[19] In the baseline model, we assumed that the issuer has an existing resource of known value W and the distribution of the project's cash flows is the issuer's private information, over which the investor faces Knightian uncertainty. In this modification, we flip the nature of private information and assume that the parties have the same information about the value added of the project but different information about the existing resource of the issuer. Specifically, suppose instead that the issuer has assets in place that generate cash flow z distributed according to f that is privately known by the issuer. The investor does not know f, but knows that $f \in B$, where B is the total variation distance neighborhood of radius ν around some base distribution $g \in \Delta(Z)$.

In addition to assets in place, there is an investment project that requires an investment of K. If K is invested, cash flow distribution f is improved to distribution $\hat{f}$, satisfying $\hat{f} = f + y$, where $y \in \mathbb{R}^N$ is commonly known and such that $f + y \in \Delta(Z)$ for all $f \in B$. Thus, the investment project augments distribution f into $\hat{f}$ by redistributing the probability mass across states in a known way. The gains from the project are $\eta \equiv \mathbb{E}_{\hat{f}}[\hat{z}] - \mathbb{E}_f[z] = \sum_{i=1}^N y_i z_i$, and its net present value (NPV) is $\eta - K$, which we assume to be positive. Thus, this modified model is the mirror image of the baseline model: The investor and the issuer share common knowledge about the new project, but the issuer is privately informed about assets in place.

As an example, consider the model with three states ($N = 2$) and $y = \begin{pmatrix} 0 & \delta \end{pmatrix}'$. In this case, the new project simply shifts probability mass $\delta > 0$ from the lowest state $z = 0$ to the highest state $z = z_2$. The project's NPV is $\delta z_2 - K > 0$. Thus, both the issuer and the investor share common knowledge about the investment project. However, the issuer has private information about the cash flow distribution from the assets in place, over which the investor faces Knightian uncertainty.

[19] With a slight abuse of notation, we use the same notation for objects analogous to the basic model.

Since there is a one-to-one mapping between f and $\hat{f}$, we can equivalently refer to $\hat{f}$ as the issuer's type. Similarly, we equivalently refer to set $\hat{B} \equiv \{\hat{f} \in \Delta(Z) : \hat{f} = f + y \text{ for some } f \in B\}$ as the uncertainty set. Intuitively, $\hat{B}$ is obtained by shifting every point $f \in B$ by the vector y. Note that $\hat{B}$ is also the total variation distance neighborhood of radius ν around the shifted base distribution $\hat{g} = g + y$.

The timing of the game is the same as in the baseline model. The issuer offers security s that pays s_n if $\hat{z} = z_n$, which the investor decides whether to accept or reject. If the investor rejects the offer, there is no investment, and the issuer's payoff is given by the cash flow realization from distribution f. If the investor accepts the offer, she pays K that the firm invests, cash flow z_n is realized from distribution $\hat{f}$, and the investor and the issuer obtain s_n and $z_n - s_n$, respectively.

The equilibrium strategies $s^*(\cdot)$ and $\sigma^*(\cdot)$, and the model updating mapping $\hat{B}(s)$ are defined analogously to the baseline model. First, security $s^*\left(\hat{f}\right)$ must maximize the expected utility of type $\hat{f}$ given the investor's acceptance strategy σ^*:

$$s^*(\hat{f}) \in \arg\max_{s \in \mathcal{S}} \{\sigma^*(s)\mathbb{E}_{\hat{f}}[z - s]\},$$

with $s^*\left(\hat{f}\right) = \mathbf{0}$ for any $\hat{f}$ such that $\max_{s \in \mathcal{S}:\sigma^*(s)=1} \mathbb{E}_{\hat{f}}\left[z - s\right] < \mathbb{E}_f\left[z\right]$. Second, the investor accepts the security if and only if she values it at K or above: $\sigma^*\left(s\right) = 1$ if and only if $P\left(s\right) \geq K$, where $P\left(s\right) = \min_{\hat{f} \in \hat{B}(s)} \mathbb{E}_{\hat{f}}\left[s\right]$. Finally, for each $s \in \mathcal{S}$, $\hat{B}\left(s\right)$ is the set of justifiable models defined by

$$\hat{B}(s) \equiv \left\{\hat{f} \in \hat{B} : \mathbb{E}_{\hat{f}}[\hat{z} - s] - K \geq U^*(\hat{f})\right\}, \tag{9}$$

whenever this set is non-empty, and $\hat{B}\left(s\right) = \hat{B}$, otherwise. In (9), $U^*\left(\hat{f}\right)$ is the equilibrium utility of the issuer type $\hat{f}$:

$$U^*(\hat{f}) \equiv \begin{cases} \mathbb{E}_{\hat{f}}[\hat{z} - s^*(\hat{f})], & \text{if } \sigma^*\left(s^*\left(\hat{f}\right)\right) = 1, \\ \mathbb{E}_{\hat{f}}[\hat{z}] - \eta, & \text{otherwise.} \end{cases}$$

The main result of this section, presented in the following theorem, shows that the equilibrium in this model is conceptually different from the equilibrium in the baseline model:

Theorem 3. *The equilibrium of the model in this section is generically unique and is characterized as follows. For any $s \in \mathcal{S}$, $P(s) = \mathbb{E}_{\underline{\hat{f}}}[s]$, where $\underline{\hat{f}} \equiv \arg\min_{\hat{f} \in \hat{B}} \mathbb{E}_{\hat{f}}[\hat{z}]$ is the distri-*

bution in $\hat{B}$ with the lowest value. If $\min_{s\in\mathcal{S}}\left\{\mathbb{E}_{\hat{f}}\left[s\right]\ \ s.t.\ \mathbb{E}_{\underline{\hat{f}}}\left[s\right]=K\right\}>\eta$*, then* $s^*\left(\hat{f}\right)=\mathbf{0}$. *Otherwise,*

$$s^*(\hat{f})\in\arg\min_{s\in\mathcal{S}}\{\mathbb{E}_{\hat{f}}[s]\ \ s.t.\ \mathbb{E}_{\underline{\hat{f}}}[s]=K\}. \tag{10}$$

Further, in equilibrium,

1. *any issuer type $\hat{f}$: $\hat{f}\succ_{LRD}\underline{\hat{f}}$ offers debt contract $s^*\left(\hat{f}\right)=\min\left\{z,d\right\}$ with face value $d:\mathbb{E}_{\underline{\hat{f}}}\left[\min\left\{z,d\right\}\right]=K$, if $\mathbb{E}_{\hat{f}}\left[\min\left\{z,d\right\}\right]\leq\eta$, and does not raise financing, otherwise.*

2. *no set of issuer types of positive Lebesgue measure offers equity $s=\alpha z$ for any $\alpha\in(0,1)$.*

3. *there is $\bar{K}<\eta$ such that for all $K\in(\bar{K},\eta)$, there is a positive Lebesgue measure of types that do not raise financing.*

Markedly different from the baseline model, in equilibrium here all securities are priced using the same model $\underline{\hat{f}}$. This difference arises due to the issuer's payoff without investment being sensitive to private information. If private information concerns the new project and value W is common knowledge, the issuer's security offer credibly signals that the payoff from the residual security cannot be too low, or else the issuer would be better off not doing the project. As a consequence, the worst-case justifiable model is different from $\underline{f}$ (unless set B is small enough) and depends on the the security offered by the issuer. In contrast, if private information concerns assets in place and the value added of the new project is common knowledge, the issuer cannot credibly signal that the assets in place are not too bad: if issuer type $\hat{f}\neq\underline{\hat{f}}$ finds it optimal to issuer security s, then we can find the distribution that assigns a marginally weakly lower probability to each positive cash flow realization than $\hat{f}$, and this type must also find it optimal to issue security s. Thus, model $\underline{\hat{f}}$ is in the set of justifiable models $B\left(s\right)$ for any security $s\in\mathcal{S}$. To give a specific example, suppose that type $\hat{f}\neq\underline{\hat{f}}$ could issue a fairly priced equity stake $K/\mathbb{E}_{\hat{f}}[\hat{z}]$. Then, by mimicking type $\hat{f}$, type $\underline{\hat{f}}$ would gain $K-\mathbb{E}_{\underline{\hat{f}}}[\hat{z}](K/\mathbb{E}_{\hat{f}}[z])>0$ compared to his equilibrium payoff. Thus, model $\hat{f}$ is also justifiable for equity stake $K/\mathbb{E}_{\hat{f}}[\hat{z}]$, so the investor would value it using model $\underline{\hat{f}}$, rather than model $\hat{f}$.

The main economic insight of Theorem 3 is that equity generically never arises in equilibrium when private information is about assets in place, in contrast to the case when private information is about the new project. Intuitively, equity is special when private information is about the new project, because it credibly signals that the project's value cannot be too low, and because both the investor and the issuer hold securities with the same shape, the investor does not care about the exact distribution of cash flows that achieves the project's value. Equity does not play this special signaling role when private information concerns assets in place, and because the worst-case justifiable model does not depend on the security offered, equity generically does not arise in equilibrium. By the same argument as in Proposition 1, program (10) generically has a unique solution, which implies the generic uniqueness of the equilibrium. By the same logic, each issuer type $\hat{f}$ that likelihood-ratio dominates distribution $\underline{\hat{f}}$ offers debt.

It is worth noting that if investment costs are sufficiently high, not all types raise financing and the equilibrium is inefficient, in contrast to the model in which private information is about the new project. To see this, note that the gain from investment for issuer type $\hat{f}$ if he raises financing with security s equals

$$\mathbb{E}_{\hat{f}}[\hat{z} - s^*(\hat{f})] - \mathbb{E}_f[z] = \underbrace{\eta - K}_{\text{NPV}} - \underbrace{\Big(\mathbb{E}_{\underline{\hat{f}}}[s^*(\hat{f})] - K\Big)}_{\text{mispricing}}. \tag{11}$$

For any $\hat{f} > \underline{\hat{f}}$, the mispricing term is always positive, because the investor evaluates securities using model $\underline{\hat{f}}$ rather than the actual distribution. As K increases, the first component in (11) decreases, and eventually, the negative impact of mispricing outweighs the benefits from the investment and type $\hat{f}$ prefers not to raise financing.

We finish by characterizing the equilibrium in the model in the special case of three cash flow realizations, which is also illustrated in Figure 5 :

Corollary 2. *Suppose that $N = 2$. Let d and k be solutions to $\mathbb{E}_{\underline{\hat{f}}}[\min\{z, d\}] = K$ and $\mathbb{E}_{\underline{\hat{f}}}[\max\{z - k, 0\}] = K$. Then, in equilibrium: (i) types $\hat{f}$ with $\hat{f}_2/\hat{f}_1 > \underline{\hat{f}}_2/\underline{\hat{f}}_1$ and $\mathbb{E}_{\hat{f}}[\min\{z, d\}]] \leq \eta\}$ offer risky debt with face value d; (ii) types $\hat{f}$ with $\hat{f}_2/\hat{f}_1 < \underline{\hat{f}}_2/\underline{\hat{f}}_1$ and $\mathbb{E}_{\hat{f}}[\max\{z - k, 0\}] \leq \eta\}$ offer a call option with strike price k; (iii) types $\hat{f}$ with $\mathbb{E}_{\hat{f}}[\min\{z, d\}]] > \eta$ and $\mathbb{E}_{\hat{f}}[\max\{z - k, 0\}] > \eta$ do not raise financing.*

All types with $\hat{f}_2/\hat{f}_1 > \underline{\hat{f}}_2/\underline{\hat{f}}_1$ prefer debt to any other security, while all types with $\hat{f}_2/\hat{f}_1 < \underline{\hat{f}}_2/\underline{\hat{f}}_1$ prefer a call option to any other security. The intuition for the optimality of the risky debt or call option is similar to that in the baseline model in the case of small uncertainty: there is an endogenous heterogeneity of beliefs between the issuer and the investor, and the issuer prefers to maximally shift payments from the security to the states that he considers relatively less likely.

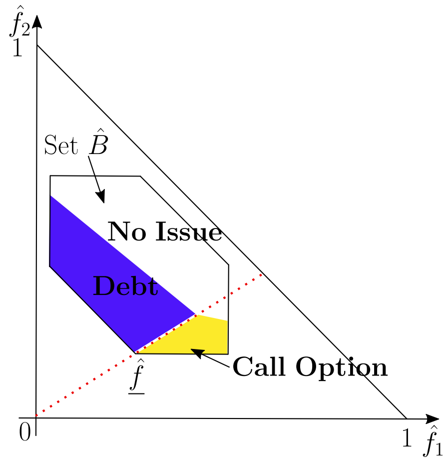


Figure 5: ***Equilibrium in the Assets in Place Model with Three States***

6 Extensions

6.1 Alternative Definition of Justifiable Models

In our model, the notion of the set of justifiable models captures the idea that the investor tries to learn as much as possible even from offers that lie out of equilibrium path. We next identify the element of investor's learning that is crucial for our results.

