

Auctions with Endogenous Initiation*

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Abstract

We study strategic initiation of auctions by potential buyers and the seller. This problem arises in auctions of companies and asset sales, among other contexts. A bidder's decision to approach the seller signals that this bidder is optimistic about the value of the asset. In common-value auctions, such as battles between financial bidders, this revelation effect disincentivizes bidders from approaching the seller. In pure common-value auctions, no bidder ever approaches and auctions are seller-initiated. By contrast, in private-value auctions, such as battles between strategic bidders, the effect is the opposite, and equilibria can feature both seller- and bidder-initiated auctions. The model generates testable implications about the link between the initiating party, bids, and sale outcomes. It also points to a role of shareholder activists as intermediaries in auctions of companies.

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1 Introduction

Over the last several decades, auction theory has developed into a highly influential field with many important practical results.¹ In particular, it has been extensively used to model transactions in the market for corporate control, such as mergers and acquisitions and intercorporate asset sales.² To focus on the insights about the auction stage, with rare exceptions, the literature examines a situation when the asset is already up for sale.

In some cases, exogeneity of a sale is an innocuous assumption. For example, the U.S. Treasury auctions off bonds at a known frequency. In many cases, however, the decision to put the asset for sale is a strategic one. For example, the board of directors of a firm (or an art collector) has a right but not an obligation to sell a division (or an art piece). In practice, an auction can be either bidder-initiated, when a potential bidder approaches the seller (e.g., the board of directors of the target company) expressing an interest, in which case the seller then decides to auction the asset off, or seller-initiated, when the seller decides to auction the asset off without being approached. To give a flavor of this heterogeneity, consider the following two deals in the M&A market. The acquisition of Taleo, a provider of cloud-based talent management solutions, by Oracle on February 9, 2012 for \$1.9 billion is an example of a bidder-initiated auction. In January 2011, a CEO of a publicly traded technology company, referred in the deal background as Party A, contacted Taleo expressing an interest in acquiring it. Following this contact, Taleo hired a financial adviser that conducted an auction, engaging four more bidders. Oracle was the winning bidder, ending up acquiring Taleo. By contrast, the acquisition of Blue Coat Systems, a provider of Web security, by a private equity firm Thoma Bravo on December 9, 2011 for \$1.1 billion is an example of a seller-initiated takeover auction. In early 2011, Elliot Associates, an activist hedge fund, amassed 9% ownership stake in Blue Coat and forced its board to auction the company. Twelve bidders participated in the auction, and Thoma Bravo was the winner. Overall, there exists a considerable heterogeneity with respect to the

¹The formal analysis of auctions goes back to Vickrey (1961). The overview of results on auction theory can be found, for example, in Krishna (2010).

²Among others, see Fishman (1988), Bulow, Huang, and Klemperer (1999), Hansen (2001), and Povel and Singh (2006). Dasgupta and Hansen (2007) provide a review of application of auction theory to corporate finance.

initiator of the contest, which does not appear to be random. For example, acquisitions by strategic buyers are more likely to be bidder-initiated, while acquisitions by private equity firms are more likely to be target-initiated (Fidrmuc et al., 2012).³

In this paper, we develop a theory of how potential buyers and the seller choose to initiate auctions. In particular, we ask the following questions: Which characteristics of auctions and the economic environment determine whether auctions are bidder- or seller- initiated? How do bidding strategies and auction outcomes differ depending on how the auction was initiated? What are the implied inefficiencies and what are the potential remedies, if any? We show that strategic incentives of potential buyers to approach the seller have first-order effects on the auction timing and outcomes and that they crucially depend on the valuation environment, i.e., whether bidder have private or common values.

To study these questions, we consider a dynamic framework, in which a seller owns an asset and faces two potential buyers. Each buyer has a signal about her valuation of the asset. Either potential buyer may leave the market following an exogenous event, in which case she is replaced by another buyer with a different signal. Thus, the valuations of potential buyers change over time. We do not assume that the auction takes place at an exogenous date and instead treat it as a strategic decision of the seller. Specifically, at any time each potential buyer decides whether to send a message to the seller indicating the interest in buying the asset. In turn, the seller decides whether to put his asset up for sale. Thus, the auction can be initiated by a bidder when the seller auctions the asset off after receiving an indication of interest, or by the seller when the seller auctions the asset off without being approached by a bidder. The benefit of waiting for the seller is that with some likelihood, a bidder with a high valuation will appear and approach the seller, resulting in a higher expected price. Conversely, the benefit of selling without being approached is the lack of delay. We focus on stationary cut-off equilibria, in which the distribution of signals, conditional on no auction in the past, stays the same over time, and each bidder communicates an indication of interest to the seller if her signal is sufficiently high.

The key driving force behind our results is that approaching the seller reveals that the

³Initiation is also related to characteristics of the seller and auction outcomes (Masulis and Simsir, 2013).

valuation of the initiating bidder is sufficiently high. In a bidder-initiated auction, ex-ante identical bidders become endogenously asymmetric at the auction stage: the signal of the initiating bidder is drawn from a more optimistic distribution. The other bidder uses this information to choose her bidding strategy and potentially re-value the asset. Similarly, the lack of an approaching bidder reveals information about valuations of all bidders: in a seller-initiated auction, each bidder knows that the valuation of the rival is sufficiently low, as she would have initiated the auction otherwise.

We show that the interplay between these information effects depends on the sources of bidders' valuations. In common-value auctions, e.g., when private equity firms compete to acquire a poorly-managed target, information effects discourage each bidder from approaching the seller. In pure common-value auctions, this effect is extreme: no bidder ever approaches, no matter how high her signal is. All auctions are initiated by the seller, if at all. By contrast, information effects work in the opposite direction in private-value auctions, e.g., when strategic bidders compete to purchase the asset they plan to integrate into their existing operations. Given the same signal, a bidder obtains a higher payoff in the auction she initiates than in an auction initiated by her rival, which gives her incentives to communicate an indication of interest to the seller.

The intuition behind these results is as follows. Consider a common-value setting: bidders have the same valuation of the asset but differ in their signals about it. Approaching the seller reveals information that the signal of the initiating bidder is sufficiently high: specifically, it is above a certain cut-off. In turn, observing that the auction is bidder-initiated, the rival bidder updates her valuation of the asset upwards. As a result, she bids aggressively not only because she competes against a strong bidder but also because of her own higher valuation. In a pure common-value setting, this logic implies that the initiating bidder with the lowest signal among those that lead to initiation wins only when the rival's signal is the lowest possible, in which case she pays the total value of the asset, obtaining no surplus. Such bidder would be better off waiting until either the rival bidder or the seller initiates the auction, as she would be able to get information rents then. Because the argument holds for any hypothetical equilibrium cut-off signal that leads to initiation, bidder-initiated auctions do not occur in equilibrium.

Next, consider a private-value setting and a bidder with a sufficiently high signal who contemplates approaching the seller. In contrast to the common-value setting, observing that the auction is bidder-initiated, the rival bidder does not update her valuation of the asset, which therefore remains low. As a result, she only bids aggressively because she competes against a strong bidder but not because of the valuation update. This logic implies that the bidder with a sufficiently high signal can take advantage of the rival's low valuation by approaching the seller immediately: she always obtains positive surplus, even if her signal is the lowest among those that lead to initiation. In contrast, waiting for an auction initiated by a rival bidder ensures that the bidder will compete against a strong rival. Even though participating in a rival-initiated auction allows the bidder with a sufficiently high valuation to hide it, competing against a weak rival who adjusts her bid upwards is better than competing against a strong rival who adjusts her bid downwards. Thus, in contrast to the common-value setting, the bidder would be worse off waiting until the rival bidder initiates the auction, implying that bidder-initiated auctions can occur in equilibrium.

In the private-value framework, multiple equilibria often arise, because initiation decisions of bidders and the seller are interdependent. If bidders perceive a seller-initiated auction to be a very unlikely event, they will have strong incentives to initiate the auction, because, as described above, a rival-initiated auction makes them worse off and is likely to occur before the seller-initiated auction. In contrast, if bidders expect the seller to auction the asset off soon, they will have weak incentives to initiate the auction, because the seller-initiated auction makes them better off by allowing a bidder with a moderately high valuation to hide it and is likely to occur before a rival bidder approaches the seller.

These effects have two implications. First, because of multiplicity of equilibria, sales of otherwise similar assets in different markets (e.g., otherwise similar companies in different countries) may have very different patterns of initiation. Second, because expectations of buyers about the likelihood of a seller-initiated auction affect their incentives to approach the seller, sales of distressed companies would almost always be seller-initiated, regardless of whether values are private or common. In contrast, sales of companies that are far from distress would be seller-initiated, if valuations are common, but are likely to be bidder-

initiated, if valuations are private.

Taken together, our results provide a benchmark with which one can compare empirical findings on initiation of auctions. For example, our results are consistent with empirical evidence on target- and bidder-initiated strategic and private-equity deals: approximately 60% (35%) of strategic (private-equity) deals are initiated by the bidders (Fidrmuc et al, 2012). Our explanation of this difference is that financial, but not strategic, bidders have a large common value component in their valuations for targets. Our analysis also has implications about how bids and auction outcomes differ depending on whether auctions are bidder- or seller-initiated. For example, bidders bid more aggressively in a bidder-initiated auction than in a seller-initiated auction; in a bidder-initiated auction, conditional on the same valuation, a non-initiating bidder bids more aggressively than the initiating bidder, while unconditionally the initiating bidder bids more aggressively.

The model generates a set of implications relating bidders' valuations to their bidding strategies and the identity of initiating bidders. For example, auctions are likely to be initiated by stronger bidders, who will submit, on average, higher bids and will be more likely to win than non-initiating bidders. However, conditional on the same valuation, non-initiating bidders will bid more aggressively than initiating bidders. Similarly, the model implies that bidders will bid less aggressively in seller-initiated auctions, even conditionally on having the same valuations.

Beyond the set of main results, the model points to a value-enhancing role of shareholder activists and investment banks in intermediating intercorporate asset sales. Consider an inefficiently-run firm followed by potential bidders, whose valuations in this case are best represented by the common-value setting. Our results show that either bidder would be reluctant to approach the seller, because it would erode her information rents from the auction. If the firm's management and board are entrenched, the seller would not initiate the auction either. The result would be the failure of the market for corporate control as a corporate governance mechanism precisely in situations when it is most needed: when the current owner manages the asset poorly. While the market alone may be insufficient to resolve such inefficiencies, an activist investor, such as Elliot Associates in an earlier example, can use it to acquire a block of shares and force the firm to auction itself off. In this respect,

shareholder activism and the market for corporate control are complements, rather than two different mechanisms for turning around poorly managed companies. Alternatively, an investment bank that intermediates a transaction and can commit (e.g., due to reputational concerns by being a long-term player) to hide the presence of buyers' indications of interest would provide strong buyers with incentives to communicate indications of interest. Lastly, the model points to a potentially value-enhancing role of toeholds, implying that in a dynamic environment the welfare effect of toeholds trades off allocative inefficiency of the auction against greater incentives of strong bidders to acquire toeholds and initiate auctions.

Our paper belongs to the large literature on auction theory. Virtually all of it only considers a stage when the auction takes place. Three exceptions are papers by Board (2007), Cong (2017), and Gorbenko and Malenko (2018), which also feature strategic timing of the auction. Board (2007) and Cong (2017) study the problem of a seller auctioning an option, such as the right to drill oil, where the timing of the sale and option exercise are decision variables. Gorbenko and Malenko (2018) assume that M&A contests are bidder-initiated and study the role of stock bids in alleviating bidders' financial constraints. These papers do not study joint initiation by bidders and the seller and restrict attention to independent private values, so the issues examined in our paper do not arise.