Let us consider two alternative specifications of the model updating mapping. First, suppose $B(s) = B$ for any off-path security s, that is, the investor does not learn from off-path offers. Then we generally can sustain a variety of equilibria in this case. To see this, consider the case when $\mathbb{E}_{\underline{f}}[z] < I$. Consider some security $\tilde{s}$ such that $\mathbb{E}_f[\tilde{s}] = I$ for some $f \in B$. Then, there is an equilibrium, in which all types f such that $\mathbb{E}_f[z - \tilde{s}] \geq W$ issue $\tilde{s}$, while the rest of types do not raise financing. Indeed, any off-path offer s will be rejected, as $\min_f \mathbb{E}_f[s] = \mathbb{E}_{\underline{f}}[z] < I$. Thus, no security other than $\tilde{s}$ could be issued, and security $\tilde{s}$ is issued only by types with $\mathbb{E}_f[z - \tilde{s}] \geq W$. This implies that the investor's learning from both on- and off-path security offers is important for our generic uniqueness result in Proposition 1.

In the second specification, suppose that

$$B(s) = \{f \in B : \mathbb{E}_f[z - s] \geq W\} \tag{12}$$

whenever this set is non-empty, and $B(s) = B$ otherwise. The intuition for this criterion is that offering security s if $\mathbb{E}_f[z - s] < W$ is dominated by not raising financing if there is even a slightest chance that such a security will be accepted. Thus, the investor discards all types for whom this is the case. One can verify that our analysis goes through under this specification of the model updating rule. The fact that our results hold for this alternative specification of model updating mapping is closely related to the conclusion of Pricing Lemma. Recall that Pricing Lemma implies that while in general the investor learns certain information from security offers, the only information that is relevant for her valuation of security offers is whether the project has positive or negative NPV. This, however, is not obvious a priori. For example, one might expect that the more refined information that the issuer signals through security offers could improve the valuation of securities and result in further separation in equilibrium. One of the important implications from Pricing Lemma is that this is not the case.

6.2 Separation of Uncertainty and Uncertainty Aversion

This subsection shows that our results are robust to an alternative robust valuation method used by investors.

In venture financing, investors often aim to limit losses in the worst-case scenario, while ensuring there is a significant upside in the best-case scenario (the "catch a unicorn"). The Hurwitz criterion captures this valuation method. Specifically, for some fixed $\omega \in (0, 1]$, the investor values security s at

$$P^{\omega}(s) = \omega \min_{f \in B(s)} \mathbb{E}_f[s] + (1 - \omega) \max_{f \in B(s)} \mathbb{E}_f[s] \tag{13}$$

instead of (3). This valuation arises when the investor lacks confidence to assign probability to all possible scenarios or lacks data to estimate these probabilities. Instead, she simplifies the problem and focuses on the weighted average of worst- and best-case scenarios. When $\omega = 1$, (13) reduces to our baseline model, while when $\omega \to 0$, the investor becomes non-prudent and only focuses on the best-case scenario. Ghirardato et al. (2004) provide axiomatic foundations for the Hurwitz criterion.

We next extend our analysis to this more general model of the valuation of securities by investors. For tractability, we deviate from the base model and follow the previous section by assuming that $B(s) = \{f \in B : \mathbb{E}_f[z - s] \geq W\}$.

By the same logic as for $\underline{f}$, we can show that if $\overline{f}$ is the unique distribution that maximizes $\mathbb{E}_f[z]$ over $f \in B$, then $\max_{f \in B} \mathbb{E}_f[s] = \mathbb{E}_{\overline{f}}[s]$ for all $s \in \mathcal{S}$. Thus, given that $B(\cdot)$ satisfies

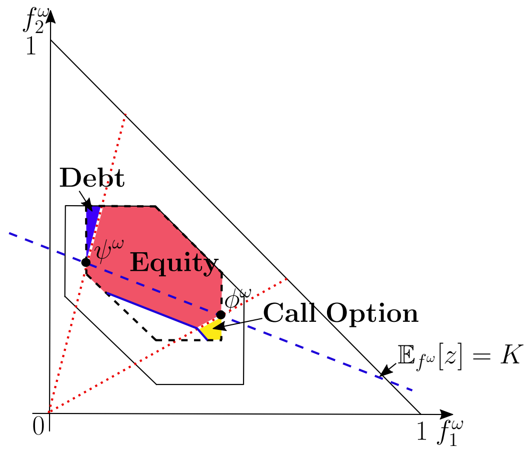


Figure 6: ***Illustration of Equilibria when Securities are Valued at*** P^ω
Pooling regions are described in terms of types f^ω inside the set B^ω (the dashed hexagon), which is a contracted version of B.

12, $P^\omega(s)$ in (13) can be rewritten as

$$P^\omega(s) = \omega \min_{f \in B(s)} \mathbb{E}_f[s] + (1-\omega)\mathbb{E}_{\overline{f}}[s] = \min_{f \in B(s)} \mathbb{E}_{\omega f + (1-\omega)\overline{f}}[s].$$

Define $B^\omega = \{f^\omega = \omega f + (1-\omega)\overline{f}, f \in B\}$, and map any $f \in B$ into corresponding distribution $f^\omega = \omega f + (1-\omega)\overline{f} \in B^\omega$. (See Figure 6). We can prove that the counter-parts of Lemmas 1 and 2 hold (Lemma 6 in the Online Appendix). In particular, it is without loss for the equilibrium analysis to restrict attention to securities such that $P^\omega(s) = I$, and for all such securities,

$$P^\omega(s) = \min_{f^\omega \in B^\omega_+} \mathbb{E}_{f^\omega}[s],$$

where $B^\omega_+ = \{f^\omega \in B^\omega : \mathbb{E}_{f^\omega}[z] \geq K\}$ is the counter-part of B_+ in the baseline model. By the same argument as in the baseline model, the issuer of type f chooses among securities $s \in \mathcal{S}$ such that $P^\omega(s) = I$ the one that minimizes $\mathbb{E}_f[s]$. Since for any securities s and s' in $\mathcal{S}$, $\mathbb{E}_f[z-s] \geq \mathbb{E}_f[z-s']$ if and only if $\mathbb{E}_{f^\omega}[z-s] \geq \mathbb{E}_{f^\omega}[z-s']$, this minimization is equivalent to minimizing $\mathbb{E}_{f^\omega}[s]$ subject to $P^\omega(s) = I$.

The solution to this problem is similar to the baseline model, and we describe it in the case of large uncertainty. Denote $\psi^\omega = \arg\min_{f^\omega \in B^\omega}\{f_1^\omega \text{ s.t. } f_1^\omega z_1 + f_2^\omega z_2 = K\}$ and $\phi^\omega = \arg\max_{f^\omega \in B^\omega}\{f_1^\omega \text{ s.t. } f_1^\omega z_1 + f_2^\omega z_2 = K\}$ the counter-parts of ψ and ϕ in the baseline model. Then for types f^ω with $f_2^\omega/f_1^\omega > \psi_2^\omega/\psi_1^\omega$ the most preferred security is risky debt d such that $\mathbb{E}_{\psi^\omega}[\min\{d, z\}] = I$; for types f^ω with $\psi_2^\omega/\psi_1^\omega > f_2^\omega/f_1^\omega > \phi_2^\omega/\phi_1^\omega$ the most preferred security is equity $\frac{I}{K}z$; and for types f^ω with $f_2^\omega/f_1^\omega < \phi_2^\omega/\phi_1^\omega$ the most preferred security is call option with strike price k such that $\mathbb{E}_{\phi^\omega}[\min\{z-k,0\}] = I$. Whether they issue these securities in equilibrium or prefer not to raise financing depends on whether their payoff exceeds W or not, i.e., for appropriate security whether $\mathbb{E}_f[z-s] \geq W$ or not. This implies that a larger set of types compared to B^ω_+ raises financing in equilibrium.

We conclude that equilibria of the version of the model where the investor's valuation is given by (3) are equivalent to equilibria in the model where B is replaced by B^ω, types f are replaced by f^ω, with the adjustment that types f^ω with $\mathbb{E}_{\frac{f^\omega-(1-\omega)\overline{f}}{\omega}}[z-s] \geq W$ issue securities in equilibrium (rather than only types with $\mathbb{E}_{f^\omega}[z-s] \geq W$). In such a model, the set B^ω is a subset of B, and so, debt would be more prevalent compared to equity. Thus, the change in the valuation method from baseline to (3) is similar to the reduction in the investor's uncertianty about possible distributions of cash flow.

6.3 Alternative Specification of the Uncertainty Set

In the baseline model, we used the total variation distance as a measure of closeness of probability distributions. As we mentioned in Section 2, with three states this metric is equivalent to the Prokhorov metric, and both are commonly used in applications.[20] In this subsection, we use the relative entropy as an alternative measure of closeness of distributions and demonstrate that our main results are robust to this alteranative specification. We refer to this variation of the baseline model as the *relative entropy modification* of the model. The relative entropy is commonly used in economics and finance (Hansen and Sargent (2001b)).

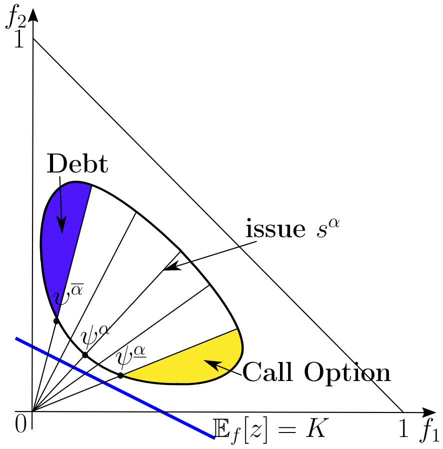
For simplicity, we focus on the model with three states. Suppose that in the baseline model, the uncertainty set B consists of all distributions f such that the relative entropy of f with respect to g is less that ν

$$B \equiv \left\{ f \in \Delta(Z) : \sum_{n=0}^{2} f_n \log(f_n/g_n) \leq \nu \right\}.$$

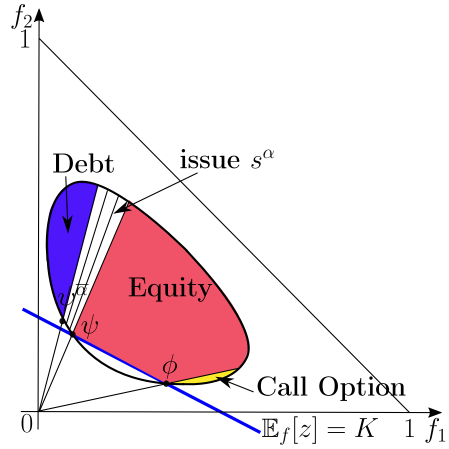
Examples of set B are depicted below in Figures 7a and 7b. First, observe that the proofs of Lemmas 1 and 2 in Section 3 are valid irrespective of the specification of set B. Thus, as before, for any $s \in \mathcal{S}$, the equilibrium pricing $P(s)$ of securities s such that $P(s) = I$ is given by the expression (7).

Let us introduce the following functions. We can represent any security $s \in \mathcal{S}$ by two

[20]It is fairly straightforward to show that for general N, all our results carry through to the case when B is a neighborhood around g in the Prokhorov metric.



(a) **Small Uncertainty**



(b) **Large Uncertainty**

Figure 7: ***Equilibrium Characterization for the Relative Entropy Modification of the Model***

parameters: $\alpha_s = s_1/s_2 \in [0,1]$ and $\beta_s = s_2/z_2 \in [0,1]$.[21] Parameter α_s reflects the curvature of the security s and β_s is a scaling parameter. For any $\alpha \in [0,1]$, define$\psi^{\alpha} \equiv \arg\min_{f\in B}\{\alpha f_1 + f_2\}$. Graphically, the minimization boils down to finding the point, at which the line $\{(f_1, f_2) : \alpha(f_1 - \psi_1^{\alpha}) + (f_2 - \psi_2^{\alpha}) = 0\}$ is tangent to the south-western boundary of B. (See Figure 7*a* for an example of ψ^{α}.) Since B is a strictly convex set and $\alpha f_1 + f_2$ is linear, ψ^{α} is unique. Define s^{α} to be the security such that $\mathbb{E}_{\psi^{\alpha}}[s^{\alpha}] = I$ and $s_1^{\alpha}/s_2^{\alpha} = \alpha$, whenever such a security exists. In the Online Appendix, we show that there is a maximal set $[\underline{\alpha}, \overline{\alpha}]$ such that s^{α} is well-defined for all $\alpha \in [\underline{\alpha}, \overline{\alpha}]$. We can now characterize the (generically unique) equilibrium.