Second, the paper is related to the literature that studies takeover contests as auctions. They have been modeled using both the common-value (e.g., Bulow, Huang, and Klemperer, 1999) and private-value framework (e.g., Fishman, 1988; Burkart, 1995; Povel and Singh, 2006).⁴ Like us, Bulow, Huang, and Klemperer (1999) interpret competition between strategic (or financial) bidders as a private-value (or common-value) auction. However, these papers do not study endogenous initiation of takeover contests. Our extension for shareholder activism relates to recent papers that study interactions between activism and the market for corporate control focusing on other aspects of the interaction – Burkart and Lee (2018) focus on the free-rider problem in tender offers, while Corum and Levit (2018) focus on the commitment problem of the bidder in a proxy fight. Jiang, Li, and Mei (2016) and Boyson, Gantchev, and Shivdasani (2017) provide empirical evidence on interactions

⁴Bulow and Klemperer (1996, 2009) provide motivations why running a simple auction is often a good way for the seller to sell the asset.

between activism and the market for corporate control, which is broadly consistent with shareholder activists creating value in the M&A market.

Finally, the paper is related to models of auctions with asymmetric bidders. Most literature on auction theory assumes that bidders are symmetric in the sense that their signals are drawn from the same distribution. Some recent literature (e.g., Maskin and Riley, 2000, 2003; Campbell and Levin, 2000; Lebrun, 2006; Kim, 2008) examines issues that arise when bidders are asymmetric. The novelty of our paper is that asymmetries at the auction stage are not assumed: they arise endogenously and are driven by incentives to approach the seller which differ with the bidder's information. While bidders are ex-ante symmetric, at the auction stage they are not: the decision of one bidder to approach the previously unapproached seller makes it commonly known that bidders' signals come from different partitions of the same ex-ante distribution of signals.⁵

The remainder of the paper is organized as follows. Section 2 describes the setup of the model. Section 3 studies the common-value framework. Section 4 studies the private-value framework. Section 5 focuses on additional features of the market for corporate control, and discusses model assumptions. Section 6 lists empirical predictions. Section 7 concludes.

2 The Model Setup

The economy consists of one risk-neutral seller (male) and a set of potential risk-neutral buyers (female). At each point in time, only two buyers are present, so we index them by $i = 1, 2$.⁶ The seller has an asset for sale. In the context of application to mergers and intercorporate asset sales, the asset can be the whole company or a business unit. The seller's valuation of the asset is normalized to zero. Time is continuous and indexed by $t \geq 0$.

At the initial date $t = 0$, each potential bidder $i \in \{1, 2\}$ randomly draws a private signal. Bidders' signals are independent draws from the uniform distribution over $[0, \hat{s}_0]$,

⁵While on a different topic, the learning effect in common-value auctions is related to Ely and Siegel (2013) who develop a static model of firms interviewing and hiring workers and show that in equilibrium only the highest-ranked firm interviews the applicant.

⁶The model can be extended to $N \geq 2$ bidders present at each time with the main qualitative effects intact.

where $\hat{s}_0 \in [0, 1]$.⁷ Conditional on all signals, the value of the asset to bidder i is $v(s_i, s_{-i})$, where s_{-i} is the signal of the rival bidder.

Assumption 1. *Function $v(s_i, s_{-i})$ is continuous in both variables, strictly increasing in s_i , and satisfies $v(0, 0) = 0$.*

Assumption 1 is standard. Continuity means that there are no gaps in possible valuations of the asset. Strict monotonicity in the first variable means that a higher private signal is always good news about the bidder's valuation. This valuation structure follows the general symmetric model of Milgrom and Weber (1982). It covers two valuation structures commonly used in the literature:

- **The private-value framework:** $v(s_i, s_{-i}) = v(s_i)$. A bidder's signal provides information only about her own valuation, but not about the valuation of her competitors.
- **The common-value framework:** $v(s_i, s_{-i}) = v(s_{-i}, s_i)$, which is symmetric in both variables. Conditional on both signals, bidders have the same valuation of the asset. However, bidders can differ in their assessments of it, because their private signals can be different.

We focus on these two valuation structures. There are two natural interpretations of common versus private values in the context of auctions of companies and business units. The first interpretation deals with different types of bidders: we can interpret the common-value (private-value) auction as a battle between two financial (strategic) bidders. Intuitively, financial bidders use similar strategies after they acquire the target (i.e., have "common" value), but may have different estimates of potential gains (i.e., have different signals about the common value). In contrast, because synergies that strategic bidders expect to achieve from acquiring the target are often bidder-specific, they provide little information about valuation of the target to the other bidder. The second interpretation

⁷Because we assume a general functional form that maps signals into valuations, uniform distribution is largely a normalization.

deals with different types of targets rather than bidders. Broadly, value in an acquisition can be created either because the incumbent target management is inefficient or because the target and the acquirer have synergies that cannot be realized by the stand-alone acquirer. To the extent that inefficiency can be resolved by any bidder, acquisitions of the first type are common-value deals. At the same time, because synergies are bidder-specific, acquisitions of the second type tend to be private-value deals.

In practice, the environment changes over time, as either the business nature or management of a bidder or a target changes, or external economic shocks arrive. To capture this feature, we assume that at each time the buy side (bidders) and the sell side (the seller) may experience shocks of the following form.

First, with Poisson intensity $\lambda > 0$ each bidder “dies,” in which case she leaves the market and obtains the payoff of zero. These shocks are independent across bidders. If a bidder leaves the market, she is replaced by a new potential buyer with a new signal s'_i , which is an independent draw from uniform distribution over $[0, 1]$. The valuation of the asset of the new potential bidder becomes $v(s'_i, s_{-i})$, and the valuation of the rival bidder becomes $v(s_{-i}, s'_i)$.⁸ Thus, at any point in time, only two current signals are relevant for bidders’ valuations, which makes the model tractable.

Second, with Poisson intensity $\nu > 0$ the seller experiences a liquidity shock and has no choice but to sell the asset immediately. Examples of such liquidity shocks are an arrival of an attractive investment opportunity that is mutually exclusive with existing assets, a change in the strategy of the firm, or a bankruptcy in which case the judge liquidates the target by auctioning its assets among potential bidders.⁹ As we shall see, the role of this assumption will be to give the bidder some incentives to wait, because its expected payoff will be endogenously higher in the auction triggered by a liquidity shock than when she approaches the seller.¹⁰

⁸We define $v(s_i, s_{-i})$ as the discounted expected flow of utility to the bidder (e.g., the present value of discounted cash flows), which already incorporates possible future shocks.

⁹As an example of an investment opportunity triggering sale of existing asset, consider the case of the seller acquiring another firm in a horizontal merger. As a condition for approval, it is common for antitrust authorities to require a spin-off of some of the existing assets to ensure that the product market does not become too concentrated.

¹⁰In the previous versions of the paper, we considered two alternative formulations of the model that gave bidders incentives to wait for different reasons. In the first formulation, bidders do not get replaced

Additionally to the exogenous reason outlined above, the seller has the right to auction the asset off to the bidders at any time. Turning to the buy side, prior to the auction, each bidder communicates with the seller by sending a private message to the seller signaling her interest in acquiring the asset. Communication is costless and follows Crawford and Sobel (1982) with the binary message space (0 or 1). Without loss of generality, message $m_{i,t} = 1$ from bidder i at time t is interpreted as signaling interest in acquiring the asset, and message $m_{i,t} = 0$ is interpreted as the lack of communication. As we shall see, we will examine responsive equilibria in which upon receiving message $m_{i,t} = 1$, the seller auctions the asset off immediately.¹¹ We refer to such an event as a “*bidder-initiated*” auction, capturing the fact that the auction is triggered by a bidder communicating her interest to the seller. We refer to an event of the seller auctioning the asset off without receiving message $m_{i,t} = 1$ as a “*seller-initiated*” auction.

Importantly, whether the auction is bidder- or seller-initiated is known to its participants. This assumption can be justified as follows. The seller may voluntarily disclose whether the auction is bidder- or seller-initiated. In many contexts, the disclosed auction type can be verified ex post – for example, any public U.S. target is required to report the deal background as part of its SEC filings, and lying there has legal consequences. By the standard reasoning (Grossman, 1981; Milgrom, 1981), because it is common knowledge that the seller knows the auction type and this information is verifiable, he will always disclose it.

Once either party (the seller or one of the two currently present bidders) initiates the auction, a sealed-bid first-price auction with no reserve price takes place.¹² Specifically, each bidder simultaneously submits a bid to the seller in a concealed fashion. The two bids are compared, and the bidder with the highest bid acquires the asset and pays her bid. Once the asset is sold, the game is over. The winning bidder obtains the payoff that equals

and a bidder’s signal changes over time. In the second formulation, an existing bidder obtains a positive exit payoff $X > 0$ if she “dies” without acquiring the target. These alternative assumptions do not affect our results qualitatively. We selected the present formulation because we find it more natural than the alternatives and because it is slightly more tractable.

¹¹As in any cheap-talk model, there are also “babbling equilibria” in which the seller never reacts to messages, and all types of buyers adopt the same communication strategy.

¹²In Section 5.1, we discuss the practical motivation for our choice of the first- versus second-price auction, and how results would be affected in the latter.

to her valuation less the price she pays. The losing bidder obtains zero payoff. The seller obtains the payoff that equals to the winning bid.¹³

2.1 The equilibrium concept

The equilibrium concept is Markov Perfect Bayesian Equilibrium (MPBE). In the auction, the strategy of each bidder is a mapping from her own signal s_i and the knowledge of whether the bidder, the rival bidder, or the seller initiated the auction, captured by set of messages $(m_{i,t}, m_{-i,t})$, into a non-negative bid. Prior to the auction, the communication strategy of each bidder is a mapping from her own signal s_i into message $m_{i,t} \in \{0, 1\}$, i.e., to send an indication of interest to the seller or not. Because bidders are ex-ante symmetric, we look for equilibria in which the bidders follow symmetric strategies prior to the auction. Furthermore, we look for equilibria in which at any time t prior to the auction a bidder follows the cut-off communication strategy, such that she sends message $m_{i,t} = 1$ if and only if her signal is above some cut-off \hat{s}_t .¹⁴ Finally, we are interested in “responsive” equilibria which we define as equilibria in which the seller reacts to message $m_{i,t} = 1$ by initiating the auction.¹⁵

For the remainder of the paper, we consider the stationary case, defined as the situation in which the cut-off \hat{s}_t stays constant over time at some level $\hat{s} \in (0, 1]$. This requires that the upper bound on the initial signal is $\hat{s}_0 = \hat{s}$. Because in practice there is often

¹³The seller’s private valuation of the asset can also be important for his decision to offer it for sale. Lauermaun and Wolinsky (2016) study common-value first-price auctions in which the seller obtains a private signal about his value and solicits a different number of bidders at a cost depending on the signal’s value. Being solicited thus discloses some information about the seller’s signal to bidders. Interestingly, this solicitation effect can result in non-competitive bids and inefficient information aggregation. While modeling the two-sided private information is beyond the scope of this paper, it is potentially interesting to examine interactions between seller and bidder initiation in the presence of the solicitation effect.

¹⁴Because arbitrarily low types always obtain an arbitrarily low surplus from the auction, it is straightforward to show that there is no equilibrium in which low types send the message that triggers the auction, while high types do not. What is less clear, however, is whether there are equilibria in which communication strategies are not described by a cut-off (e.g., if there are multiple cut-offs). Because the analysis of first-price auctions when distributions of valuations have an arbitrary number of gaps is, to our knowledge, an open problem, we cannot say anything about the possible existence of such equilibria.