Let us start with the case of small uncertainty, i.e., $\mathbb{E}_f[z] \geq K$ for all $f \in B$. Then, by the analogue of Lemma 1, only securities in $\{s^{\alpha}, \alpha \in [\underline{\alpha}, \overline{\alpha}]\}$ are issued in equilibrium, which are priced in equilibrium at $P(s^{\alpha}) = \mathbb{E}_{\psi^{\alpha}}[s^{\alpha}]$. By analogy with (6), each issuer type f chooses the security $s^*(f)$ that minimizes the mispricing $\mathbb{E}_f[s] - \mathbb{E}_{f(s)}[s]$ subject to $\mathbb{E}_{f(s)}[s] = I$, or equivalently, solves

$$\alpha \in \arg\min_{\alpha\in[\underline{\alpha},\overline{\alpha}]} \{\mathbb{E}_f[s] - \mathbb{E}_{\psi^{\alpha}}[s^{\alpha}]\}. \tag{14}$$

The solution to this problem gives us the following characterization of equilibria in the case of small uncertainty:

Proposition 5. *Consider the relative entropy modification of the model and suppose* $\mathbb{E}_f[z] \geq$

[21]Observe that there is a one-to-one correspondence between s and (α_s, β_s) given by $s_1 = \alpha_s\beta_s z_2$ and $s_2 = \beta_s z_2$.

K for all $f \in B$. Then, in equilibrium, (i) for any $\alpha \in (\underline{\alpha}, \overline{\alpha})$, types f with $f_2/f_1 = \psi_2^\alpha/\psi_1^\alpha$ offer security s^α; (ii) types f with $f_2/f_1 > \psi_2^{\overline{\alpha}}/\psi_1^{\overline{\alpha}}$ offer the risky debt d such that $\mathbb{E}_{\psi^{\overline{\alpha}}}[\min\{z, d\}] = I$; (iii) types f with $f_2/f_1 < \psi_2^{\underline{\alpha}}/\psi_1^{\underline{\alpha}}$ offer the call option with the strike price k such that $\mathbb{E}_{\psi^{\underline{\alpha}}}[\min\{z - k, 0\}] = I$.

Figure 7a depicts a typical equilibrium in the case of small uncertainty. Similarly to the baseline model, there are two regions of types with sufficiently large and sufficiently small f_2/f_1 that pool on the risky debt $s^{\overline{\alpha}}$ or call option $s^{\underline{\alpha}}$, respectively. However, unlike the baseline model, in the relative entropy modification, there is a range of types with $f_2/f_1 \in (\psi_2^{\underline{\alpha}}/\psi_1^{\underline{\alpha}}, \psi_2^{\overline{\alpha}}/\psi_1^{\overline{\alpha}})$ that partially separate. Specifically, all issuer types with $f_2/f_1 = \psi_2^\alpha/\psi_1^\alpha$ offer security s^α. Graphically, these are the types that lie on the part of the line with direction vector ψ^α inside the set B. This separation is possible, because the slope of the south-western boundary of B continuously changes from $-\infty$ to 0, which is in contrast to the baseline model, where it can be only $-\infty$, -1, or 0.

Observe that the actual distribution of cash flows f likelihood raio dominates any worst-case justifiable model that the investor uses in equilibrium if and only if $f_2/f_1 > \psi_2^{\overline{\alpha}}/\psi_1^{\overline{\alpha}}$, and the issuer offers debt in this case. This result is analogous to Proposition 2.

We turn now to the case of large uncertainty. The formal characterization of the equilibrium in this case is given in Proposition 6 in the Online Appendix, and here, we simply use Figure 7b to illustrate how the equilibrium looks like in this case. The equilibrium structure is quite similar to that in the baseline model with the difference that there is an additional separation of certain types that offer securities s^α. (Compare Figures 3a and 7b). Specifically, let us define ϕ and ψ as in Section 4.1. By the same logic as in Section 4.1, the issuer's types with $f_2/f_1 \in (\phi_2/\phi_1, \psi_2/\psi_1)$ offer equity. The rest of the types issue the same securities that they issued in the case of small uncertainty whenever it is possible, otherwise, they pool on the risky debt or call option. Observe that similarly to the baseline model, as K or ν increase, the equity still becomes the dominant source of financing, as the cone $\{f \in B : f_2/f_1 \in (\phi_2/\phi_1, \psi_2/\psi_1)\}$, in which it is issued, expands.

Remark 3. *In this section, we carried the analysis for $N = 2$ and the relative entropy specification of set B. One can verify that the results for general N also hold for the relative entropy modification. Concerning more general specifications of B, our analysis reveals that the slope of the south-wastern boundary of B determines whether the issuer offers securities different from debt, equity, and call option to partially separate. Our analysis can be carried out for general convex B with non-empty interior. While the shape of B affects the details of equilibrium strategies, the general message is robust to the specification of set B. Namely, 1)* *equity becomes dominant as K or ν increase, and 2) debt is optimal in the case of small uncertainty whenever the actual distribution likelihood ratio dominates all investor's worst-case justifiable models that arise in equilibrium*

7 Discussion

In this section, we relate our results to the empirical tests of the pecking order theory, compare our model of financing under asymmetric information and Knightian uncertainty to modelf of financing under asymmetric information and Bayesian uncertainty and to models of moral hazard.

Empirical Implications This subsection connects our predictions to recent empirical tests of the pecking order theory. Let us first summarize main empirical predictions of our model:

1. When the uncertainty about the new project is large, equity is the optimal security (Theorem 1).
2. When the uncertainty about the new project is small, debt is the optimal under the MLRP ordering of beliefs (Theorem 2).
3. When the private information is about assets in place, debt is optimal under the MLRP ordering of beliefs (Theorem 3).

The classical pecking-order theory proposed by Myers and Majluf (1984) states that information asymmetry leads to the issuer's preference for financing through raising debt rather than issuing equity. There is at best mixed evidence about the validity of the pecking order theory. Shyam-Sunder and Myers (1999) show that for a sample of mature firms, there is a strong relation between the financial deficit and net debt issuance. Based on this evidence, they conclude that the data support the pecking order theory. Frank and Goyal (2003) show that for small, high-growth firms, this relationship is no longer present. They reason that for such firms the information asymmetries should be a significant concern, and thus, one should expect the support for the pecking order theory to be more pronounced. Based on the fact that they find the opposite, they reject the pecking order theory.

This evidence, however, is in line with our theory, which stresses the nature of the private information for the ordering of securities. For mature firms, the value comes mostly from assets in place and our theory is in accord with the pecking order theory. For young, high-growth firms, the private information is more likely to be about the new project, and equity is optimal particularly when the uncertainty is large. Thus, results by Shyam-Sunder and Myers (1999) and Frank and Goyal (2003) are in line with our predictions.

Relation to Bayesian Signaling In this subsection, we compare our model of security design under Knightian uncertainty to models of security design under Bayesian uncertainty.

Formally, our model belongs to a general class of signalling games with multidimensional types (in our case, distributions of cash flow) and multidimensional signals (in our case, mappings from the future cash flow into the security payment). In general, this problem with Bayesian receivers is considered to be very hard, and the characterization of the equilibrium set remains an open question. The existing theoretical work on multidimensional signalling (see Quinzii and Rochet (1985), Engers (1987)) provides sufficient conditions for separating equilibria. The existing applied work, which includes models of signalling with securities, makes strong assumptions on the ordering of types to essentially reduce the analysis to single-dimensional types. The closest to our paper in the Bayesian signalling literature is the classical paper by Nachman and Noe (1994), and we use it next to illustrate the difference of our robust approach.[22]

There are two key differences of our model. First, is the robust approach to security valuation by the investor. Second, Nachman and Noe (1994) impose a strong ordering on the issuer's types, which allows them to essentially reduce the problem to one-dimensional signalling. In contrast, we put very little structure on the possible distributions of cash flows, but only require that the belong to set B.

The robust approach allows us to provide the sharp characterization of equilibria of the cash raising game under weaker assumptions about the possible distributions of cash flow. We will next demonstrate that relaxing the assumption about the structure of the issuer's private information allows us to uncover novel features in financing under asymmetric information, and in particular, to show that equity may be optimal when the uncertainty is large. To see this, let us first consider the model of Nachman and Noe (1994) with three states. In their model, issuer's type θ belongs to a finite set $\theta \in \Theta$, and each type is associated with the distribution $f(\theta)$ of future cash flows from the new project. Nachman and Noe (1994) assume that types in Θ are ordered by the strict conditional stochastic dominance (SCSD) ordering. With three states, SCSD ordering of types implies that for all $\theta, \theta' \in \Theta$ such that $\theta < \theta'$,

[22] In fact, an attentive reader may notice that in the title we intentionally mirrored their title.

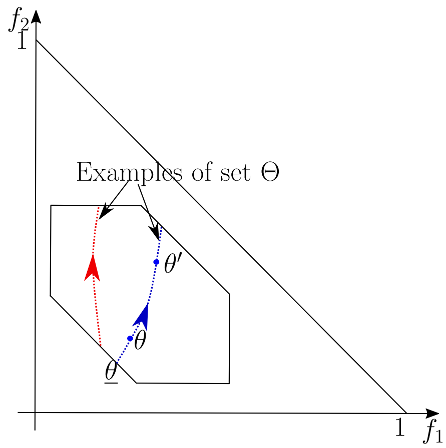


Figure 8: ***Examples of set*** Θ ***in*** *Nachman and Noe (1994)*
Arrows indicate the direction of the increase of types.

$f_1(\theta')/f_1(\theta) < f_2(\theta')/f_2(\theta)$ and $f_1(\theta) + f_2(\theta) < f_1(\theta') + f_2(\theta')$. Figure 8 provides examples of SCSD-ordered set Θ, and in particular, demonstrates that types in Θ are non-generic in our model. Nachman and Noe (1994) additionally impose the D1 refinement and assume that all types have positive NPV. Their main result is that all types in Θ pool on the risky debt. By following the same argument as in the case of small uncertainty in Subsection 4.2, we can show that the same result also obtains in our model, when types are SCSD-ordered.[23] The important insight of our paper is that without strong restrictions on the set of issuer's types, equity naturally arises as an optimal source of financing, and it is closely related to the level of the investor's uncertainty.

Relation to Robust Contracting under Moral Hazard Our theory generates standard risky debt and standard outside equity as equilibrium securities based on asymmetric information between the issuer and the investor. An alternative explanation for these securities comes from models of moral hazard. In a classic paper, Innes (1990) shows that selling debt is the optimal way to finance a project when the entrepreneur faces a moral hazard problem under risk neutrality and limited liability. In a similar moral hazard setting but assuming that the principal faces nonquantifiable uncertainty and requires robustness, Carroll (2015) shows that the optimal contract is linear. Related ideas that linear contracts, in particular, equity, are robust contracts to moral hazard problems also appear in Holmstrom

[23]The sketch of the argument is as follows. First, let $\underline{\theta}$ be the lowest type according to the SCSD ordering. (See Figure 8 for the illustration). Similarly to the first statement in Theorem 2, all securities are priced using $f(\underline{\theta})$. Note that with three states the likelihood ratio dominance and the SCSD orderings coincide. By the argument analogous to Proposition 4, all types issue debt in equilibrium.

and Milgrom (1987), Admati and Pfleiderer (1994), and Ravid and Spiegel (1997). Thus, our theory shares a common prediction with moral hazard models that one should observe financing via equity when Knightian uncertainty is very high and via debt when it is very low.

There are two conceptual differences in implications of our model based on asymmetric information from the moral hazard model of Innes (1990) and Carroll (2015). The first conceptual difference concerns the predictions of the model when Knightian uncertainty is present (unlike Innes (1990)) but not extreme (unlike Carroll (2015)). The moral hazard problem with these features is analyzed recently by Antic (2015), and the optimal contract in this case is neither standard risky debt nor standard outside equity. In contrast, standard risky debt and standard outside equity arise in non-extreme versions of the model based on asymmetric information - as we saw, standard outside equity is issued by some types whenever the investor contemplates that the project has a negative NPV, and standard risky debt arises is issued by some types whenever uncertainty is not too high. That is, the specialness of these securities in our model does not rely on Knightian uncertainty being extremely high or absent.

The second conceptual difference concerns the importance of the nature of private information. Our model implies a big difference between the case when private information of the issuer and uncertainty of the investor concern cash flows of the new project with the case when they concern existing assets in place. In particular, standard outside equity arises in equilibrium in the former case, but never in the latter case, no matter how high Knightian uncertainty is. In contrast, moral hazard models do not imply this difference.

8 Conclusion

The objective of the paper is to analyze the classical problem of optimal financing under asymmetric information when the investor is uncertain about the cash flow distribution in the Knightian, rather than Bayesian, sense. The investor only knows that the cash flow distribution is within a neighborhood of a certain base distribution and demands robustness evaluating any security by the worst-case distribution at which the investor could justify the issuer offering that security.