¹⁵There also exist unresponsive equilibria. First, there can be “babbling” equilibria in which bidders’ messages are completely uninformative. Second, there can be equilibria in which communication is informative but the seller initiates the auction only upon receiving indications of interest from both bidders. Equilibria of the first type are clearly uninteresting, so we do not consider them. Equilibria of the second type are similar in spirit to the ones we consider, but are more complicated to analyze.

no clear starting date, at least, in the applications we look at, focusing on the stationary solution is reasonable. In Section 5.4 and the online appendix, we provide an analysis of the non-stationary dynamics, starting at $\hat{s}_0 = 1$.

3 The Case of Common Values

In this section, we consider the case of pure common value, $v(s_i, s_{-i}) = v(s_{-i}, s_i)$.

3.1 Equilibria in bidder- and seller-initiated contests

First, we solve for the equilibrium at the auction stage.

3.1.1 A bidder-initiated takeover contest

Consider a bidder-initiated auction with an exogenous cut-off type \hat{s} . The equilibrium cut-off type will be determined at the initiation stage. Denote the initiating and non-initiating bidder by I and N . Then, from the point of view of bidder N (or bidder I) and the seller, the type of bidder I (or bidder N) is distributed uniformly over $[\underline{s}_I, \bar{s}_I] = [\hat{s}, 1]$ (or over $[\underline{s}_N, \bar{s}_N] = [0, \hat{s}]$). Thus, even though all bidders are ex-ante symmetric, initiation based on a cut-off type endogenously creates an asymmetry between them.

Conjecture that there is an equilibrium in pure strategies. Denote the equilibrium bid of bidder I and N with signal s by $a_I(s, \hat{s})$, $s \geq \hat{s}$ and $a_N(s, \hat{s})$, $s \leq \hat{s}$, respectively, and conjecture that each is strictly increasing in s in the relevant range. Denote the corresponding inverses in s by $\phi_I(b, \hat{s})$ and $\phi_N(b, \hat{s})$. The expected payoffs of bidders I and N with signal s and bid b are

$$\Pi_j(b, s, \hat{s}) = \mathbb{E}[v(s, x) - b | x \in [\underline{s}_k, \phi_k(b, \hat{s})]] = \int_{\underline{s}_k}^{\phi_k(b, \hat{s})} (v(s, x) - b) \frac{1}{\bar{s}_k - \underline{s}_k} dx, \quad (1)$$

where $j \neq k \in \{I, N\}$. The intuition behind the system of equations (1) is as follows. Consider the initiating bidder who bids b . She wins the auction if and only if the bid of the non-initiating bidder is below b , which happens if such bidder's signal is below $\phi_N(b, \hat{s})$. Conditional on winning when the rival's signal is $x \in [0, \phi_N(b, \hat{s})]$, the value of the asset

to the initiating bidder is $v(s, x)$. Integrating over $x \in [0, \phi_N(b, \hat{s})]$ yields (1) for $j = I$. The same intuition explains the expected payoff of the non-initiating bidder. Taking the first-order conditions of (1), we obtain

$$\frac{\partial \phi_j(b, \hat{s})}{\partial b} (v(s, \phi_j(b, \hat{s})) - b) - (\phi_j(b, \hat{s}) - \underline{s}_j) = 0 \quad (2)$$

for $j \in \{I, N\}$. The first and second terms of equations (2) represent the trade-off between the marginal benefit and the marginal cost of increasing a bid by a small amount. The marginal benefit is that bidder k wins a marginal event in which the signal of the rival bidder j is exactly $\phi_j(b, \hat{s})$. The marginal cost is that bidder k must pay more in case she wins. In equilibrium, $b = a_j(s, \hat{s})$ must satisfy (2) for $j \in \{I, N\}$, implying $s = \phi_j(b, \hat{s})$. Plugging in and rearranging the terms, we obtain

$$\frac{\partial \phi_j(b, \hat{s})}{\partial b} = \frac{\phi_j(b, \hat{s}) - \underline{s}_j}{v(\phi_k(b, \hat{s}), \phi_j(b, \hat{s})) - b}. \quad (3)$$

The system of equations (3) is solved subject to the following boundary conditions. First, in equilibrium, the highest bid submitted by both bidders must be the same: $a_j(\bar{s}_j, \hat{s}) \equiv \bar{a}(\hat{s})$ for $j \in \{I, N\}$. $a_I(1, \hat{s}) > a_N(\hat{s}, \hat{s})$ cannot occur in equilibrium, because then types of bidder I close enough to 1 would reduce their bids and still win the auction with probability 1. Similarly, $a_I(1, \hat{s}) < a_N(\hat{s}, \hat{s})$ cannot occur in equilibrium. Second, the lowest bid submitted by both bidders must be the same: $a_j(\underline{s}_j, \hat{s}) \equiv \underline{a}(\hat{s})$ for $j \in \{I, N\}$. Suppose instead that $a_I(\hat{s}, \hat{s}) > a_N(0, \hat{s})$. From (1), for type \hat{s} of bidder I to get non-negative rents, $a_I(\hat{s}, \hat{s})$ cannot exceed $\mathbb{E}[v(\hat{s}, x) | x \leq \phi_N(a_I(\hat{s}, \hat{s}), \hat{s})]$. Consider bidder N with type $\phi_N(a_I(\hat{s}, \hat{s}), \hat{s}) > \phi_N(a_N(0, \hat{s}), \hat{s}) = 0$, i.e., a bidder who bids exactly $a_I(\hat{s}, \hat{s})$. Her payoff is zero, because the initiating bidder never bids below $a_I(\hat{s}, \hat{s})$. However, if this bidder deviated to bidding $b \in (a_I(\hat{s}, \hat{s}), v(\hat{s}, \phi_N(a_I(\hat{s}, \hat{s}), \hat{s})))$, her payoff would be positive, because her bid would win with positive probability and, conditional on winning, the payoff would be positive, as $b < v(\hat{s}, \phi_N(a_I(\hat{s}, \hat{s}), \hat{s}))$. Because $a_I(\hat{s}, \hat{s}) \leq \mathbb{E}[v(\hat{s}, x) | x \leq \phi_N(a_I(\hat{s}, \hat{s}), \hat{s})] < v(\hat{s}, \phi_N(a_I(\hat{s}, \hat{s}), \hat{s}))$ when $\phi_N(a_I(\hat{s}, \hat{s}), \hat{s}) > 0$, set $(a_I(\hat{s}, \hat{s}), v(\hat{s}, \phi_N(a_I(\hat{s}, \hat{s}), \hat{s})))$ is non-empty. Therefore, $a_I(\hat{s}, \hat{s}) > a_N(0, \hat{s})$ cannot occur in equilibrium. Similarly, $a_I(\hat{s}, \hat{s}) < a_N(0, \hat{s})$ cannot occur in equilibrium. Hence, the support of possible equilibrium bids for

both bidders is given by $[\underline{a}(\hat{s}), \bar{a}(\hat{s})]$.

The upper boundary implies $1 = \phi_I(\bar{a}(\hat{s}), \hat{s})$ and $\hat{s} = \phi_N(\bar{a}(\hat{s}), \hat{s})$. Consider the lower boundary $\underline{a}(\hat{s})$. First, it must be that $\underline{a}(\hat{s}) \geq v(\hat{s}, 0)$, as otherwise either type \hat{s} of bidder I or type 0 of bidder N would find it optimal to deviate and submit a marginally higher bid. By doing this she can increase the probability of winning from zero to a positive number, and thus get a positive expected surplus instead of zero. Second, it must be that $\underline{a}(\hat{s}) \leq v(\hat{s}, 0)$, otherwise a too high lower boundary would imply that low enough types get a negative surplus in equilibrium. Thus, we have proved the following lemma:

Lemma 1 (equilibrium in the bidder-initiated CV auction). *The equilibrium bidding strategies of the initiating and the non-initiating bidders, $a_j(s, \hat{s})$, $j \in \{I, N\}$, are increasing functions, such that their inverses satisfy (3), with boundary conditions*

$$1 = \phi_I(\bar{a}(\hat{s}), \hat{s}), \quad \hat{s} = \phi_N(\bar{a}(\hat{s}), \hat{s}), \quad \hat{s} = \phi_I(v(\hat{s}, 0), \hat{s}), \quad 0 = \phi_N(v(\hat{s}, 0), \hat{s}). \quad (4)$$

Example 1 in the appendix provides the closed-form solution for the case of additive valuations, $v(s_i, s_{-i}) = \frac{1}{2}(s_i + s_{-i})$, which is linear in a bidder's signal. Figure 1 illustrates the equilibrium bids and expected bidder surpluses of Example 1 for the case $\hat{s} = 0.5$.

The equilibrium in the auction implies the payoff of type \hat{s} of bidder I is zero:

Lemma 2. *The equilibrium payoff of the initiating bidder of type \hat{s} is zero.*

Proof. $\Pi_I(\underline{a}(\hat{s}), \hat{s}, \hat{s}) = 0$ follows immediately from $\underline{a}(\hat{s}) = v(\hat{s}, 0)$.

The intuition behind this result is simple. The non-initiating bidder knows that the initiating bidder communicates her interest to the seller if and only if her signal is at least \hat{s} . Therefore, the non-initiating bidder with signal s knows that the lowest possible valuation is $v(s, \hat{s}) \geq v(0, \hat{s})$. Similarly, the initiating bidder with signal s knows that the lowest possible valuation is $v(s, 0) \geq v(\hat{s}, 0)$. Because both bidders cannot possibly value the asset below $v(\hat{s}, 0)$, no bidder bids less than this amount. Consequently, the initiating bidder with signal \hat{s} wins the auction only when the non-initiating bidder's signal is zero and at price $\underline{a}(\hat{s}) = v(\hat{s}, 0)$, leaving her with zero surplus. This argument holds for any

cut-off type \hat{s} .

The above result extends Engelbrecht-Wiggans, Milgrom, and Weber (1983), who show that a bidder with access to public information only obtains zero surplus in equilibrium. An important difference is that bidder I here does retain private information, because her decision to approach the seller only reveals that her signal cannot be below \hat{s} . Hence, bidder I with almost any signal s obtains a positive expected surplus. This can be seen on the right panel of Figure 1. Only bidder I with the *marginal* signal, \hat{s} , obtains zero surplus. In the common-value framework, to reveal her higher signal through a higher bid, the bidder must be compensated with a higher surplus, which takes the form of a higher probability of winning. However, bidder I with signal \hat{s} has no information rent: in equilibrium, the cut-off \hat{s} is known, so type \hat{s} has no lower types to separate from; hence, she does not get compensated with rents.

3.1.2 A seller-initiated takeover contest

Suppose that the seller initiates the auction. Conditional on no bidder approaching the seller, all parties believe that each bidder's signal is distributed uniformly over $[0, \hat{s}]$ for some cut-off signal \hat{s} . Because the auction is seller-initiated, bidders are symmetric, so the solution is standard (see, e.g., Chapter 6.4 in Krishna, 2010). Denote the equilibrium bid by a bidder with signal s by $a_S(s, \hat{s})$. In the appendix, we prove the following lemma:

Lemma 3 (equilibrium in the seller-initiated CV auction). *The symmetric equilibrium bidding strategies of the non-initiating bidders, $a_S(s)$, are increasing functions that are independent of \hat{s} and solve (19) with boundary condition $a_S(0) = v(0, 0)$. Specifically,*

$$a_S(s) = \int_0^s v(x, x) \frac{1}{s} dx = \mathbb{E}[v(x, x) | x \leq s]. \quad (5)$$

In contrast to bidder I in the bidder-initiated auction, bidders with all but the lowest signal 0 obtain positive expected payoff in the seller-initiated auction. In particular, the cut-off type \hat{s} obtains a positive payoff. For the case of additive valuations, $v(s_i, s_{-i}) = \frac{1}{2}(s_i + s_{-i})$, $a_S(s) = \mathbb{E}[x | x \leq s] = \frac{s}{2}$.