Our analysis generates two insights. First, the model rationalizes two most common financial contracts, standard outside equity and standard risky debt, as usual equilibrium outcomes. Outside equity is special because it serves as a very credible signal that the project is "good enough", because the investor and the issuer both hold security with the same shape. In contrast, any other security only sends the message that the residual security (i.e., the one kept by the issuer) is "good enough". Standard risky debt is special, because it gives the lowest possible sensitivity of the payoff to the cash flow, which is valued by a cautious investor. While there are many models providing foundations for standard risky debt, rationalizing outside equity has been more difficult, and it usually relies on very different arguments than models rationalizing debt. For example, Fluck (1998) and Myers (2000) rationalize outside equity as the optimal relational contract between insiders and outside investors. In our model, both securities are rationalized with one simple market imperfection, private information of the issuer, provided that the investor faces uncertainty in the Knightian sense.

The second insight is to relate the equilibrium security (risky debt or outside equity) to economic environment. In our view, the most interesting implication is that outside equity arises in equilibrium only if the issuer's private information concerns a new project. In contrast, if the issuer's private information is about assets in place, then outside equity never arises in equilibrium, because it is a credible signal that the project is good enough, but not that the assets in place are good enough. Another implication is that when private information concerns a new project, outside equity arises in equilibrium when uncertainty is high, while risky debt arises when it is low. While refined empirical tests need to be conducted, at first glance these implications appear to be in line with the existing empirical evidence on the validity of the pecking order theory.

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A Appendix

A.1 Proofs for Section 3

Lemma 1 is proven in the Online Appendix.

Before proving Proposition 1, we first prove the following two auxiliary lemmas. The first lemma constructs "the worst distribution" in set B.

Lemma 3. *Let $\underline{f}$ be given by $\underline{f}_N = \max\{g_N - \nu, 0\}$, $\underline{f}_n = \max\left\{g_n - \max\{0, \nu - \sum_{m=n+1}^{N} g_m\}, 0\right\}$ for all $n = 1, \ldots, N-1$, and $\underline{f}_0 = \min\{g_0 + \nu, 1\}$. Then, $\underline{f} \in \arg\min_{f \in B} \mathbb{E}_f[s]$ for any $s \in \mathcal{S}$. Further, if $B_+ = B$, then for any $s \in \mathcal{S}$ such that $P(s) = I$, it holds $P(s) = \mathbb{E}_{\underline{f}}[s]$.*

Proof. Consider the following problem:

$$\min_f \sum_{n=1}^{N} f_n z_n,$$

$$\text{s. to} \quad f_n \in [0,1] \; \forall n \in \{0, ..., N\}, \quad \sum_{n=0}^{N} f_n = 1, \quad \sum_{n=0}^{N} |f_n - g_n| \leq 2\nu.$$

Since z_n is strictly increasing in n, it is optimal to shift distribution from the highest possible to the lowest possible states. If $g_0 + \nu \geq 1$, then the problem is solved by $f_0 = 1$, and $f_n = 0 \, \forall n \neq 0$. If $g_0 + \nu < 1$, then $f_0 = g_0 + \nu$ and the distribution mass $\nu$is taken from the highest possible states in the pecking order: $f_N = \max\{g_N - \nu, 0\}$, $f_{N-1} = g_{N-1} - \max\{0, \nu - g_N\}$, $f_{N-2} = g_{N-2} - \max\{0, \nu - g_N - g_{N-1}\}$, etc. Combining both cases implies that $\underline{f}$ solves this problem.

By construction, distribution $\underline{f}$ first-order stochastically dominates any $f \in B$. Hence, for any $s \in \mathcal{S}$, $\mathbb{E}_f[s] \geq \mathbb{E}_{\underline{f}}[s]$, which proves the first statement of the lemma. The first statement in conjunction with Pricing Lemma gives the second statement of the lemma. $\square$

The next lemma shows that Pricing Lemma can be strengthened as follows.

Lemma 4. *Suppose $B_+ \subset B$. Then, for any security $s \in \mathcal{S}$ such that $P(s) = I$,*

$$P(s) = \min_{f \in B_0} \mathbb{E}_f[s]. \tag{15}$$

The argument for Lemma 4 is particularly simple if $N = 2$. By Pricing Lemma, $f^*(s) \in \arg\min_{f \in B_+} \mathbb{E}_f[s]$. In Figure 2, the slope of the south-west boundary of B_+ (the dashed region) is either 0, -1, or below -1, where the latter corresponds to the part of the boundary of B_+ that coincides with B_0. The iso-line for $f_1 s_1 + f_2 s_2$ has slope in $[-1, 0]$, and so the minimum of $\mathbb{E}_f[s]$ over $f \in B_+$ is attained at some point in B_0.

Proof of Lemma 4. By contradiction, suppose that there exists security $s \in \mathcal{S} :\ P(s) < \min_{f \in B_0} \mathbb{E}_f[s]$. This inequality and Pricing Lemma imply that there exists $h \in B_+ \backslash B_0 :\ \mathbb{E}_h[s] < \min_{f \in B_0} \mathbb{E}_f[s]$. For any $\alpha \in [0,1]$, define distribution $h(\alpha) \equiv \alpha \underline{f} + (1-\alpha)h$. Note that $h(\alpha) \in B$ because $\underline{f} \in B$ and $h \in B$. By linearity, $\mathbb{E}_{h(\alpha)}[s] = \alpha \mathbb{E}_{\underline{f}}[s] + (1-\alpha)\mathbb{E}_h[s]$for any $\alpha \in [0,1]$. Since $\mathbb{E}_{\underline{f}}[s] \leq \mathbb{E}_h[s]$, $\mathbb{E}_{h(\alpha)}[s]$ is weakly decreasing in α. Since $h \in B_+ \backslash B_0$, $\underline{f} \in B_-$ (from $B_+ \subset B$ and Lemma 3), and $h(\cdot)$ is continuous in α, there exists $\tilde{\alpha} \in (0,1)$ for which $h(\tilde{\alpha}) \in B_0$. Thus, we found $h(\tilde{\alpha}) \in B_0$ such that $\mathbb{E}_{h(\tilde{\alpha})}[s] \leq \mathbb{E}_h[s]$, which contradicts $\mathbb{E}_h[s] < \min_{f \in B_0} \mathbb{E}_f[s]$. Therefore, $P(s) = \min_{f \in B_0} \mathbb{E}_f[s]$, which proves (15). □

We can now prove Proposition 1.

Proof of Proposition 1. By Corollary 1, types in B_- do not offer any security. Thus, we focus on determining equilibrium securities for types in B_+. We consider separately two cases.

Case 1: $B_+ \subset B$. Denote by $H_0 \equiv \{f \in \Delta(Z) : \mathbb{E}_f[z] = K\}$ the hyperplane of zero-NPV distributions, and by $H_+ \equiv \{f \in \Delta(Z) : \mathbb{E}_f[z] \geq K\}$ the half-space of non-negative NPV distributions. Observe that it follows from the definition of set B in equation (1) that

$$B = \left(\bigcap_{I \in 2^N} \left\{ f \in \Delta(z) : \sum_{n \in I} f_n \leq \sum_{n \in I} g_n + \nu \right\} \right) \bigcap \left(\bigcap_{I \in 2^N} \left\{ f \in \Delta(z) : \sum_{n \in I} f_n \geq \sum_{n \in I} g_n - \nu \right\} \right). \tag{16}$$

Further, $B_0 = B \cap \{f \in \Delta(Z) : \mathbb{E}_f[z] = K\}$. Thus, B is a convex polyhedron in $\Delta(Z)$, and B_0 is a convex polyhedron in H_0.

For any $f \in \Delta(Z)$ (i.e., including but not limiting to B), let $S^*(f)$ define the set of securities that solve program (8). We will show that the set of types in H_+ for whom $S^*(f)$ is not a singleton has a dimension of less than or equal to $N-1$, which implies that it has Lebesgue measure zero.[24] Since $B_+ \subseteq H_+$, this will imply that the set of types in B_+ for whom $S^*(f)$ is not a singleton also has a dimension of less than or equal to $N-1$, which will prove the proposition.

Observe that if for some type $f \in H_0$, $S^*(f)$ is a singleton, then for any $\gamma \geq 1$ such that $\gamma f \in H_+$, $S^*(\gamma f) = S^*(f)$, and hence, $S^*(\gamma f)$ is also a singleton. This follows from the fact that for any s and s' in $\mathcal{S}$, $\mathbb{E}_f[s] \geq (>) \mathbb{E}_f[s']$ if and only if $\mathbb{E}_{f'}[s] \geq (>) \mathbb{E}_{f'}[s']$). Since set H_0 is $N-1$ dimensional, it is sufficient to show the following claim:

Claim 1. *The set of types $f \in H_0$, for which $S^*(f)$ is not a singleton, has dimension less than or equal to $N-2$.*

Next, we prove Claim 1. Since B_0 is a convex polyhedron (of dimension $N-1$) in H_0, it has a finite number of extreme points $E \equiv \{h^1, \ldots, h^I\} \subset B_0$. Since the minimum of a linear function on a convex polyhedron is attained at extreme points, we have by Lemma 4 that for any $s \in \mathcal{S}$ such

[24]Recall the dimensionality of $\Delta(Z)$ is N.

that $P(s) = I$, it holds

$$P(s) = \min_{f \in B_0} \mathbb{E}_f[s] = \min_{i \in \{1,\ldots,I\}} \mathbb{E}_{h^i}[s] \tag{17}$$

This implies that the program (8) for finding $S^*(f)$ can be equivalently rewritten as follows:

$$\min_{s \in \mathcal{S}} \left\{ \begin{array}{l} \mathbb{E}_f[s] - \min\limits_{i \in \{1,\ldots,I\}} \mathbb{E}_{h^i}[s] \\ \text{s.t.} \quad \min\limits_{i \in \{1,\ldots,I\}} \mathbb{E}_{h^i}[s] = I \end{array} \right\} = \min_{s \in \mathcal{S}} \left\{ \begin{array}{l} \mathbb{E}_f[s] \\ \text{s.t.} \quad \min\limits_{i \in \{1,\ldots,I\}} \mathbb{E}_{h^i}[s] = I \end{array} \right\}.$$

Further, if we replace equality $\min_{i \in \{1,\ldots,I\}} \mathbb{E}_{h^i}[s] = I$ with inequality $\min_{i \in \{1,\ldots,I\}} \mathbb{E}_{h^i}[s] \geq I$, the solution to the problem will not change.

Let us recursively construct the following sets of securities $\mathcal{S}^i$, $i \in \{1, \ldots I\}$. For $i = 1, \ldots, I$, let $\mathcal{S}^i$ be the set of all securities in $\{s \in \mathcal{S} : P(s) \geq I\}$ such that $h_i \in \arg\min_{j \in \{1,\ldots,I\}} \mathbb{E}_{h^j}[s]$. Each set $\mathcal{S}^i$ is a convex polyhedron defined by the finite set of inequalities, which consists of a finite set of inequalities defining set $\mathcal{S}$, inequalities $\mathbb{E}_{h^i}[s] \leq \mathbb{E}_{h^j}[s]$, $j \in \{1, \ldots, I\} \backslash i$, and $\mathbb{E}_{h^i}[s] \geq I$. Hence, set $\mathcal{S}^i$ has a finite number of extreme points $\bar{S}^i \equiv \{s^1, \ldots, s^{J_i}\}$.

Fix $f \in H_0 \backslash E$. We can equivalently find $S^*(f)$ in two steps as follows. In the first step, for each $i = 1, \ldots, I$, we find

$$S^i(f) \equiv \arg\min_{s \in \mathcal{S}^i} \{\mathbb{E}_f[s] - \mathbb{E}_{h^i}[s]\}. \tag{18}$$

The minimized function in (18) is linear in s and $\mathcal{S}^i$ is defined by the finite number of inequalities. Therefore, by $f \notin E$, the solution to (18) is attained at one of extreme points $\bar{S}^i$ of $\mathcal{S}^i$. In the second step, the issuer type f chooses the set of indices $I(f) \subseteq \{1, \ldots, I\}$ that minimize (18). Then,

$$S^*(f) = \cup_{i \in I(f)} \mathrm{conv}(S^i(f)),$$

where $\mathrm{conv}(\cdot)$ denotes the convex hull of the set.

Now, we can show that the set of $f \in H_0$, for which $S^*(f)$ is not a singleton, is of dimension less than or equal to $N-2$. Since E is a finite set, we focus on $f \in H_0 \backslash E$. If there are different securities s and s' that are both in $\cup_{i \in I(f)} S^i(f)$, then type f is indifferent between them. Hence, $\mathbb{E}_f[s - s'] = 0$, and in particular, vector $s - s'$ has some negative elements. Thus, z is not proportional to $s - s'$, and so, the set of types in H_0 who are indifferent between securities s and s' has dimension $N-2$. Since s and s' belong to $\cup_{i \in \{1,\ldots,I\}} \bar{S}^i$, the set of all types in H_0, who are indifferent between issuing several securities, is a finite union of sets of dimension $N-2$. Thus, it itself has dimension at most $N-2$, which is the desired conclusion.

Case 2: $B_+ = B$. Denote by $\tilde{H}_0 \equiv \{f \in \Delta(Z) : \mathbb{E}_f[z] = \min_{f \in B} \mathbb{E}_f[z]\}$ the hyperplane of distributions with NPVs equal to the minimal NPV in set B, and by $\tilde{H}_+ \equiv \{f \in \Delta(Z) : \mathbb{E}_f[z] \geq \min_{f \in B} \mathbb{E}_f[z]\}$ the half-space of distributions with NPVs greater than or equal to the minimal NPV in set B. As in Case 1, for any $f \in \Delta(Z)$ (i.e., including but not limiting to B), let $S^*(f)$ define the set of securities that solve program (8). By the analogous argument as in Case 1, we can show that the set of types in $\tilde{H}_+$ for whom $S^*(f)$ is not a singleton has a dimension of less than or equal to $N-1$, which implies that it has Lebesgue measure zero.[25] Since $B = B_+ \subseteq H_+ \subseteq \tilde{H}_+$, this implies that the set of types in B for whom $S^*(f)$ is not a singleton also has a dimension of less than or equal to $N-1$, which proves the proposition. $\square$

A.2 Proofs for Section 4

In order to prove Theorem 1, we start with the following lemma, which shows that the set of types that issue the same security is a cone.

Lemma 5. *The following hold:*

1. *If $\mathbb{E}_f[s] \le \mathbb{E}_f[\tilde{s}]$ ($\mathbb{E}_f[s] < \mathbb{E}_f[\tilde{s}]$), then for any $\gamma > 0$ such that $\gamma f \in \Delta(Z)$, $\mathbb{E}_{\gamma f}[s] \le \mathbb{E}_{\gamma f}[\tilde{s}]$ (respectively, $\mathbb{E}_{\gamma f}[s] < \mathbb{E}_{\gamma f}[\tilde{s}]$).*
2. *If $\mathbb{E}_f[s] \le \mathbb{E}_f[\tilde{s}]$ and $\mathbb{E}_{f'}[s] \le \mathbb{E}_{f'}[\tilde{s}]$ ($\mathbb{E}_f[s] < \mathbb{E}_f[\tilde{s}]$ and $\mathbb{E}_{f'}[s] < \mathbb{E}_{f'}[\tilde{s}]$), then for any $\alpha \in [0,1]$, $\mathbb{E}_{f''}[s] \le \mathbb{E}_{f''}[\tilde{s}]$ ($\mathbb{E}_{f''}[s] < \mathbb{E}_{f''}[\tilde{s}]$), where $f'' = \alpha f + (1-\alpha) f'$.*
3. *For any $s \in \mathcal{S}$ that can be issued on- or off-path, $B(s) = B_+ \cap C(s)$ where $C(s)$ is a convex cone.*

Proof. 1) Since $s_0 = \tilde{s}_0 = 0$, $\mathbb{E}_f[s-\tilde{s}] = \sum_{n=1}^N f_n(s_n - \tilde{s}_n)$, and so, for any $\gamma > 0$, $\mathbb{E}_f[s-\tilde{s}] \le (<)0$ if and only if $\mathbb{E}_{\gamma f}[s-\tilde{s}] \le (<)0$.

2) $\mathbb{E}_{f''}[s-\tilde{s}] = \mathbb{E}_{\alpha f}[s-\tilde{s}] + \mathbb{E}_{(1-\alpha)f'}[s-\tilde{s}] = \alpha\mathbb{E}_f[s-\tilde{s}] + (1-\alpha)\mathbb{E}_{f'}[s-\tilde{s}] \le (<)0$ whenever $\mathbb{E}_f[s-\tilde{s}] \le (<)0$ and $\mathbb{E}_{f'}[s-\tilde{s}] \le (<)0$.

3) The first two statements imply that the set of types who weakly prefer to issue s to any $s^* \in S^*$ is a convex cone, which we denote by $C(s)$, and so, $B(s) = C(s) \cap B_+$ by the definition of $B(s)$. $\square$

Proof of Theorem 1.

Proof of Part 1: Let $\hat{\mathcal{E}} \equiv \{f \in B : s^*(f) = (I/K)z\}$, and let $\hat{\mathcal{C}}$ be the convex cone generated by vectors in $\hat{\mathcal{E}}$. Denote by $H_0 \equiv \{f \in \Delta(Z) : \mathbb{E}_f[z] = K\}$ the zero-NPV hyperplane. Recall that we defined in the main text $\mathcal{C}$ to be the convex cone generated by vectors in the relative interior of B_0, and $\mathcal{E} = \mathcal{C} \cap B$. By part 1 of Lemma 5, to prove that $\hat{\mathcal{E}} = \mathcal{E}$ it is sufficient to show that $\hat{\mathcal{C}} \cap H_0 = \mathcal{C} \cap H_0$.

We first show that $\hat{\mathcal{C}} \cap H_0 \supseteq \mathcal{C} \cap H_0$. By Pricing Lemma, for type $f \in \mathcal{C} \cap H_0$, the mispricing from issuing equity $s = (I/K)z$ equals $\mathbb{E}_f[s] - I = 0$. Again, by Pricing Lemma, the mispricing

[25] Indeed, since set $\tilde{H}_0$ is $N-1$ dimensional, it is sufficient to show that the set of types $f \in \tilde{H}_0$, for which $S^*(f)$ is not a singleton, has dimension less than or equal to $N-2$. This, in turn, follows from the argument in Claim 1 if we replace there B_0 with $\{\underline{f}\}$.

from any other security s with $P(s) = I$ equals $\mathbb{E}_f[s] - \min_{f \in B_+} \mathbb{E}_f[s] \geq 0$. Moreover, since f is in the relative interior of B_0 (by the construction of cone $\mathcal{C}$) and vector $(s_1, \ldots, s_N)$ is not co-linear to vector $(z_1, \ldots, z_N)$, $\mathbb{E}_f[s] - \min_{f \in B_+} \mathbb{E}_f[s] > 0$. Therefore, any type $f \in \mathcal{C} \cap H_0$ strictly prefers to issue equity to any other security, and so, we have shown that $\hat{\mathcal{C}} \cap H_0 \supseteq \mathcal{C} \cap H_0$.

We next show that $\mathcal{C} \cap H_0 \supseteq \hat{\mathcal{C}} \cap H_0$. Suppose to contradiction that there is $f \in (\hat{\mathcal{C}} \cap H_0) \backslash (\mathcal{C} \cap H_0)$ that issues equity $s = (I/K)z$ in any equilibrium. Then, there is $f \in \partial B_0$ that strictly prefers to offer equity $s = (I/K)z$ to any other security in any equilibrium, where ∂B_0 denotes the boundary of set B_0. To obtain a contradiction, we construct another security that issuer type f weakly prefers to equity.

Since $f \in B$, by equation (16) distribution f satisfies the following inequalities:

$$\sum_{n \in I} f_n \leq \sum_{n \in I} g_n + \nu \text{ for some } I \in 2^{\{1,\ldots,N\}};$$
$$\sum_{n \in I} f_n \geq \sum_{n \in I} g_n - \nu \text{ for some } I \in 2^{\{1,\ldots,N\}}.$$

Equivalently, these inequalities can be written as[26]

$$\eta \cdot (f - g) \leq \nu \text{ for some } \eta \in \mathbb{R}^n \text{ with coordinates 0 or 1}$$
$$\eta \cdot (f - g) \leq \nu \text{ for some } \eta \in \mathbb{R}^n \text{ with coordinates 0 or -1}$$

Since $f \in \partial B_0$, f satisfies exactly J of such inequalities as equalities and the rest as strict inequalities. We denote corresponding norm vectors η of binding inequalities by $\eta^j, j = 1, \ldots, J$. Note that the norm vector of the hyperplane $\{f \in \Delta(Z) : \mathbb{E}_f[z] \geq K\}$ is z. Let

$$\bar{s} \equiv \left(1 - \sum_{j=1}^{J} \varepsilon_j\right)(I/K)z + \sum_{j=1}^{J} \varepsilon_j \eta^j,$$

and consider security

$$s' \equiv \frac{I}{\bar{s} \cdot f} \bar{s}.$$

Since equity security $s = (I/K)z$ belongs to the interior of $\mathcal{S}$, we can choose $\varepsilon^j, j = 1, \ldots, \eta^j$ sufficiently small so that $s' \in \mathcal{S}$. By construction of s', $\mathbb{E}_f[s'] = I$ and s' belongs to the convex cone generated by vectors $\eta_1, \ldots, \eta_J$, and z. Thus, $\min_{h \in B_+} \mathbb{E}_h[s]$ is attained at distribution f. Hence, there is no mispricing for type f from issuing security s', and so, type f is indifferent between offering equity $s = (I/K)z$ and security s'. This gives the desired contradiction, and so, $\mathcal{C} \cap H_0 \supseteq \hat{\mathcal{C}} \cap H_0$.

To summarize, we showed that $\hat{\mathcal{C}} \cap H_0 = \mathcal{C} \cap H_0$, and by the argument above, $\mathcal{E} = \hat{\mathcal{E}}$.

Proof of Part 2: Fix $K > 0$. As ν increases, set B expands, and hence, set $B_0 = B \cap H_0$ also

[26] Here and further, $x \cdot y \equiv \sum_{n=1}^{N} x_i y_i$ denotes the dot product of vectors x and y in $\mathbb{R}^N$.

expands. Thus, set $\mathcal{C}$ expands. Further, this expansion is continuous in ν. As ν increases, the set B_0 converges in the Hausdorff metric to the set $\{f \in \Delta(Z) : \mathbb{E}_f[z] = K\}$, and so, cone $\mathcal{C}$ converges to $\{f \in \Delta(Z) : \mathbb{E}_f[z] \geq K\}$. Together with the fact that B_+ also converges to $\{f \in \Delta(Z) : \mathbb{E}_f[z] \geq K\}$, this implies the limit $\lim_{\nu \to \overline{\nu}} \Lambda(\mathcal{E})/\Lambda(B_+) = 1$.

Proof of Part 3: Fix $\nu > 0$. Observe that sets B_0, B_+, $\mathcal{E}$, and $\mathcal{C}$ depend on K. In this proof, we stress this dependence on K with superscript K. In order to show that set $B_+^K \backslash \mathcal{C}^K$ continuously shrinks as K increases, we will show that if some type switches to issuing equity in equilibrium for some K, then he continues to issue equity for all larger K until the NPV of his project becomes negative. Formally, we show that

Claim 2. *For any $f \in B^K$, if $f \in \mathcal{E}^{\tilde{K}}$ for some $\tilde{K}$, then $f \in \mathcal{E}^K$ for all $K \in [\tilde{K}, \mathbb{E}_f[z])$.*

Proof: Fix $f \in B$ and let $f^\gamma \equiv \gamma f$ for $\gamma \in (0, 1]$. Suppose that for some $\tilde{K}$, $f \in \mathcal{E}^{\tilde{K}}$. This implies that there is $\underline{\gamma} \leq 1$ such that $f^{\underline{\gamma}}$ belongs to the relative interior of $B_0^{\tilde{K}}$, which, in turn, implies that $f^{\underline{\gamma}}$ belongs to the interior of B. By the convexity of set B, for all $\gamma \in [\underline{\gamma}, 1)$, f^γ belongs to the interior of B. For any $K \in [\tilde{K}, \mathbb{E}_f[z])$, let $\gamma(K)$ be such that $f^{\gamma(K)} \in B_0^K$. Then, $\gamma(K) \in [\underline{\gamma}, 1)$. Since $f^{\gamma(K)}$ belongs to the interior of B, $f^{\gamma(K)}$ belongs to the relative interior of B_0^K. By the first statement in Theorem 1, $f \in \mathcal{E}^K$, which is the desired conclusion. *q.e.d.*

We next show that there is $\underline{K} < \overline{K}$ such that $\mathcal{E}^K = B_+^K$ for all $K \in [\underline{K}, \overline{K}]$. Let $\overline{f}$ be given by $\overline{f}_0 = \max\{g_0 - \nu, 0\}$, $\overline{f}_n = g_n - \max\{0, \nu - \sum_{m=0}^{n-1} g_m\}$ for all $n = 1, \ldots, N-1$, and $\overline{f}_N = \min\{g_N + \nu, 1\}$. By the argument analogous to the proof of Lemma 3, $\overline{f}$ is also the unique solution to $\max_{f \in B} \mathbb{E}_f[z]$. Let distribution $\overline{f}^\gamma \equiv \gamma \overline{f}$ for $\gamma < 1$. We first prove the following auxiliary claim:

Claim 3. *There is $\tilde{\gamma}$ such that $\overline{f}^\gamma$ belongs to the interior of set B for $\gamma \in (\tilde{\gamma}, 1)$.*