3.2 The initiation game

Because the marginal type of the initiating bidder always obtains zero surplus in equilibrium, it is straightforward to show that pure common-value auctions are never bidder-initiated. Suppose, by contradiction, that there is an equilibrium with $\hat{s} < 1$. Then, a bidder with a signal just above \hat{s} obtains a positive but infinitesimal surplus by initiating the auction. In contrast, if a bidder waits until either the other bidder or the seller initiates the auction, her expected payoff will be bounded away from zero. Thus, any $\hat{s} < 1$ is inconsistent with equilibrium. Because the seller does not expect a bidder to ever initiate, there is no value for him in delaying the auction. Thus, the auction is initiated by the seller with no delay:

Proposition 1. *There exists a unique equilibrium cut-off $\hat{s} = 1$. That is, no bidder ever communicates her interest to the seller, and the seller initiates the auction immediately at date 0.*

It is straightforward to extend the model by assuming that the seller gets some disutility $C > 0$ from selling the asset. If C is below the expected revenues of the seller from the auction when both bidders' signals are distributed uniformly over $[0, 1]$, then he initiates the auction at $t = 0$. In contrast, if C is above this value, then the sale never happens, as neither the seller nor the initiating bidder with a signal close to the cut-off benefits from it. In the application to auctions of companies, it is natural to interpret C as the degree of entrenchment of the management and board of the target: they will not contemplate a voluntary (seller-initiated) auction unless the expected revenues exceed C . The following corollary summarizes this result:

Corollary 1. *Let $\bar{C} = \mathbb{E} \left[a \left(\max_{i \in \{1, 2\}} s_i \right) \mid s_i \in [0, 1] \ \forall i \right]$. If $C < \bar{C}$, the seller initiates the auction immediately at $t = 0$. If $C > \bar{C}$, the seller does not initiate the auction unless he is hit by a liquidity shock and no bidder communicates her interest to the seller.*

4 The Case of Private Values

In this section, we consider the case of private values, $v(s_i, s_{-i}) = v(s_i)$. In the following analysis, we impose the following natural restriction on equilibrium bids, which pins down the unique equilibrium in the auction:

Assumption 2. *No bidder bids above her valuation in equilibrium.*

The rationale behind this assumption is that bidding above one's valuation is a dominated strategy.¹⁶

4.1 Equilibria in bidder- and seller-initiated contests

First, we solve for the equilibrium at the auction stage.

4.1.1 A bidder-initiated takeover contest

Consider a bidder-initiated auction with a fixed cut-off type \hat{s} . As before, denote the equilibrium bid of bidder I and N with signal s by $a_I(s, \hat{s})$, $s \geq \hat{s}$ and $a_N(s, \hat{s})$, $s \leq \hat{s}$, respectively. Denote their inverses in s by $\phi_I(b, \hat{s})$ and $\phi_N(b, \hat{s})$. The expected payoffs of bidders I and N with signal s and bid b are now

$$\Pi_j(b, s, \hat{s}) = \mathbb{E}[v(s) - b | x \in [\underline{s}_k, \phi_k(b, \hat{s})]] = (v(s) - b) \frac{\phi_k(b, \hat{s}) - \underline{s}_k}{\bar{s}_k - \underline{s}_k}, \quad (6)$$

where $j \neq k \in \{I, N\}$. Intuitively, bidder I 's (or bidder N 's) bid exceeds the bid of her rival with probability $\frac{\phi_N(b, \hat{s})}{\hat{s}}$ (or $\frac{\phi_I(b, \hat{s}) - \hat{s}}{1 - \hat{s}}$). Taking the first-order conditions of (6), we obtain

$$\frac{\partial \phi_j(b, \hat{s})}{\partial b} (v(s) - b) - (\phi_j(b, \hat{s}) - \underline{s}_j) = 0 \quad (7)$$

¹⁶As Kaplan and Zamir (2015) show, without this restriction, multiple equilibria in the first-price auction with asymmetric bidders arise, in which some bidders submit “non-serious” bids (i.e., bids that win with probability zero) above their valuations. Such equilibria are implausible, because even though “non-serious” bidders obtain zero surplus in equilibrium, a deviation by the rival results in their negative payoff. Thus, it is reasonable to rule out these strategies. Assumption 2 pins down the unique equilibrium in the auction (Lebrun, 2006).

for $j \in \{I, N\}$. In equilibrium, $b = a_j(s, \hat{s})$ must satisfy (7), implying $s = \phi_j(b, \hat{s})$. Thus,

$$\frac{\partial \phi_j(b, \hat{s})}{\partial b} = \frac{\phi_j(b, \hat{s}) - \underline{s}_j}{v(\phi_k(b, \hat{s})) - b}. \quad (8)$$

The system of equations (8) is solved subject to the following boundary conditions. Similarly to the common-value case, the equilibrium maximum and minimum bids that win with a positive probability, or “serious” bids of both bidders must be the same: $a_j(\bar{s}_j, \hat{s}) \equiv \bar{a}(\hat{s})$ and $a_j(\underline{s}_j, \hat{s}) \equiv \underline{a}(\hat{s})$. If the maximum bids are not the same, the bidder whose maximum bid is higher can increase her payoff by reducing her bid by a small amount: Doing so does not affect the probability of winning, which is one, and reduces the payment conditional on winning. If the minimum serious bid of bidder N is below that of bidder I , bidder N never wins with such bid, which violates the definition of a serious bid. If the minimum bid of bidder I is below that of bidder N , there must be discontinuity in the expected payoff of bidder N at the signal that results in the minimum serious bid. However, this would imply that bidder I with signals resulting in non-serious bids just below the minimum serious bid of bidder N would benefit from a deviation to such bid. Hence, the support of possible equilibrium bids for both bidders is $[\underline{a}(\hat{s}), \bar{a}(\hat{s})]$.

The upper boundary implies $1 = \phi_I(\bar{a}(\hat{s}), \hat{s})$ and $\hat{s} = \phi_N(\bar{a}(\hat{s}), \hat{s})$. Next, the lowest type of bidder I submits the lowest serious bid: $\hat{s} = \phi_I(\underline{a}(\hat{s}), \hat{s})$. This lowest bid, in turn, determines the cut-off on the signal of bidder N , who submits a serious bid: the cut-off is equal to the lowest bid. If the minimum bid is above such cut-off, bidder N with the cut-off signal would bid above her valuation, which would violate Assumption 2. If the minimum bid is below the cut-off, she would profitably deviate to increasing her bid by a small amount, which would result in a positive expected payoff, exceeding her equilibrium payoff of zero. Formally, $v^{-1}(\underline{a}(\hat{s})) = \phi_N(\underline{a}(\hat{s}), \hat{s})$.

Assumption 2 uniquely pins down the minimum “serious” bid (Lebrun, 2006):

$$\underline{a}(\hat{s}) = \arg \max_b \frac{v^{-1}(b)}{\hat{s}} (v(\hat{s}) - b) \quad \Rightarrow \quad \text{F.O.C.:} \quad \frac{v(\hat{s}) - \underline{a}(\hat{s})}{v'(v^{-1}(\underline{a}(\hat{s})))} = v^{-1}(\underline{a}(\hat{s})). \quad (9)$$

The following lemma summarizes the unique equilibrium in the bidder-initiated first-price auction. Existence and uniqueness results follow from Lebrun (2006):

Lemma 4 (equilibrium in the bidder-initiated PV auction). *The equilibrium is unique (up to the non-serious bids of types $s < v^{-1}(\underline{a}(\hat{s}))$ of non-initiating bidders). The equilibrium bidding strategies of the initiating and non-initiating bidders, $a_j(s, \hat{s})$, $j \in \{I, N\}$, are increasing functions, such that their inverses satisfy (8), with boundary conditions*

$$1 = \phi_I(\bar{a}(\hat{s}), \hat{s}), \quad \hat{s} = \phi_N(\bar{a}(\hat{s}), \hat{s}), \quad \hat{s} = \phi_I(\underline{a}(\hat{s}), \hat{s}), \quad v^{-1}(\underline{a}(\hat{s})) = \phi_N(\underline{a}(\hat{s}), \hat{s}) \quad (10)$$

and the lowest serious bid is given by (9).

Example 2 in the appendix provides the closed-form solution for the case $v(s_i) = s_i$. Figure 2 illustrates the equilibrium bids and expected bidder surpluses for the case $\hat{s} = 0.5$.

Denote $\Pi_j^*(s, \hat{s}) = \Pi_j(a_j(s, \hat{s}), s, \hat{s})$, $j \in \{I, N\}$. The next lemma shows that the payoff of the bidder with the cut-off signal, \hat{s} , is higher when she is an initiating bidder rather than a non-initiating bidder:

Lemma 5. *For any \hat{s} , $\Pi_I^*(\hat{s}, \hat{s}) \geq \Pi_N^*(\hat{s}, \hat{s})$. The inequality is strict if $\hat{s} < 1$.*

This result is in stark contrast with the case of common value, in which bidder I with the cut-off signal always obtains zero expected payoff, which, in particular, is strictly lower than a payoff of bidder N with the same signal, resulting in unraveling in initiation. In the private-value framework, all else equal, the bidder with the cut-off signal is better off initiating the auction rather than being the non-initiating bidder. Intuitively, because bidders do not update valuations, the strength of competition is endogenous on who the initiator is and favors the initiator. If a bidder waits until the rival approaches the seller, she will compete against a rival with a strong signal (above \hat{s}). In contrast, if a bidder initiates the auction today, she will compete against a rival with a weak signal (below \hat{s}). In turn, the rival adjusts her bid upwards (or downwards) upon the bidder's (or rival's) initiation compared to the symmetric bidder case, believing that she competes against a strong (or weak) bidder. However, this adjustment, in the absence of valuation updating, is

second-order on the bidder's profits and cannot eliminate the benefit of being the initiator.

4.1.2 A seller-initiated takeover contest

Suppose that the seller initiates the auction. Conditional on no bidder approaching the seller, all parties believe that each bidder's signal is distributed uniformly over $[0, \hat{s}]$. Bidders are symmetric, so the solution is standard (see, e.g., Krishna, 2010). Denote the equilibrium bid by a bidder with signal s by $a_S(s, \hat{s})$. In the appendix, we prove the following lemma:

Lemma 6 (equilibrium in the seller-initiated PV auction). *The symmetric equilibrium bidding strategies of the non-initiating bidders, $a_S(s)$, are increasing functions that are independent of \hat{s} and solve (23) with boundary condition $a_S(0) = v(0)$. Specifically,*

$$a_S(s) = \int_0^s v(x) \frac{1}{s} dx = \mathbb{E}[v(x)|x \leq s]. \quad (11)$$

For the case $v(s_i) = s_i$, $a_S(s) = \mathbb{E}[x|x \leq s] = \frac{s}{2}$.

Denote the equilibrium expected payoff of a bidder with signal s in a seller-initiated auction as $\Pi_S^*(s, \hat{s}) = \Pi_S(a_S(s), s, \hat{s})$. The next lemma shows that a seller-initiated auction leads to a higher expected payoff to a bidder with signal \hat{s} than an auction initiated by this bidder:

Lemma 7. *For any $\hat{s} > 0$, $\Pi_S^*(\hat{s}, \hat{s}) > \Pi_I^*(\hat{s}, \hat{s})$.*

The intuition behind Lemma 7 is as follows. Consider a bidder with signal \hat{s} . If the auction is initiated by the seller, the rival believes that the bidder's signal is weak (below \hat{s}). In contrast, if the auction is initiated by the bidder, the rival believes that her signal is strong (above \hat{s}). In response, the rival adjusts her bid upwards upon the bidder's initiation compared to the symmetric bidder case. In turn, the seller-initiated auction leaves a higher expected payoff to the bidder.

Together, Lemmas 5 and 7 imply that incentives of a bidder to approach the seller depend on whether her best outside option is to wait for another bidder to approach or for

the seller to put the asset up for sale. When a bidder expects the seller to sell soon, she benefits from waiting. In the extreme case of an immediate sale, no bidder approaches the seller, as $\Pi_S^*(\hat{s}, \hat{s}) > \Pi_I^*(\hat{s}, \hat{s})$ for any $\hat{s} > 0$. When a bidder expects the rival to approach the seller soon, she benefits from initiating the deal herself. A practical implication for the market of distressed assets is as follows. Bidders are reluctant to approach the seller when they expect him to put the asset up for sale soon regardless of the demand for it, such as when the seller is close to bankruptcy. This intuition holds regardless of whether the asset is commonly or privately valued by market participants.

4.2 The initiation game

Having solved for the equilibria in bidder- and seller-initiated auctions for any cut-off \hat{s} , we next solve for the equilibrium cut-off. Our analysis proceeds in four steps. First, we solve a bidder's problem taking the initiation strategy of the seller and the other bidder as given. Applying the symmetry condition, we obtain the cut-off on the bidder's signal, at which it is indifferent between sending message $m_{it} = 1$ (an indication of interest) and $m_{it} = 0$ (wait) to the seller. Second, we check that bidders with signals above the cut-off prefer to send message $m_{it} = 1$, while bidders with signals below the cut-off prefer to send message $m_{it} = 0$. The first two steps thus characterize equilibrium initiation strategies of both bidders for any given initiation strategy of the seller. Third, we solve the seller's problem of whether to put the auction up for sale upon first receiving $m_{it} = 1$, taking the equilibrium strategy of bidders as given. Fourth, we check that the seller does not prefer to wait for simultaneous indications of interest from both bidders, $m_{it} = 1$ and $m_{-it} = 1$. The last two steps thus characterize an equilibrium initiation strategy of the seller for any given initiation strategy of both bidders. We combine the two strategies to obtain equilibria.

4.2.1 A bidder's problem

Recall that we focus on the stationary case where the distribution of bidders' signals, conditional on no auction having taken place, is uniform over $[0, \hat{s}]$ for some \hat{s} . Stationarity and the restriction to Markov strategies imply that the initiation strategy of the seller is the same at any time t . Let μdt denote the probability with which the seller initiates the

auction during any short period of time $(t, t + dt)$, $\mu \in [\nu, \infty]$. Here, $\mu = \nu$ means that the seller only initiates the auction if he is hit by the shock and has no choice but to sell the asset; $\mu = \infty$ means that the seller initiates the auction over the next instant with probability one; and $\mu \in (\nu, \infty)$ means that the seller auctions the asset off with some intensity even if he is not hit by the shock or approached by a bidder.

First, we fix μ and solve for the symmetric equilibrium initiation strategy of bidders. Suppose that a bidder believes that the rival approaches the seller if and only if her signal exceeds \hat{s} . Consider the bidder with signal s . Denote the expected continuation value of this bidder by $V_B(s, \hat{s}, \mu)$. This value satisfies

$$V_B(s, \hat{s}, \mu) = \max \left\{ \Pi_I^*(s, \hat{s}), \frac{\lambda(1 - \hat{s}) \Pi_N^*(s, \hat{s}) + \mu \Pi_S^*(s, \hat{s})}{r + \lambda(1 - \hat{s}) + \lambda + \mu} \right\}. \quad (12)$$

The intuition behind (12) is as follows. The continuation value is the maximum of the expected payoff from immediate initiation, which yields $\Pi_I^*(s, \hat{s})$, and waiting, which yields the second term of (12). Three independent events can occur if the bidder chooses to wait. First, with intensity $\lambda(1 - \hat{s})$, a rival bidder with signal above \hat{s} indicates the interest to the seller, and the seller reacts by putting the asset for sale. The non-initiating bidder obtains $\Pi_N^*(s, \hat{s})$ in this case. Second, with intensity μ , the seller puts the asset up for sale, and the bidder obtains $\Pi_S^*(s, \hat{s})$. Third, with intensity λ , the bidder experiences a shock and leaves the market.

By continuity of $\Pi_I^*(\cdot)$, $\Pi_N^*(\cdot)$, and $\Pi_S^*(\cdot)$ in s , the cut-off type must satisfy

$$\Pi_I^*(\hat{s}, \hat{s}) = \frac{\lambda(1 - \hat{s}) \Pi_N^*(\hat{s}, \hat{s}) + \mu \Pi_S^*(\hat{s}, \hat{s})}{r + \lambda(1 - \hat{s}) + \lambda + \mu} \Rightarrow \quad (13)$$

$$(r + \lambda) \Pi_I^*(\hat{s}, \hat{s}) + \lambda(1 - \hat{s}) (\Pi_I^*(\hat{s}, \hat{s}) - \Pi_N^*(\hat{s}, \hat{s})) = \mu (\Pi_S^*(\hat{s}, \hat{s}) - \Pi_I^*(\hat{s}, \hat{s})).$$

This transformed equation is intuitive. For the cut-off type \hat{s} , the cost of waiting equals the benefit. The cost of waiting (the left-hand side) comes from two sources. The first source, represented by the first term on the left-hand side of (13), is discounting due to a positive discount rate and the possibility that the bidder leaves the market. The second source, represented by the second term on the left-hand side of (13), is the possibility that a strong

rival bidder appears and initiates the auction, lowering the expected rents of the bidder (as implied by Lemma 5).¹⁷ The benefit of waiting (the right-hand side) comes from the possibility that the seller initiates the auction without being approached by a bidder. A seller-initiated auction leads to a less aggressive bidding by the rival bidder, because she expects the competing bidder to be weaker than in a bidder-initiated auction (as implied by Lemma 7).

Existence of the indifferent type \hat{s} , given by (13), is only a necessary condition for the equilibrium. We also need to verify that if type \hat{s} is indifferent, then all types above \hat{s} find it optimal to approach the seller, while all types below \hat{s} find it optimal to wait. The next proposition shows that this condition holds if and only if μ is below some cut-off level, denoted by $\hat{\mu}(\hat{s})$:

Proposition 2. *Fix $\mu \geq \nu$ and let $\hat{s} < 1$ denote the equilibrium cut-off signal for bidder-initiated auctions. Then, \hat{s} satisfies (13) and*

$$\mu < \hat{\mu}(\hat{s}) \equiv \frac{v^{-1}(\underline{a}(\hat{s}))}{\hat{s} - v^{-1}(\underline{a}(\hat{s}))} (r + \lambda) - \lambda(1 - \hat{s}). \quad (14)$$

If \hat{s} that satisfies (13) and (14), then a bidder finds it optimal to indicate her interest to the seller if and only if its signal exceeds cut-off \hat{s} .

4.2.2 The seller's problem

The previous subsection considered the bidders' problem for a fixed selling strategy of the seller. Next, we fix \hat{s} and consider the seller's problem. The seller's problem is two-fold. First, the seller needs to decide whether to initiate the auction upon receiving messages $m_{i,t} = 0$, $i \in \{1, 2\}$ (no indications of interest). Second, the seller needs to decide whether to auction the asset off upon receiving message $m_{i,t} = 1$ from one of the bidders (an indication of interest).

First, consider the former problem. If the seller follows the strategy of selling voluntarily with intensity $\mu - \nu$ (making the total probability of a seller-initiated auction over an instant

¹⁷Recall that this term is a benefit rather than cost in the case of common values: there, the bidder benefits when a rival bidder with a high signal appears and initiates the auction.

μdt), then the seller's expected payoff is:

$$\frac{\mu R_S(\hat{s}) + 2\lambda(1 - \hat{s}) R_B(\hat{s})}{r + \mu + 2\lambda(1 - \hat{s})},$$

where $R_S(\hat{s})$ and $R_B(\hat{s})$ are the expected revenues in a seller-initiated and bidder-initiated auction, respectively. They admit simple integral representations:

$$R_S(\hat{s}) = \int_0^{a_S(\hat{s})} bd \left(\left(\frac{\phi_S(b, \hat{s})}{\hat{s}} \right)^2 \right), \quad R_B(\hat{s}) = \int_{\underline{a}(\hat{s})}^{\bar{a}(\hat{s})} bd \left(\frac{\phi_I(b, \hat{s}) - \hat{s} \phi_N(b, \hat{s})}{1 - \hat{s}} \frac{1}{\hat{s}} \right). \quad (15)$$

The derivative of this payoff in μ is positive if and only if $R_S(\hat{s}) > \frac{2\lambda(1-\hat{s})}{r+2\lambda(1-\hat{s})} R_B(\hat{s})$, i.e., if and only if it takes a long enough time for a bidder to indicate her interest to the seller. Because $R_B(\hat{s}) > R_S(\hat{s})$, the seller faces the trade-off between delay and lower expected revenues.

Second, consider the seller's problem whether to auction the asset off upon receiving an indication of interest. In the responsive equilibrium of the cheap-talk game, the seller's optimal strategy is to put the asset up for sale upon receiving message $m_{i,t} = 1$ from one of the bidders. In this case, the seller obtains $R_B(\hat{s})$. Alternatively, the seller could deviate and not auction the asset off upon receiving this message but instead wait until both bidders present at that time send indications of interest ($m_{1,t} = m_{2,t} = 1$). By doing this, the seller could obtain higher revenues from the auction at the cost of additional waiting. In the appendix (see the next proposition), we show that the condition that the seller is better off holding an auction upon receiving an indication of interest simplifies to (43).

The next proposition summarizes the best response of the seller:

Proposition 3. *Fix \hat{s} that satisfies condition (43) in the appendix. Then, the seller's optimal strategy is: (a) to never initiate the auction unless he is hit by a liquidity shock ($\mu = \nu$) if $\frac{2\lambda(1-\hat{s})}{r+2\lambda(1-\hat{s})} R_B(\hat{s}) \geq R_S(\hat{s})$; (b) to initiate the auction immediately ($\mu = \infty$) if $\frac{2\lambda(1-\hat{s})}{r+2\lambda(1-\hat{s})} R_B(\hat{s}) \leq R_S(\hat{s})$; (c) to randomize between initiating the auction and waiting (any $\mu \in (\nu, \infty)$) if $\frac{2\lambda(1-\hat{s})}{r+2\lambda(1-\hat{s})} R_B(\hat{s}) = R_S(\hat{s})$.*

4.2.3 Equilibria

Combining the previous derivations, we can characterize all equilibrium cut-off signals \hat{s} . Because initiation decisions of bidders are strategic complements, there can be multiple equilibria. If a bidder expects other bidders to only approach the seller if s is very high (i.e., $\hat{s} \uparrow 1$), then payoffs of the initiating and non-initiating bidder with the cut-off signal are close. In turn, a bidder has weak incentives to approach the seller. In contrast, if \hat{s} is sufficiently low so that $\Pi_I^*(\hat{s}, \hat{s}) - \Pi_N^*(\hat{s}, \hat{s})$ is high, a bidder has strong incentives to approach. Note that this argument is the opposite of that in the common-value model. There, initiation decisions of bidders are strategic substitutes: if a bidder expects her rival to approach the seller, she has weak incentives to approach herself. The next proposition shows the existence of at least one equilibrium:

Proposition 4 (equilibrium with only seller-initiated auctions). *There always exists an equilibrium in which all auctions are seller-initiated: $\hat{s} = 1$ and $\mu = \infty$.*

Intuitively, if the seller believes that bidders will not approach, then delaying the seller-initiated sale is costly. Similarly, if bidders believe that the seller will put the asset up for sale immediately, approaching the seller earlier is costly.