Proof: By construction of $\overline{f}$ and the fact that g belongs to the interior of B, there is $M \in \{1, \ldots, N-1\}$ such that $g_n - \overline{f}_n > 0$ for all $n = 0, \ldots, M-1$, $g_n - \overline{f}_n = 0$ for all $n = M, \ldots, N-1$, and $\overline{f}_N - g_N > 0$. Hence,

$$\sum_{n=0}^{N} |\overline{f}_n - g_n| = 1 - \sum_{n=1}^{N} g_n - \left(1 - \sum_{n=1}^{N} \overline{f}_n\right) + \sum_{n=1}^{N-1} (g_n - \overline{f}_n) + \overline{f}_N - g_N = 2(\overline{f}_N - g_N) \leq 2\nu,$$

where the inequality is by $\overline{f} \in B$. Suppose that γ is sufficiently close to 1 so that $g_0 - \overline{f}^\gamma > 0$ and $\overline{f}_N^\gamma - g_N > 0$. Then, $g_n - \gamma \overline{f}_n \geq 0$ for all $n = 1, \ldots, N-1$, and $\gamma \overline{f}_N - g_N > 0$. Hence,

$$\sum_{n=0}^{N} |\overline{f}_n^\gamma - g_n| = 1 - \sum_{n=1}^{N} g_n - \left(1 - \gamma \sum_{n=1}^{N} \overline{f}_n\right) + \sum_{n=1}^{N-1} (g_n - \gamma \overline{f}_n) + \gamma \overline{f}_N - g_N = 2(\gamma \overline{f}_N - g_N) < 2\nu.$$

This together with the fact that B is contained in the interior of simplex $\Delta(Z)$ implies that $\overline{f}^\gamma$ belongs to the interior of set B for γ sufficiently close to 1. Denoting the set of such γs by $(\tilde{\gamma}, 1)$, we get the desired conclusion. *q.e.d.*

Next, choose $\underline{K} \in (\mathbb{E}_{\overline{f}^\gamma}[z], \overline{K})$ such that for $K \geq \underline{K}$, the only extreme point of set B in the half-space $\{f \in \Delta(Z) : \mathbb{E}_f[z] \geq K\}$ is $\overline{f}$. Since B is a convex polyhedron, it has a finite number of extreme points, and so, we can indeed choose $\underline{K} < \overline{K}$. Then, for any $K > \underline{K}$, set B_+^K is a pyramid with base B_0^K and apex $\overline{f}$. Thus, any $f \in B_+^K \backslash \{B_0, \overline{f}\}$ is a non-trivial convex combination of some point in the base B_0^K and apex $\overline{f}$. Since $\overline{f}^\gamma$ belongs to the interior of B (by Claim 3), it also belongs to the relative interior of B_0^K. Hence, for any $f \in B_+^K \backslash \{B_0, \overline{f}\}$ there is $\alpha < 1$ such that αf belongs to the relative interior of B_0^K. Thus, by the first statement in Theorem 1, $f \in \mathcal{E}^K$, which completes the proof of the third statement. □

Proof of Proposition 2. Consider any type $f \in \mathcal{D}$ and any $s' \in S^*(f)$. Suppose to contradiction that s' is not a risky debt security, i.e., $s' \neq \min\{z, d\}$ for any $d \in (0, z_N)$. Consider the highest state $\bar{n}$ such that $s'_{\bar{n}} < s'_N$. Since s' is not a risky debt security, $s'_{\bar{n}} < z_{\bar{n}}$. Let us construct the following modification s'' of security $\tilde{s}$: $s''_n = s'_n$ for all $n < \bar{n}$; $s''_{\bar{n}} = s'_{\bar{n}} + \varepsilon$; and $s''_n = s'_n - c\varepsilon$ for all $n > \bar{n}$. Next, we determine ε and c

There exists a finite number of extreme points of B_0, call them $h^1, \ldots, h^I$, such that $\min_{h \in B_0} \mathbb{E}_h[s'] = \mathbb{E}_{h^i}[s']$ for $i = 1, \ldots, I$. Define $c^i \equiv h^i_{\bar{n}}/(h^i_{\bar{n}+1} + \cdots + h^i_N)$ for all $i = 1, \ldots, I$. Since the mapping $\arg\min_{h \in B_0} \mathbb{E}_h[s]$ is upper-hemicontinuous in s, for sufficiently small ε, the worst-case justifiable model for s' and s'' is the same. Call it h^i. Then, we let $c = c^i$ and choose ε sufficiently small so that $s'' \in \mathcal{S}$.

By construction, $P(s'') = I$. The gain for issuer type f from issuing security s'' rather than security s' is

$$\mathbb{E}_f[s''] - \mathbb{E}_f[s'] = -\varepsilon f_{\bar{n}} + \varepsilon c^i (f_{\bar{n}+1} + \cdots + f_N),$$

which is positive if and only if

$$\frac{f_{\bar{n}+1}}{f_{\bar{n}}} + \cdots + \frac{f_N}{f_{\bar{n}}} > \frac{h^i_{\bar{n}+1}}{h^i_{\bar{n}}} + \cdots + \frac{h^i_N}{h^i_{\bar{n}}}.$$

This inequality indeed holds, because $f \succ_{LRD} h^i$. Therefore, type f is strictly better off issuing security s'', which is a contradiction. Thus, s' is necessarily risky debt. □

Proof of Proposition 3. We start with the proof of the following claim that determines the worst-case justifiable model for each security that makes the investor break even.

Claim 4. *Suppose $N = 2$ and $B_+ \subset B$. Then, for any $s \in \mathcal{S}$ such that $P(s) = I$,*

$$f^*(s) = \begin{cases} \psi & \text{if } s_1 > \frac{z_1}{z_2} s_2, \\ \phi & \text{if } s_1 < \frac{z_1}{z_2} s_2, \\ \alpha\psi + (1-\alpha)\phi, \forall \alpha \in (0,1) & \text{if } s_1 = \frac{z_1}{z_2} s_2; \end{cases} \tag{19}$$

where $\psi \equiv \arg\min_{f\in B}\{f_1$ *s.t.* $f_1z_1+f_2z_2=K\}$ *and* $\phi \equiv \arg\max_{f\in B}\{f_1$ *s.t.* $f_1z_1+f_2z_2=K\}$. *Further,* $P(s)=\mathbb{E}_\psi[s]$ *if* $s_1 \geq \frac{z_1}{z_2}s_2$, *and* $P(s)=\mathbb{E}_\phi[s]$ *if* $s_1 \leq \frac{z_1}{z_2}s_2$.

Proof: By Lemma 4, for any $s\in\mathcal{S}$ such that $P(s)=I$, there exists $f\in B_0$ such that $P(s)=\mathbb{E}_f[s]$. That is, it is without loss of generality to restrict $f(s)\in B_0$. Let us solve the Nature's program

$$\begin{aligned}&\min_{f\in B} f_1s_1+f_2s_2\\ &\text{s.t. } f_1z_1+f_2z_2=K.\end{aligned}$$

Using the constraint, we express $f_2=\frac{K}{z_2}-f_1\frac{z_1}{z_2}$ and plug this expression for f_2 into the minimized function, we get

$$f_1s_1+(K-f_1z_1)\frac{s_2}{z_2}=K\frac{s_2}{z_2}+f_1\left(s_1-z_1\frac{s_2}{z_2}\right).$$

The solution has a bang-bang property: $s_1>z_1\frac{s_2}{z_2}$ implies $f_1=\phi_1$ and $s_1<z_1\frac{s_2}{z_2}$ implies $f_1=\psi_1$. When $s_1=z_1\frac{s_2}{z_2}$, the solution is the segment connecting ϕ and ψ. This completes the proof of the claim. *q.e.d.*

Consider the problem of the issuer of type f who chooses s so that $\mathbb{E}_{f^*(s)}[s]=I$ to minimize the mispricing. First, suppose s is concave. Then by Lemma 4, $f^*(s)=\psi$ and the mispricing equals

$$\begin{aligned}\mathbb{E}_f[s]-\mathbb{E}_\psi[s] &= (f_1-\psi_1)s_1+(f_2-\psi_2)s_2\\ &= (f_1-\psi_1)s_1+(\tfrac{f_2}{\psi_2}-1)(I-\psi_1s_1)\\ &= \left(\tfrac{f_1}{f_2}-\tfrac{\psi_1}{\psi_2}\right)f_2s_1+\left(\tfrac{f_2}{\psi_2}-1\right)I,\end{aligned}$$

and so, types $f\in B_+$ with $f_2/f_1>\psi_2/\psi_1$ prefer to issue debt to maximize s_1, while types $f\in B_+$ with $f_2/f_1<\psi_2/\psi_1$ prefer equity among all concave securities. Similarly, if s is convex, then by Lemma 4, $f^*(s)=\phi$ and the mispricing equals

$$\mathbb{E}_f[s]-\mathbb{E}_\phi[s]=\left(\tfrac{f_1}{f_2}-\tfrac{\phi_1}{\phi_2}\right)f_2s_1+\left(\tfrac{f_2}{\phi_2}-1\right)I,$$

and so, types $f\in B_+$ with $f_2/f_1<\phi_2/\phi_1$ prefer to issue the call option to minimize s_1, while types $f\in B_+$ with $f_2/f_1>\phi_2/\phi_1$ prefer equity among all convex securities. Therefore, types $f\in B_+$ with $f_2/f_1>\psi_2/\psi_1$ prefer to issue debt, types $f\in B_+$ with $f_2/f_1<\phi_2/\phi_1$ prefer to issue call option, and types $f\in B_+$ with $\psi_2/\psi_1>f_2/f_1>\phi_2/\phi_1$ prefer to issue equity. □

Proof of Theorem 2. The first statement in Theorem 2 follows from Lemma 3. The argument for the last statement in Theorem 2 is analogous to that in the proof of Proposition 2. In fact, it is simpler, because F^* is a singleton.

To prove the second statement in Theorem 2, suppose to contradiction that there is a set of types of positive Lebesgue measure that issue s such that $s_n \in (s_{n-1}, s_{n-1} + z_n - z_{n-1})$ for some $n \in \{1, \ldots, N\}$. Fix some such type f and $s = s^*(f)$. Consider the problem that the issue of type f solves:

$$\min_{s \in \mathcal{S}} \mathbb{E}_f[s] \text{ s.t. } \mathbb{E}_{\underline{f}}[s] = I.$$

Consider the following two modifications s' and s'' of security s. First, s' is given by $s'_m = s_n$ for $m = 1, \ldots, n-1$; $s'_n = s_n - \varepsilon$; and $s'_m = s_n + \varepsilon c$ for $m = n+1, \ldots, N$. Second, s'' is given by $s''_m = s_n$ for $m = 1, \ldots, n-1$; $s''_n = s_n + \varepsilon$; and $s'_m = s_n - \varepsilon c$ for $m = n+1, \ldots, N$. Constants $c \equiv \underline{f}_n/(\underline{f}_{n+1} + \cdots + \underline{f}_N)$ so that the investor, who evaluates both securities using worst-case justifiable model $\underline{f}$ (which does not vary with security by the first statement of the theorem), still breaks even for both s' and s''. Choose ε sufficiently small so that s' and s'' still belong to $\mathcal{S}$. This is possible because $s_n \in (s_{n-1}, s_{n-1} + z_n - z_{n-1})$. Then, the change in the expected utility of issuer type f from issuing s' instead of s equals

$$\varepsilon \left[f_n - (f_{n+1} + \cdots + f_N) \frac{\underline{f}_n}{\underline{f}_{n+1} + \cdots + \underline{f}_N} \right],$$

and it equals the minus of that when he issues s'' instead of s. Unless, $\underline{f}_n/(\underline{f}_{n+1} + \cdots + \underline{f}_N) = f_n/(f_{n+1} + \cdots + f_N)$, issuer type f strictly prefers to deviate to either s' or s''. However, $\underline{f}_n/(\underline{f}_{n+1} + \cdots + \underline{f}_N) = f_n/(f_{n+1} + \cdots + f_N)$ only holds for the zero Lebesgue measure of types, which contradicts our conjecture that a positive Lebesgue measure of types issues s in equilibrium. This completes the proof of the second statement of the theorem. $\square$

Proof of Proposition 4. By Lemma 1, we can consider only securities s such that $\underline{f}_1 s_1 + \underline{f}_2 s_2 = I$. The mispricing of type f from issuing s equals

$$\mathbb{E}_f[s] - \mathbb{E}_{\underline{f}}[s] = (f_1 - \underline{f}_1)s_1 + (f_2 - \underline{f}_2)s_2 = \left(\frac{f_1}{f_2} - \frac{\underline{f}_1}{\underline{f}_2}\right) f_2 s_1 + \left(\frac{f_2}{\underline{f}_2} - 1\right) I.$$