To have both seller- and bidder-initiated auctions in equilibrium at the same time, the seller must play mixed strategies. Equivalently, the seller must be indifferent between initiating the auction himself and waiting until he is approached by a bidder.

Proposition 5 (equilibria with seller- and bidder-initiated auctions). *A pair (\hat{s}, μ) with $\hat{s} < 1$ and $\mu \geq \nu$ constitutes an equilibrium if and only if it satisfies (13), $\mu < \hat{\mu}(\hat{s})$, (43), and: (a) $\frac{2\lambda(1-\hat{s})}{r+2\lambda(1-\hat{s})}R_B(\hat{s}) = R_S(\hat{s})$, if $\mu > \nu$; (b) $\frac{2\lambda(1-\hat{s})}{r+2\lambda(1-\hat{s})}R_B(\hat{s}) \geq R_S(\hat{s})$, if $\mu = \nu$.*

Condition (a) of Proposition 5 covers the case when the seller sometimes voluntarily auctions the asset off without being approached by the bidder. The indifference condition there states that the seller must be indifferent between a seller-initiated auction and a

delayed bidder-initiated auction. This condition is independent of μ and pins down the equilibrium cut-off \hat{s} . Then, given \hat{s} , (13) pins down equilibrium μ . Condition (b) of Proposition 5 covers the case when the seller only auctions the asset off upon either receiving an indication of interest from the bidder or being hit by a liquidity shock.

Example 3 in the appendix considers one set of parameters that gives a single equilibrium with seller initiation only ($v(s_i) = s_i$) and another set of parameters that gives an additional equilibrium with both bidder and seller initiation ($v(s_i) = s_i^2$). Figure 3 illustrates equilibria for the second case, $\hat{s} = 1$ and $\mu = \infty$, and $\hat{s} = 0.977$ and $\mu = 0.752$, when $r = 0.05$ and $\lambda = 0.5$.

Our analysis of the base model shows that initiation of private- and common-value auctions can be drastically different. While both auctions, in equilibrium, can be voluntarily initiated by sellers, only private-value auctions can be initiated by bidders. Furthermore, as Figure 3 illustrates, in the private-value setting the initiation game can have multiple equilibria, some of which feature infrequent voluntary initiation by the seller or no voluntary initiation at all. This result suggests that different markets with otherwise similar characteristics can feature very different asset sale initiation patterns and, as a consequence, different allocation and welfare properties.

5 Extensions and Discussion

The base model is designed to explain broad initiation patterns of endogenous auctions with our lead application being the market for corporate control. In this section, we discuss additional features of this market and analyze whether they can alleviate the problem that bidders are not willing to approach sellers in common-value auctions. Additionally, we examine robustness of our results to various model assumptions.

5.1 Robustness to auction format

We assume that the sale proceeds as a first-price auction for two reasons. First, under regularity conditions, it has a unique equilibrium even when signals of bidders are distributed asymmetrically. In contrast, the difficulty with analyzing the second-price auction in the

common value setting is that there exists a continuum of equilibria (see Milgrom, 1981b). When bidders are symmetric, it is natural to focus on the equilibrium in which both bidders play identical strategies, which is unique. However, when bidders are asymmetric, as is the case here, there is no well-established equilibrium selection criterion that selects one equilibrium from the continuum. Second, a first-price auction highlights how bidders respond to the perceived aggressiveness of the rival bidder. Such strategic considerations are common to a variety of auction formats but are often absent from the second-price auction.¹⁸

In this section, we extend the model by assuming that the sale may proceed as a second-price auction. We consider the cases of common values and private values in sequence.

The case of common values. Consider a bidder-initiated second-price auction, in which the initiating bidder's signal is drawn from $[\hat{s}, 1]$ and the non-initiating bidder's signal is drawn from $[0, \hat{s}]$. Among a continuum of equilibria, we restrict our attention to natural equilibria in which bidders only use undominated bids. We show that all such equilibria have the same key property as the equilibrium in our base model: the equilibrium payoff of the initiating bidder with the cut-off signal is zero, regardless of the cut-off. Thus, no bidder communicates her interest to the seller, and the seller puts the asset for sale immediately at date 0:

Proposition 6 (common values, second-price auction). *Suppose that bidders only submit undominated bids at the auction stage. Then: (a) in any equilibrium of the bidder-initiated auction, type $s \leq \hat{s}$ of the non-initiating bidder does not bid below $v(s, \hat{s})$; (b) $\Pi_I^*(\hat{s}, \hat{s}) = 0 \forall \hat{s}$, i.e., the expected payoff of the cut-off type of the initiating bidder \hat{s} is zero in any equilibrium of the bidder-initiated auction; (c) in the initiating game, there exists a unique equilibrium cut-off $\hat{s} = 1$. That is, no bidder communicates her interest to the seller, and the seller puts the asset for sale immediately at date 0.*

¹⁸In practice, the actual format differs across auctions. Typically, there are multiple rounds of informal bidding, in which bids are contingent on further due diligence and acquisition of financing and can be retracted, followed by few (often one) rounds of formal bidding, which proceeds as a first-price auction (see Hansen (2001) for a description). Avery (1998) shows that the outcomes of such a format are close to those in the first-price auction.

The case of private values. Each bidder i has a weakly dominant strategy to bid $v(s_i)$. First, consider a bidder-initiated auction. As in the base model, let $\Pi_j^*(s, \hat{s})$, $j \in \{I, N\}$ denote the expected payoff of the initiating and non-initiating bidder with signal s when the types \hat{s} and above initiate the auction. Clearly, $\Pi_N^*(s, \hat{s}) = 0$ for any $s \leq \hat{s}$ and any $\hat{s} \in [0, 1]$. At the same time, $\Pi_I^*(s, \hat{s}) = v(s) - \mathbb{E}[v(x) | x \in [0, \hat{s}]]$. Second, consider a seller-initiated auction. The expected payoff of a bidder is $\Pi_S^*(s, \hat{s}) = \int_0^s \frac{x}{\hat{s}} v'(x) dx$. For the cut-off type \hat{s} , we have the following relationship:

$$\Pi_S^*(\hat{s}, \hat{s}) = v(\hat{s}) - \mathbb{E}[v(x) | x \in [0, \hat{s}]] = \Pi_I^*(\hat{s}, \hat{s}) > \Pi_N^*(\hat{s}, \hat{s}) = 0.$$

Unlike in the first-price auction, we do not have the result that $\Pi_S^*(\hat{s}, \hat{s}) > \Pi_I^*(\hat{s}, \hat{s})$ (Lemma 7). Intuitively, in the first-price auction the non-initiating bidder competes more aggressively in a bidder-initiated auction than in a seller-initiated auction, because she expects the rival to be stronger. In contrast, in the second-price auction, a bidder's bidding strategy is independent of her expectations of how strong the rival bidder is. The equation that determines the equilibrium at the initiating game becomes:

$$\Pi_I^*(\hat{s}, \hat{s}) = \frac{\mu \Pi_S^*(\hat{s}, \hat{s})}{r + \lambda(1 - \hat{s}) + \lambda + \mu}, \quad (16)$$

which holds only for $\hat{s} = 0$.

Consider the private-value setting in which the seller randomizes between the two mechanisms. Specifically, assume that the seller holds the first-price auction with probability q and the second-price auction with probability $1 - q$. We denote $\Pi_j^{SPA}(s, \hat{s})$ and $\Pi_j^{FPA}(s, \hat{s})$ for $j \in \{I, N, S\}$ to be the equilibrium expected payoffs of type s from the second-price and first-price auctions, respectively. Then, the expected payoff from the auction prior to learning its format is: $\Pi_j^*(s, \hat{s}) = q\Pi_j^{FPA}(s, \hat{s}) + (1 - q)\Pi_j^{SPA}(s, \hat{s})$. By analogies with Lemma 5 and 7, we have that for any $q > 0$ and $\hat{s} < 1$, $\Pi_S^*(\hat{s}, \hat{s}) > \Pi_I^*(\hat{s}, \hat{s}) > \Pi_N^*(\hat{s}, \hat{s})$. It follows that the equilibrium cut-off type \hat{s} is still characterized by Propositions 4 and 5 of the base model.

To sum up, this section establishes that our main results, which were established as-

suming that the seller holds the first-price auction, are robust to the possibility of the second-price auction. First, in the case of the common values and under a weak restriction on equilibrium bids, the implications for initiation are identical. Second, in the case of private values, the implications for initiation are identical provided that bidders contemplate a possibility that the seller will hold the first-price auction, $q > 0$. We only get qualitatively different implications for initiation when $q = 0$. However, this belief is unnatural for two reasons. First, there is significant heterogeneity and likely unpredictability of selling mechanisms of different firms (Boone and Mulherin, 2007). Second, theoretically, neither the first-price nor the second-price auction dominates the other in seller's expected revenues when bidders are asymmetric (Krishna, 2010).

5.2 Shareholder activists as facilitators of auctions of companies

Corollary 1 shows that if a company is underperforming (inducing common value in potential bidders), its management is entrenched, and the likelihood of a liquidity shock is low, such company can remain without a change in ownership for a long time, even when the value added from the change is large. Thus, the role of takeovers as a corporate governance mechanism can be limited.

The lack of an incentive for bidders to initiate common-value auctions gives rise to alternative ways of promoting takeovers, in particular, to shareholder activism. If an activist finds it beneficial to buy a fraction of the company's shares and undertake an activism campaign, which results in putting the target up for sale, seller-initiated takeovers can occur in equilibrium even in the presence of an entrenched management.

In the appendix, we extend the base model by allowing a profit-maximizing activist to enter the market and characterize the activist's choice in an equilibrium with seller initiation. Our analysis shows that shareholder activism and the market for corporate control are not two unrelated mechanisms for disciplining the management but rather complement each other: activists use the market for corporate control to facilitate transactions of targets, inefficiencies in which would not be corrected otherwise.

5.3 Toeholds

In common-value auctions of companies, the initiating bidder with the cut-off signal can obtain a positive expected profit if she secretly acquires a toehold in the target prior to the auction. Such a bidder can thus find it optimal to indicate her interest to the target in the first place.

In the appendix, we extend our base model to allow for endogenous toehold acquisition, derive equilibrium bidding strategies of an initiating and non-initiating bidder, and show that $\Pi_I^*(\hat{s}, \hat{s}) > 0$ for $\hat{s} > 0$, implying the possibility of equilibria with bidder initiation. Example 4 in the appendix provides a quasi-closed form solution for equilibrium bidding strategies for the case $v(s_i, s_{-i}) = \frac{1}{2}(s_i + s_{-i})$. Additionally, for $r = 0.05$ and $\lambda = 0.5$, the example illustrates the initiation-stage equilibrium with bidder initiation only. Our analysis shows that while toeholds are often considered to be a source of inefficiency, because in static models of auctions they can result in the acquisition of an asset by a bidder with a lower valuation, they help bidders initiate positive-value deals that would not occur otherwise in our dynamic model. Because toeholds can be valuable, it is important to further study optimal disclosure requirements of blocks of shares in a dynamic setting.

5.4 Preemption and participation costs

The base model assumes that upon initiation of the auction, both bidders enter it. In practice, participating in the auction can be costly, and thus bidders with sufficiently low signals may prefer to avoid entering the auction. In general, analyzing the effects of participation costs properly is a difficult problem, because the implications can be quite specific to the assumptions of the setting, in particular to whether the seller can reimburse the participation costs of the bidders¹⁹ and whether the entry of other bidders is observable. In this section, we analyze one particular extension of the model: common values, no reimbursement of participation costs, and observable entry. The complete analysis of the effects of participation costs on initiation across many different environments is beyond the scope of

¹⁹Because participation has a positive externality on the seller, the seller benefits from subsidizing participation. Compensation of due diligence costs is quite common in M&A transactions - see Wang (2018) for examples.

this work.

Suppose that a bidder has to pay a cost $C > 0$ to participate in the auction. A positive participation cost implies a possibility of a single-bidder auction. This leads to the question of how the transaction price is determined in this case, and whether it is different upon the seller receiving a liquidity shock, at which point it has to involuntarily sell the asset.²⁰ It is natural to assume that the transaction price equals to the seller's reservation value and that this reservation value declines when he is hit by the shock. Let the seller's reservation value in these two cases be $v_s > 0$ and 0. We keep the same assumptions for the bidder's signals and valuations but shift the valuation by v_s , so that $v(0, 0) = v_s$. The next proposition shows the main result of the extension:

Proposition 7. *Suppose that $C < \mathbb{E}[v(1, x)] - v_s$. Then, the only equilibrium of the initiation game has cut-off $\hat{s} = 1$. That is, no bidder ever communicates her interest to the seller, and the seller puts the asset for sale immediately at date 0. Each bidder enters this seller-initiated auction if and only if her signal exceeds cut-off \hat{s}_S , defined as:*

$$\int_0^{\hat{s}_S} (v(\hat{s}_S, x) - v_s) dx = C. \quad (17)$$

The restriction on C is weak. It simply means that the participation cost does not exceed the surplus from the auction when one bidder's signal is the highest possible. The intuition behind this result is as follows. Because of positive participation costs, a bidder-initiated auction results in the rival bidder not entering the auction by the unraveling argument similar to that in the base model. Because of low entry, a bidder-initiated auction results in low seller's revenues, and thus from the seller's point of view there is no benefit from waiting until he gets approached by the bidder. Thus, in equilibrium the auction is initiated immediately by the seller.

²⁰These questions were irrelevant in the base model, because both bidders always participated and the seller's reservation value was $0 \leq v(s_1, s_2) \forall s_1, s_2$.

5.5 Investment banks

Our results in the common-value framework rely on the assumption that whether the auction is bidder- or seller-initiated is known by auction participants. While it is in the ex-post interest of the seller to disclose the interest of the initiating bidder to other bidders, such disclosure can be ex-ante suboptimal for all parties combined as it impedes initiation. This inconsistency between ex-ante and ex-post objectives can create a role for an intermediary, such as an investment bank, to alleviate the lack-of-commitment problem. The investment bank can centralize communication among all participating parties, and, because it is a long-run player that interacts with buyers and sellers over time across a range of different services, can incentivize the seller to not disclose the ex-post beneficial but ex-ante harmful, for all parties combined, information in the context of a single transaction.²¹

5.6 Non-stationary dynamics

The stationary solution restriction of Section 4 does not inform on how quickly the initiation game reaches stationarity. If the process is slow, the agents are likely to resolve the game using considerably different strategies. To examine non-stationarity, we specialize to the case of private values (the case of common values still results in no bidder initiation). It is reasonable to assume that at time $t = 0$, both currently present bidders start with unrestrictedly distributed signals on $[0, 1]$. In the online appendix, we show that there exists an equilibrium of the initiation game, in which the cut-off signal, \hat{s}_t , is decreasing over time, such that $\hat{s}_0 = 1$ and $\lim_{t \rightarrow \infty} \hat{s}_t = \hat{s}$, guaranteeing convergence to the stationary solution. Examples 6 and 7 in the online appendix illustrate the solution. For $r = 0.05$ and $\lambda = 0.5$ used throughout the paper, convergence is fast: for example, when the frequency of involuntary seller initiation is $\nu = 0.75$, 50% of the distance between \hat{s}_0 and \hat{s} is covered in 7.5 years. For larger ν , this number can be substantially lower. Hence, it appears that the focus on the stationary solution in most cases is not overly restrictive.

²¹Hiding whether the auction is bidder- or seller-initiated gives positive rents to the initiating bidder with the cut-off signal, because the rival believes that, with some probability, the auction is seller-initiated and the initiating bidder's signal is below the cut-off.

6 Empirical Implications

The model delivers many empirical implications. In particular, they are testable in the context of auctions of companies, because U.S. public targets must file deal backgrounds, which include information on initiation, with the SEC. We split implications into two groups: about bidding in bidder- versus seller-initiated auctions and about initiation.

6.1 Implications about bidding

Consider a private-value setting, in which auctions can be either bidder- or seller-initiated (i.e., $\hat{s} < 1$ and $\mu \in (0, \infty)$). How is bidding by the initiating bidder different from bidding by the non-initiating bidder? And how is bidding in a seller-initiated auction different from bidding in a bidder-initiated auction? The model delivers the following implications:

1. *In bidder-initiated deals, the initiating bidder is stronger (has, on average, higher valuations) than the other bidders.*
2. *In bidder-initiated deals, conditional on the same valuation, the non-initiating bidder bids more aggressively than the initiating bidder: $a_N(\hat{s}, \hat{s}) > a_I(\hat{s}, \hat{s})$.*
3. *In bidder-initiated deals, unconditionally on the exact valuation, the initiating bidder bids more aggressively and wins more often: $\mathbb{E}[a_I(s, \hat{s}) | s \geq \hat{s}] > \mathbb{E}[a_N(s, \hat{s}) | s \leq \hat{s}]$.*
4. *All else equal, bidders in seller-initiated auctions are weaker (have lower valuations) than bidders in bidder-initiated auctions.*
5. *Conditional on the same valuation, bidders bid less aggressively in seller-initiated deals.*

6.2 Implications about initiation

The model links the valuation structure to whether the auction is likely to be seller- or bidder-initiated. In particular, assuming that a takeover battle between strategic (financial) bidders is represented by the private-value (common-value) framework, the model delivers the following implications:

1. *Contests among financial bidders are more likely to be seller-initiated than contests among strategic bidders.*
2. *Contests among financial bidders for a target with the entrenched board and management are often initiated by an activist investor pressuring the board to sell the company.*
3. *In bidder-initiated acquisitions by financial bidders, the initiating bidder is likely to have a toehold.*
4. *If due diligence is costly (e.g., if the target is complex) and the target does not reimburse the costs, a bidder-initiated common-value auction is likely to be uncontested.*

7 Conclusion

In this paper, we theoretically examine endogenous initiation of a first-price auction by potential buyers and the seller. Our model aims to capture many real-world environments in which initiation of an auction is a strategic choice. Our key application is to corporate takeovers and intercorporate asset sales. We show that in common-value auctions, such as battles between financial bidders for the target company, bidders are reluctant to approach the seller, because this decision erodes their information rents. In pure common-value auctions, this effect is extreme: no bidder ever approaches, and auctions are initiated by the seller, if at all. By contrast, in private-value auctions, the effect can be opposite: observing that no bidder has approached the seller yet reveals information that rivals are weak, which incentivizes sufficiently strong bidders to approach. Bidder and seller initiation become substitutes, which leads to multiple equilibria with different initiation patterns. We extend the model to include various features of the market for corporate control and show that activist investors and, surprisingly, toeholds held by bidders in the target can alleviate the inefficiencies in common-value auctions and result in more positive-value deals.

Two applied extensions of the paper could be interesting. First, it can be useful to consider bids in securities, such as a bidder's stock, and the interaction of securities used in a bid and initiation decisions. Second, the asset for sale can be made divisible: for

example, bankrupt companies are often sold piecemeal in a liquidation auction. A more general extension is to consider multiple sellers with similar assets for sale and allow bidders to choose which asset to pursue and sellers to choose which bidders to invite to an auction in a dynamic model of matching.

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Appendix

7.1 Additional proofs

Proof of Lemma 3. Denote the inverse of $a_S(s, \hat{s})$ in s by $\phi_S(b, \hat{s})$. The expected payoff of a bidder with signal s and bid b is

$$\Pi_S(b, s, \hat{s}) = \mathbb{E}[v(s, x) - b | x \in [0, \phi_S(b, \hat{s})]] = \int_0^{\phi_S(b, \hat{s})} (v(s, x) - b) \frac{1}{\hat{s}} dx. \quad (18)$$

The logic behind (18) and (1) is similar. A bidder with bid b wins the auction if and only if the signal of the rival bidder is below $\phi_S(b, \hat{s})$. Conditional on winning when the rival's signal is $x \in [0, \phi_S(b, \hat{s})]$, the value of the asset to the bidder is $v(s, x)$. Integrating over $x \in [0, \phi_S(b, \hat{s})]$ yields (18). Taking the first-order condition and using the fact that in equilibrium $b = a_S(s, \hat{s})$ (or, equivalently, $s = \phi_S(b, \hat{s})$), we obtain

$$\left(\frac{\partial a_S(s, \hat{s})}{\partial s} \right)^{-1} (v(s, s) - a_S(s, \hat{s})) - s = 0. \quad (19)$$

This equation is solved subject to the boundary condition $a_S(0, \hat{s}) = v(0, 0)$, or, equivalently, $0 = \phi_S(v(0, 0), \hat{s})$. Intuitively, a bidder with the lowest signal only wins against the rival with the lowest signal. Upon winning, she re-evaluates the asset to $v(0, 0)$. Because $v(0, 0)$ is the lowest possible asset value, bidders with the lowest signal bid exactly $v(0, 0)$ in equilibrium.

Re-write (19) as

$$s \frac{\partial a_S(s, \hat{s})}{\partial s} + a_S(s, \hat{s}) = v(s, s) \quad \Rightarrow \quad \frac{d(a_S(s, \hat{s}))}{ds} = v(s, s). \quad (20)$$

Integrating and applying the initial value condition $a_S(0, \hat{s}) = v(0, 0)$ yields (5).

Proof of Lemma 5. The equilibrium payoff of bidder I with signal \hat{s} is

$$\begin{aligned}\Pi_I^*(\hat{s}, \hat{s}) &= \frac{\phi_N(\underline{a}(\hat{s}), \hat{s})}{\hat{s}} (v(\hat{s}) - \underline{a}(\hat{s})) = \max_{b \in [\underline{a}(\hat{s}), \bar{a}(\hat{s})]} \frac{\phi_N(b, \hat{s})}{\hat{s}} (v(\hat{s}) - b) \\ &\geq \frac{\phi_N(\bar{a}(\hat{s}), \hat{s})}{\hat{s}} (v(\hat{s}) - \bar{a}(\hat{s})) = v(\hat{s}) - \bar{a}(\hat{s}) = \Pi_N^*(\hat{s}, \hat{s}).\end{aligned}\quad (21)$$

Therefore, $\Pi_I^*(\hat{s}, \hat{s}) \geq \Pi_N^*(\hat{s}, \hat{s})$. Moreover, if $\hat{s} < 1$, then $a_I(\hat{s}, \hat{s}) = \underline{a}(\hat{s}) < \bar{a}(\hat{s})$. Therefore, the inequality in (21) is strict.

Proof of Lemma 6. Denote the inverse of $a_S(s, \hat{s})$ in s by $\phi_S(b, \hat{s})$. The expected payoff of a bidder with signal s and bid b is

$$\Pi_S(b, s, \hat{s}) = \mathbb{E}[v(s) - b | x \in [0, \phi_S(b, \hat{s})]] = (v(s) - b) \frac{\phi_S(b, \hat{s})}{\hat{s}}. \quad (22)$$

Taking the first-order condition and using the fact that in equilibrium, $b = a_S(s, \hat{s})$ (or, equivalently, $s = \phi_S(b, \hat{s})$), we obtain

$$\left(\frac{\partial a_S(s, \hat{s})}{\partial s} \right)^{-1} (v(s) - a_S(s, \hat{s})) - s = 0. \quad (23)$$

This equation is solved subject to the boundary condition $a_S(0, \hat{s}) = v(0)$. Applying the transformation similar to that in Lemma 3 yields (11).