If $f_2/f_1 > \underline{f}_2/\underline{f}_1$, then type f prefers to issue s that maximizes s_1, and so, issues debt in equilibrium. If $f_2/f_1 < \underline{f}_2/\underline{f}_1$, then type f prefers to issue s that minimizes s_1, and so, issues call option. Since $\mathbb{E}_f[z - s] \geq \mathbb{E}_{\underline{f}}[z - s]$ for all $s \in \mathcal{S}$ and $\mathbb{E}_{\underline{f}}[z - s] = W$ for $s \in S^*$, $\mathbb{E}_f[z - s^*(f)] - W \geq 0$ for all $f \in B$. $\square$

B Online Appendix (Not for Publication)

B.1 Proofs for Section 3

Proof of Lemma 1. *Case 1: s such that set* (5) *is non-empty.* First, consider any s at which set (5) is non-empty. Hence, for any security γs with $\gamma \in (0, 1)$, set (5) is also non-empty. Thus,

$P(\gamma s) = \min_{f \in B} \mathbb{E}_f [\gamma s]$ subject to $\mathbb{E}_f [z - \gamma s] \geq U^* (f)$. By individual rationality of the issuer, $U^*(f) = \max \{W, \mathbb{E}_f [z] - \min_{\tilde{s} \in \mathcal{S}^*} [\tilde{s}]\}$. Therefore, $P(\gamma s)$ equivalently solves

$$\min_{f \in B} \mathbb{E}_f [\gamma s] \tag{20}$$

$$\text{s.t. } \mathbb{E}_f [z - \gamma s] \geq W, \tag{21}$$

$$\mathbb{E}_f [\gamma s] \leq \mathbb{E}_f [\tilde{s}], \forall \tilde{s} \in \mathcal{S}^*. \tag{22}$$

The objective function (20) is continuous in γ and f . The set of constraints, (21)-(22), is continuous in γ and compact for all γ. By the maximum theorem, $P(\gamma s)$ is continuous in γ. Since $P(s) > I$ and $P(0) = 0$, by continuity there exists $\gamma \in (0, 1)$ such that $P(\gamma s) = I$.

Case 2: s such that set (5) *is empty.* In this case, $B(s) = B$, so $P(s) = \min_{f \in B} \mathbb{E}_f [s]$. Consider set $\{f \in B : \mathbb{E}_f [z - \gamma s] \geq U^*(f)\}$ for a fixed s while varying γ. It is empty for $\gamma = 1$. It is non-empty for $\gamma = 0$, because $\{f \in B : \mathbb{E}_f [z] \geq K\} \neq \emptyset$. Therefore, by continuity of $\{f \in B : \mathbb{E}_f [z - \gamma s] \geq U^*(f)\}$ in γ, it is empty if and only if $\gamma > \gamma^*$ for some $\gamma^* \in (0, 1)$. Consider the valuation of security γs in the range $\gamma \in (\gamma^*, 1)$. In this case, $B(\gamma s) = B$, so $P(\gamma s) = \min_{f \in B} \mathbb{E}_f [\gamma s] = \gamma P(s)$, which is continuous in $\gamma \in (\gamma^*, 1)$. If $\gamma^* P(s) < 1$, then $\gamma = \frac{1}{P(s)} \in (\gamma^*, 1)$ yields $P(\gamma s) = I$. If $\gamma^* P(s) \geq 1$, then $P(\gamma^* s) = \min_{f \in B(\gamma^* s)} \mathbb{E}_f [s] \geq I$, and for any $\gamma \in (0, \gamma^*)$, $P(\gamma s) = \min_{f \in B(\gamma s)} \mathbb{E}_f [\gamma s]$ with $B(\gamma s)$ given by (5). Since set (5) is non-empty, the proof of case 1 applies. Hence, there exists $\gamma \in (0, \gamma^*)$ satisfying $P(\gamma s) = I$.

Given that $P(\gamma s) = I$ for $\gamma \in (0, 1)$ whenever $P(s) > I$, the individual rationality of the issuer and the investor imply that $P(s) = I$ for any $s \in \mathcal{S}^*$. □

Proof of Lemma 2. We first show that $f^*(s) \in B_+$. From $f^*(s) \in B(s)$ and $\{f \in B : \mathbb{E}_f [z - s] \geq U^*(f)\} \neq \emptyset$, it follows that $\mathbb{E}_{f^*(s)} [z - s] \geq W$. If $\mathbb{E}_{f^*(s)} [z] < K$, then $P(s) = \mathbb{E}_{f^*(s)} [s] \leq \mathbb{E}_{f^*(s)} [z] - W < K - W = I$, which contradicts $P(s) = I$. Thus, $f^*(s) \in B_+$. Now, suppose that for some $s \in \mathcal{S}$, it holds that $P(s) = I$, but $\mathbb{E}_{f^*(s)} [s] > \mathbb{E}_{\tilde{f}} [s]$ for some $\tilde{f} \in \arg \min_{f \in B_+} \mathbb{E}_f [s]$. Observe that $\tilde{f} \in \arg \min_{f \in B_+} \mathbb{E}_f [s]$ implies that type $\tilde{f}$ issues some security $\tilde{s} = s^*(\tilde{f}) \neq \mathbf{0}$ in equilibrium.[27] We will show that type $\tilde{f}$ prefers to deviate to s in this case. Since $P(s) = I$ and $P(\tilde{s}) = I$ (by Lemma 1), it is sufficient to show that the mispricing of s ($\mathbb{E}_{\tilde{f}} [s] - P(s)$) is smaller than the mispricing of $\tilde{s}$ ($\mathbb{E}_{\tilde{f}} [\tilde{s}] - P(\tilde{s})$). The mispricing from issuing security $\tilde{s}$ for type $\tilde{f}$ equals

$$\mathbb{E}_{\tilde{f}}[\tilde{s}] - P(\tilde{s}) \geq \min_{f \in B(\tilde{s})} \mathbb{E}_f[\tilde{s}] - P(\tilde{s}) = 0,$$

[27]Indeed, suppose it were not the case and type $\tilde{f}$ did not raise financing. For equity $s = \frac{I}{K} z$, $\mathbb{E}_f[z - s] \geq W$ if and only if $\mathbb{E}_f[s] \geq I$ and so such offer would be accepted by the investor. The expected utility of type $\tilde{f}$ from such security is $\mathbb{E}_f[z - s] - I + K = (1 - \frac{I}{K})(\mathbb{E}_f[z] - K) \geq 0$ and so, type $\tilde{f}$ weakly prefers to issue it, which is a contradiction.

where the inequality holds from $\tilde{f} \in B(\tilde{s})$ and the equality holds from the pricing of $P(\tilde{s})$. At the same time, the mispricing from issuing security s for type $\tilde{f}$ equals $\mathbb{E}_{\tilde{f}}[s]-P(s) < \mathbb{E}_{f^*(s)}[s]-P(s) = 0$. Since $P(s) = P(\tilde{s}) = I$, these two inequalities imply that $\mathbb{E}_{\tilde{f}}[s] < \mathbb{E}_{\tilde{f}}[\tilde{s}]$. Thus, type $\tilde{f}$ is better off deviating to issuing s, which contradicts the premise that type $\tilde{f}$ issues $\tilde{s}$ in equilibrium.

Next, we prove the second statement of the lemma. Observe that type $\tilde{f} \in \arg\min_{f \in B_+} \mathbb{E}_f[s]$ satisfies $\mathbb{E}_{\tilde{f}}[z-s] \geq U^*\left(\tilde{f}\right)$. This follows from $I = \mathbb{E}_{\tilde{f}}[s] \leq \mathbb{E}_{\tilde{f}}[\tilde{s}]$ for any $\tilde{s} \in \mathcal{S}^*$. Therefore $\tilde{f} \in B(s)$, so it cannot be that $P(s) > I$. By contradiction, suppose that $P(s) < I$. Then, there exists $f \in B_-$ for which $\mathbb{E}_f[s] < I$ and $\mathbb{E}_f[z-s] \geq W$. The first inequality follows from $P(s) < I$. The second inequality follows from $\{f \in B : \mathbb{E}_f[z-s] \geq U^*(f)\} \neq \emptyset$, because type $\tilde{f}$ belongs to it. Consider type $f' \in B_0$ for which $\mathbb{E}_{f'}[z-s] > \mathbb{E}_f[z-s]$. Because $f' \in B_+$ and $\min_{f \in B_+} \mathbb{E}_f[s] = I$, $\mathbb{E}_{f'}[s] \geq I$. At the same time, $\mathbb{E}_{f'}[z-s] > W$. Together, they imply $\mathbb{E}_{f'}[z] > K$, which contradicts $f' \in B_0$. Therefore, $P(s) = I$. □

B.2 Proofs for Section 5

***Proof of Theorem* 3.** We first show the following claim.

Claim 5. *$P(s) = \mathbb{E}_{\underline{\hat{f}}}[s]$ for any $s \in \mathcal{S}$.*

Proof: We first make two preliminary observations. First, by the analogous argument as in Lemma 1, one can show that for any s such that $P(s) > K$, there exists $\gamma \in (0,1)$ such that $P(\gamma s) = K$. In particular,

$$P(s) = K \text{ for any } s \in \mathcal{S}^*. \tag{23}$$

Second, by the analogous argument as in Lemma 3,

$$\underline{\hat{f}} \in \arg\min_{\hat{f} \in \hat{B}} \mathbb{E}_{\hat{f}}[s] \text{ for any } s \in \mathcal{S}. \tag{24}$$

Since $\eta > K$, type $\underline{\hat{f}}$ issues some security on the equilibrium path, because he does not suffer from mispricing. Denote this security by $s^\dagger$.

We now proceed to the proof of the claim. Fix an equilibrium. Suppose to contradiction that for some security s it holds that $\mathbb{E}_{\hat{f}(s)}[s] > \mathbb{E}_{\underline{\hat{f}}}[s]$. This together with (24) implies that $\hat{B}(s)$ does not contain $\underline{\hat{f}}$, and so, type $\underline{\hat{f}}$ strictly prefers issuing $s^\dagger$ to s. Then,

$$K = \mathbb{E}_{\underline{\hat{f}}}[s^\dagger] < \mathbb{E}_{\underline{\hat{f}}}[s] \leq \mathbb{E}_{\hat{f}(s)}[s] = P(s),$$

where the first equality is by (23) and the fact that type $\underline{\hat{f}}$ issues security $s^\dagger$ in equilibrium; the first inequality is by the fact that type $\underline{\hat{f}}$ strictly prefers issuing $s^\dagger$ to s; the second inequality is by (24); and the second equality is by the definition of $P(s)$. Thus, $P(s) > K$. If type $\underline{\hat{f}}$ deviates to s, he gets

$$\mathbb{E}_{\underline{\hat{f}}}[\hat{z}-s] > \mathbb{E}_{\underline{\hat{f}}}[\hat{z}] - \mathbb{E}_{\hat{f}(s)}[s] = \mathbb{E}_{\underline{\hat{f}}}[\hat{z}] - K,$$

while in equilibrium, type $\underline{\hat{f}}$ issues $s^{\dagger}$ and gets

$$\mathbb{E}_{\underline{\hat{f}}}[\hat{z}-s^{\dagger}] = \mathbb{E}_{\underline{\hat{f}}}[\hat{z}] - \min_{\hat{f}\in\hat{B}(s^{\dagger})} \mathbb{E}_{\hat{f}}[s^{\dagger}] = \mathbb{E}_{\underline{\hat{f}}}[\hat{z}] - K,$$

where the last equality is by (23). Thus, deviation to s is profitable for type $\underline{\hat{f}}$, which is a contradiction. Therefore, we conclude that $P(s) = \mathbb{E}_{\underline{\hat{f}}}[s]$ for any $s \in \mathcal{S}$. *q.e.d.*

Given Claim 5 and equality (23), for any $\hat{f} \in B$ such that $s^*(\hat{f}) \neq \mathbf{0}$, it is necessary that $s^*(\hat{f})$ solves

$$\begin{aligned}
&\arg\max_{s\in\mathcal{S}}\{\mathbb{E}_{\hat{f}}[\hat{z}-s] \text{ s.t. } \mathbb{E}_{\underline{\hat{f}}}[s]=K\}\\
=&\arg\min_{s\in\mathcal{S}}\{\mathbb{E}_{\hat{f}}[s] \text{ s.t. } \mathbb{E}_{\underline{\hat{f}}}[s]=K\}\\
=&\arg\min_{s\in\mathcal{S}}\{\mathbb{E}_{\hat{f}}[s]-\mathbb{E}_{\underline{\hat{f}}}[s] \text{ s.t. } \mathbb{E}_{\underline{\hat{f}}}[s]=K\},
\end{aligned}$$

which gives (10).

Observe that program (10) is the same as program (8) in the small uncertainty case (when $B_+ = B$, and so, $\min_{h\in B_+}\mathbb{E}_h[s] = \mathbb{E}_{\underline{f}}[s]$). Thus, we can follow the same argument as in Theorem 2 to prove (numbered) statements 1 and 2 in Theorem 3.

To prove (numbered) statement 3 of Theorem 3, define

$$V(K,\nu,\hat{f}) \equiv \max_{s\in\mathcal{S}}\{\mathbb{E}_{\hat{f}}[\hat{z}-s] - \mathbb{E}_f[z] \text{ s.t. } \mathbb{E}_{\underline{\hat{f}}}[s]=K\}.$$

Fix $\gamma > 1$ and consider type $\hat{f}' = \gamma\underline{\hat{f}}$ such that $\hat{f} \in \hat{B}$. By the Berge's maximum theorem, function V is continuous in K, ν, and $\hat{f}$. Since $V(\eta,\nu,\hat{f}') < 0$ (because the issuer raises $K=\eta$ from the investor that underprices the offered securities), for sufficiently large $K<\eta$, $V(K,\nu,\hat{f}')<0$. Since the inequality is strict and V is continuous in $\hat{f}$, $V(K,\nu,\hat{f})<0$ in some neighborhood of $\hat{f}'$ for sufficiently large $K<\eta$, which proves statement 3 of Theorem 3. □

B.3 Proofs for Section 6

Proofs for Subsection 6.2

Lemma 6. *1. For any s such that $P^{\omega}(s) > I$, there exists $\gamma \in (0,1)$ such that $P^{\omega}(\gamma s) = I$. In particular, $P^{\omega}(s) = I$ for any $s \in S^*$.*

2. Define $f^{\omega}(s) = \omega f(s) + (1-\omega)\overline{f}$. For any security $s \in \mathcal{S}$ such that $P^{\omega}(s) = I$, it holds

$f^\omega(s) \in \underset{f^\omega \in B^\omega_+}{\arg\min} \mathbb{E}_{f^\omega}[s]$ *and* $P^\omega(s) = \min_{f^\omega \in B^\omega_+} \mathbb{E}_{f^\omega}[s]$.

Proof. The first part follows by an identical argument to Lemma 1. To prove the second part, we follow the argument as in the proof of Pricing Lemma. We first show that $f^\omega(s) \in B^\omega_+$ whenever $P^\omega(s) = I$. By the definition of the model-updating mapping $B(\cdot)$, $f(s) \in B(s)$ implies $\mathbb{E}_{f(s)}[z-s] \geq W$, and so, $\mathbb{E}_{f^\omega(s)}[z-s] \geq W$. If $\mathbb{E}_{f^\omega(s)}[z] < K$, then

$$\begin{aligned} P^\omega(s) &= \mathbb{E}_{f^\omega(s)}[s] \\ &= \mathbb{E}_{f^\omega(s)}[z] - \mathbb{E}_{f^\omega(s)}[z-s] \\ &\leq \mathbb{E}_{f^\omega(s)}[z] - W \\ &= \mathbb{E}_{f^\omega(s)}[z] - K + I < I, \end{aligned}$$

which contradicts to $P^\omega(s) = I$. Thus, $f^\omega(s) \in B^\omega_+$.

Now, suppose that for some $s \in \mathcal{S}$, it holds that $P^\omega(s) = I$, but $P^\omega(s) = \mathbb{E}_{f^\omega(s)}[s] > \mathbb{E}_{\tilde{f}^\omega}[s]$ for some $\tilde{f}^\omega \in \arg\min_{f^\omega \in B^\omega_+} \mathbb{E}_{f^\omega}[s]$. Observe that $\tilde{f}^\omega \in \arg\min_{f^\omega \in B^\omega_+} \mathbb{E}_{f^\omega}[s]$ implies that corresponding issuer type $\tilde{f} = \frac{\tilde{f}^\omega - (1-\omega)\overline{f}}{\omega}$ issues some security $\tilde{s} = s^*(\tilde{f})$ in equilibrium. This implies that $\mathbb{E}_{\tilde{f}^\omega}[\tilde{s}] \leq \mathbb{E}_{\tilde{f}^\omega}[s]$ and so,

$$I = P^\omega(\tilde{s}) \leq \mathbb{E}_{\tilde{f}^\omega}[\tilde{s}] \leq \mathbb{E}_{\tilde{f}^\omega}[s] < \mathbb{E}_{f^\omega(s)}[s] = I,$$

which is a contradiction. □

Proofs for Subsection 6.3 The analysis of the relative entropy modification proceeds in a series of lemmas.

Lemma 7. *There is a maximal set* $[\underline{\alpha}, \overline{\alpha}] \subseteq [0,1]$ *such that* $s^\alpha \in \mathcal{S}$ *for all* $\alpha \in [\underline{\alpha}, \overline{\alpha}]$.

Proof. We need to show that if $s^{\underline{\alpha}}, s^{\overline{\alpha}} \in \mathcal{S}$, thenfv $s^\alpha \in \mathcal{S}$ for any $\alpha \in (\underline{\alpha}, \overline{\alpha})$ (the fact that the maximal set is closed follows from the continuity argument). Consider distribution $\tilde{\psi} = \{f : \mathbb{E}_f[s^{\underline{\alpha}}] = I\} \cap \{f : \mathbb{E}_f[s^{\overline{\alpha}}] = I\}$, and let $\tilde{s} \equiv \gamma s^{\underline{\alpha}} + (1-\gamma)s^{\overline{\alpha}} \in \mathcal{S}$, where $\gamma \in (0,1)$ is such that $\alpha = \gamma\underline{\alpha} + (1-\gamma)\overline{\alpha}$. Then, $\mathbb{E}_{\tilde{\psi}}[\tilde{s}] = I$. By the strict convexity of B and the definition of $\psi^{\underline{\alpha}}$ and $\psi^{\overline{\alpha}}$, $\tilde{\psi}$ does not belong to the interior of B. This implies that $\mathbb{E}_{\psi^\alpha}[\tilde{s}] > I$, and so, for some $\delta \in (0,1]$, $\mathbb{E}_{\psi^\alpha}[\delta\tilde{s}] = I$. Since $\tilde{s} \in \mathcal{S}$, $s^\alpha \equiv \delta\tilde{s} \in \mathcal{S}$, which is the desired security. □

Lemma 8. *For any* α *and* α' *in* $[\underline{\alpha}, \overline{\alpha}]$, $\mathbb{E}_f[s^\alpha] < \mathbb{E}_f[s^{\alpha'}]$ *for all* f *such that* $f = \gamma\psi^\alpha$ *for some* $\gamma \geq 1$.

Proof. It is sufficient to show that $\mathbb{E}_{\psi^\alpha}[s^\alpha] < \mathbb{E}_{\psi^\alpha}[s^{\alpha'}]$, as it is equivalent to $\mathbb{E}_f[s^\alpha] < \mathbb{E}_f[s^{\alpha'}]$. Using the definitions of ψ^α and s^α, and the strict convexity of B, we get

$$\mathbb{E}_{\psi^\alpha}[s^\alpha] = I = \mathbb{E}_{\psi^{\alpha'}}[s^{\alpha'}] = \min_{f \in B} \mathbb{E}_f[s^{\alpha'}] < \mathbb{E}_{\psi^\alpha}[s^{\alpha'}],$$

which is the desired conclusion. $\square$

Proof of Proposition 5. Lemma 8 implies that whenever all types $f = \gamma\psi^{\alpha}, \gamma \geq 1$ prefer to issue security s^{α} to any other security $s^{\alpha'}, \alpha' \in [\underline{\alpha}, \overline{\alpha}]$. The optimality of $s^{\overline{\alpha}}$ for types with $f_2/f_1 > \psi_2^{\overline{\alpha}}/\psi_1^{\overline{\alpha}}$ and call option $s^{\underline{\alpha}}$ for types with $f_2/f_1 < \psi_2^{\underline{\alpha}}/\psi_1^{\underline{\alpha}}$ follows from the same argument as in the proof of Proposition 4.

Finally, we show that $s^{\overline{\alpha}}$ is the risky debt (by the analogous argument, $s^{\underline{\alpha}}$ is the call option). Suppose to contradiction that security $s^{\overline{\alpha}}$ is not the risky debt, that is, $s_1^{\overline{\alpha}} < \min\{z_1, s_2^{\overline{\alpha}}\}$. This implies that there exists a security $s \in \mathcal{S}$ such that $s_1 > s_1^{\overline{\alpha}}$ and $s_2 < s_2^{\overline{\alpha}}$, and $\min_{f \in B} \mathbb{E}_f[s] > I$. This, in turn, implies that for $\alpha = s_1/s_2 > \overline{\alpha}$, there exists $s^{\alpha} \in \mathcal{S}$ such that $\mathbb{E}_{\psi^{\alpha}}[s^{\alpha}] = I$, which contradicts the definition of $\overline{\alpha}$. Therefore, $s_1^{\overline{\alpha}} = \min\{z_1, s_2^{\overline{\alpha}}\}$, and $s^{\overline{\alpha}}$ is the risky debt. $\square$

The next proposition characterizes the equilibria in the case of large uncertainty in the relative entropy modification of the model.

Proposition 6. *Consider the relative entropy modification of the model and s suppose $\mathbb{E}_f[z] \geq K$ for all $f \in B$. Then, in any equilibrium of the signaling game, the following hold:*

1. *Issuer types f with $f_2/f_1 \in (\phi_2/\phi_1, \psi_2/\psi_1)$ issue equity $s = (I/K)z$.*
2. *For issuer types f with $f_2/f_1 > \psi_2/\psi_1$, it holds*
   - (a) *if $\psi_2/\psi_1 \geq \psi_2^{\overline{\alpha}}/\psi_1^{\overline{\alpha}}$, then types f with $f_2/f_1 > \psi_2/\psi_1$ issue risky debt d such that $\mathbb{E}_{\psi}[\min\{z, d\}] = I$;*
   - (b) *if $\psi_2/\psi_1 < \psi_2^{\overline{\alpha}}/\psi_1^{\overline{\alpha}}$, then for any α such that $\psi_2^{\alpha}/\psi_1^{\alpha} \in (\psi_2/\psi_1, \psi_2^{\overline{\alpha}}/\psi_1^{\overline{\alpha}})$, types f with $f_2/f_1 = \psi_2^{\alpha}/\psi_1^{\alpha}$ issue s^{α}, and types f with $f_2/f_1 > \psi_2^{\overline{\alpha}}/\psi_1^{\overline{\alpha}}$ issue risky debt d such that $\mathbb{E}_{\psi^{\overline{\alpha}}}[\min\{z, d\}] = I$;*
3. *For issuer types f with $f_2/f_1 < \phi_2/\phi_1$, it holds*
   - (a) *if $\phi_2/\phi_1 \leq \psi_2^{\underline{\alpha}}/\psi_1^{\underline{\alpha}}$, then types f with $f_2/f_1 < \phi_2/\phi_1$ issue call option with strike k such that $\mathbb{E}_{\phi}[\max\{z - k, 0\}] = I$;*
   - (b) *if $\phi_2/\phi_1 > \psi_2^{\underline{\alpha}}/\psi_1^{\underline{\alpha}}$, then for any α such that $\psi_2^{\alpha}/\psi_1^{\alpha} \in (\psi_2^{\underline{\alpha}}/\psi_1^{\underline{\alpha}}, \phi_2/\phi_1)$, types f with $f_2/f_1 = \psi_2^{\alpha}/\psi_1^{\alpha}$ issue s^{α}, and types f with $f_2/f_1 < \psi_2^{\underline{\alpha}}/\psi_1^{\underline{\alpha}}$ issue call option with strike k such that $\mathbb{E}_{\zeta_{\psi^{\underline{\alpha}}}}[\max\{z - k, 0\}] = I$.*

Proof of Proposition 6. We define for any $\alpha \in [0, 1]$,

$$\tilde{\psi}^{\alpha} \equiv \arg\min_{f \in B_+} \{\alpha f_1 + f_2\},$$

and let $\tilde{s}^\alpha$ be the security such that $\mathbb{E}_{\tilde{\psi}^\alpha}[\tilde{s}^\alpha] = I$ and $\frac{\tilde{s}_1^\alpha}{\tilde{s}_2^\alpha} = \alpha$, whenever such a security exists. By the analogous argument to Lemmas 7 and 8, there is a maximal set $[\underline{\alpha}', \overline{\alpha}']$ such that $\tilde{s}^\alpha \in \mathcal{S}$ for all $\alpha \in [\underline{\alpha}', \overline{\alpha}']$, and types $\{\gamma\tilde{\psi}^\alpha : \gamma \geq 1\}$ prefer to issue $\tilde{s}^\alpha$ to any other security. Thus, to find the equilibrium security offer for each type f, we need to determine the direction $\tilde{\psi}^\alpha$ of the line connecting 0 and f, and find corresponding $\tilde{s}^\alpha$. By the same argument as in Propositions 3 and 5, we can verify that the equilibrium offers $s^*(f)$ are as in the statement of the proposition. $\square$