Proof of Lemma 7. The equilibrium payoff of a bidder with signal \hat{s} in a seller-initiated auction is

$$\begin{aligned}\Pi_S^*(\hat{s}, \hat{s}) &= \Pi_S(a_S(\hat{s}), \hat{s}, \hat{s}) \geq \Pi_S(\underline{a}(\hat{s}), \hat{s}, \hat{s}) = \frac{\phi_S(\underline{a}(\hat{s}))}{\hat{s}} (v(\hat{s}) - \underline{a}(\hat{s})) \\ &\geq \frac{v^{-1}(\underline{a}(\hat{s}))}{\hat{s}} (v(s) - \underline{a}(\hat{s})) = \Pi_I^*(\hat{s}, \hat{s}),\end{aligned}\quad (24)$$

with the strict inequality if $\phi_S(\underline{a}(\hat{s}), \hat{s}) > v^{-1}(\underline{a}(\hat{s}))$. The last inequality follows from $a_S(s) \leq v(s)$, which implies $\phi_S(b) \geq v^{-1}(b)$. Intuitively, no bidder bids above his valuation, so if a bidder bids b , then her signal is at least $v^{-1}(b)$.

Proof of Proposition 1. First, suppose that $\hat{s} \in (0, 1)$, and consider a bidder whose signal is $\hat{s} + \varepsilon$ for an infinitesimal positive ε . By sending message $m_{i,t} = 1$ and initiating the auction immediately, the bidder obtains the expected payoff of $\Pi_I^*(\hat{s} + \varepsilon, \hat{s}) \equiv \Pi_I(a_I(\hat{s} + \varepsilon, \hat{s}), \hat{s} + \varepsilon, \hat{s})$, which converges to zero as $\varepsilon \rightarrow 0$. Consider a deviation to the following strategy: the bidder waits until either the seller or the rival initiates the auction. The expected payoff from this strategy is

$$\frac{\lambda(1 - \hat{s}) \Pi_N^*(\hat{s} + \varepsilon, \hat{s}) + \mu \Pi_S^*(\hat{s} + \varepsilon, \hat{s})}{r + \lambda(1 - \hat{s}) + \mu + \lambda}, \quad (25)$$

where $\mu dt \geq \nu dt$ is the probability with which the seller initiates the auction at any short period $(t, t + dt)$ in the stationary equilibrium, $\Pi_N^*(\hat{s} + \varepsilon, \hat{s}) \equiv \max_b \Pi_N(b, \hat{s} + \varepsilon, \hat{s}) \geq \Pi_N^*(\hat{s}, \hat{s})$ and $\Pi_S^*(\hat{s} + \varepsilon, \hat{s}) \equiv \max_b \Pi_S(b, \hat{s} + \varepsilon, \hat{s}) \geq \Pi_S^*(\hat{s}, \hat{s})$ for any $s \in [\hat{s}, 1]$. Because $\Pi_N^*(\hat{s}, \hat{s})$ and $\Pi_S^*(\hat{s}, \hat{s})$ are strictly positive for any $\hat{s} > 0$, $\Pi_N^*(\hat{s} + \varepsilon, \hat{s})$ and $\Pi_S^*(\hat{s} + \varepsilon, \hat{s})$ are strictly positive and bounded away from zero. Therefore, because r is finite, the bidder gets a strictly higher expected payoff from this deviation than from the conjectured equilibrium strategy for any $\mu \geq \nu$.

Second, consider $\hat{s} = 0$ and suppose that the seller receives message $m_{i,t} = 1$ from bidder i at time t . If a responsive equilibrium with cut-off $\hat{s} = 0$ exists, the seller must be better off auctioning the asset off immediately upon receipt of such message. If the seller chooses to auction the asset off, the non-initiating bidder bids $v(0, 0)$ with probability one (because its signal is zero). Hence, the seller's revenues from an immediate sale are $v(0, 0) = 0$. In contrast, if the seller deviates to auctioning the asset off upon receiving messages $m_{1,t} = m_{2,t} = 1$ from both bidders present at that time, then her expected revenues would be $\mathbb{E}[\mathbb{E}[v(x, x) | x \leq \max\{s_1, s_2\}]] > 0$. The profitability of the deviation shows that there is no responsive equilibrium with $\hat{s} = 0$.

Proof of Proposition 2. Let \hat{s} be the signal of a bidder indifferent between sending message $m_{i,t} = 1$ and initiating the auction and sending message $m_{i,t} = 0$ and delaying the auction (i.e., \hat{s} satisfies (13)). A bidder with signal $s > \hat{s}$ finds it optimal to send $m_{i,t} = 1$ and a bidder with signal $s < \hat{s}$ finds it optimal to send $m_{i,t} = 0$ if and only if:

$$\Pi_I^*(s, \hat{s}) \geq \frac{\lambda(1 - \hat{s}) \Pi_N^*(s, \hat{s}) + \mu \Pi_S^*(s, \hat{s})}{r + \lambda(1 - \hat{s}) + \lambda + \mu} \quad (26)$$

for any $s > \hat{s}$, and

$$\Pi_I^*(s, \hat{s}) \leq \frac{\lambda(1 - \hat{s}) \Pi_N^*(s, \hat{s}) + \mu \Pi_S^*(s, \hat{s})}{r + \lambda(1 - \hat{s}) + \lambda + \mu} \quad (27)$$

for any $s < \hat{s}$. First, consider $s > \hat{s}$. In this range,

$$\Pi_N^*(s, \hat{s}) = v(s) - \bar{a}(\hat{s}) \quad \Rightarrow \quad \frac{\partial \Pi_N^*(s, \hat{s})}{\partial s} = v'(s); \quad (28)$$

$$\Pi_S^*(s, \hat{s}) = v(s) - a_S(\hat{s}) \quad \Rightarrow \quad \frac{\partial \Pi_S^*(s, \hat{s})}{\partial s} = v'(s). \quad (29)$$

Thus, the derivative of the right-hand side of (26) in s is

$$\frac{\partial RHS}{\partial s} = \frac{\lambda(1 - \hat{s}) + \mu}{r + \lambda(1 - \hat{s}) + \lambda + \mu} v'(s) \quad \forall s > \hat{s}. \quad (30)$$

The left-hand side of (26) is equal to

$$\Pi_I^*(s, \hat{s}) = \max_b \frac{\phi_N(b, \hat{s})}{\hat{s}} (v(s) - b). \quad (31)$$

By the envelope theorem, the derivative of the left-hand side of (26) in s is

$$\frac{\partial \Pi_I^*(s, \hat{s})}{\partial s} = \frac{\phi_N(a_I(s, \hat{s}), \hat{s})}{\hat{s}} v'(s). \quad (32)$$

Whenever $LHS(s) = RHS(s)$, a bidder with signal s finds it optimal to send $m_{i,t} = 1$ if and only if

$$\frac{\phi_N(a_I(s, \hat{s}), \hat{s})}{\hat{s}} > \frac{\lambda(1 - \hat{s}) + \mu}{r + \lambda(1 - \hat{s}) + \lambda + \mu}. \quad (33)$$

The left-hand side of (33) is increasing in s ($a_I(s, \hat{s})$ is increasing in s , so $\phi_N(a_I(s, \hat{s}), \hat{s})$ is also increasing in s), while the right-hand side is constant in s . Thus, it is sufficient to verify that the inequality holds for $s \downarrow \hat{s}$. Note that $a_I(\hat{s}, \hat{s}) = \underline{a}(\hat{s})$. Then, $\phi_N(\underline{a}(\hat{s}), \hat{s}) = v^{-1}(\underline{a}(\hat{s}))$. Thus, (26) holds for all $s > \hat{s}$ if and only if

$$\frac{v^{-1}(\underline{a}(\hat{s}))}{\hat{s}} > \frac{\lambda(1 - \hat{s}) + \mu}{r + \lambda(1 - \hat{s}) + \lambda + \mu}. \quad (34)$$

Next, consider $s < \hat{s}$. In this range,

$$\begin{aligned} \Pi_I^*(s, \hat{s}) &= \max \left\{ \frac{v^{-1}(\underline{a}(\hat{s}))}{\hat{s}} (v(s) - \underline{a}(\hat{s})), 0 \right\} \\ \Rightarrow \frac{\partial \Pi_I^*(s, \hat{s})}{\partial s} &= \frac{v^{-1}(\underline{a}(\hat{s}))}{\hat{s}} v'(s) \text{ if } v(s) > \underline{a}(\hat{s}). \end{aligned} \quad (35)$$

Applying the envelope theorem,

$$\begin{aligned} \Pi_N^*(s, \hat{s}) &= \max \left\{ \max_b \frac{\phi_I(b, \hat{s}) - \hat{s}}{1 - \hat{s}} (v(s) - b), 0 \right\} \\ \Rightarrow \frac{\partial \Pi_N^*(s, \hat{s})}{\partial s} &= \frac{\phi_I(a_N(s, \hat{s}), \hat{s}) - \hat{s}}{1 - \hat{s}} v'(s) \text{ if } v(s) > \underline{a}(\hat{s}); \end{aligned} \quad (36)$$

$$\Pi_S^*(s, \hat{s}) = \max_b \frac{\phi_S(b, \hat{s})}{\hat{s}} (v(s) - b) \Rightarrow \frac{\partial \Pi_S^*(s, \hat{s})}{\partial s} = \frac{s}{\hat{s}} v'(s). \quad (37)$$

Thus, the derivative of the right-hand side of (26) in s is

$$\frac{\partial RHS}{\partial s} = \frac{\lambda(1 - \hat{s})}{r + \lambda(1 - \hat{s}) + \lambda + \mu} \frac{\phi_I(a_N(s, \hat{s}), \hat{s}) - \hat{s}}{1 - \hat{s}} + \mu \frac{s}{\hat{s}} v'(s). \quad (38)$$

Whenever $LHS(s) = RHS(s)$, a bidder with signal s finds it optimal to send $m_{i,t} = 0$ if and only if

$$\frac{v^{-1}(\underline{a}(\hat{s}))}{\hat{s}} > \frac{\lambda(1 - \hat{s})}{r + \lambda(1 - \hat{s}) + \lambda + \mu} \frac{\phi_I(a_N(s, \hat{s}), \hat{s}) - \hat{s}}{1 - \hat{s}} + \mu \frac{s}{\hat{s}}, \quad (39)$$

The right-hand side of (39) is strictly increasing in s , while the left-hand side is constant in s . Thus, it is sufficient to verify that the inequality holds for $s \uparrow \hat{s}$. Note that $\lim_{s \uparrow \hat{s}} a_N(s, \hat{s}) = \bar{a}(\hat{s})$. Then, $\lim_{s \uparrow \hat{s}} \phi_I(a_N(s, \hat{s}), \hat{s}) = \phi_I(\bar{a}(\hat{s}), \hat{s}) = 1$. Thus, the equilibrium condition on \hat{s} and μ , again, simplifies to (34), which is equivalent to (14). Thus we have the “if and only if” result: if μ is below the cut-off in (14), then \hat{s} satisfying (13) yields a cut-off equilibrium (given μ), while if μ is above the cut-off in (14) (except $\mu = \infty$), then no cut-off equilibrium exists.

Proof of Proposition 3. It is instructive to first specify bidders’ information sets when the

seller is approached by a bidder. First, we assume that if the seller is approached but decides not to put the auction up for sale, the non-initiating bidder remains unaware of the rival bidder's indication of interest. Second, if the seller is approached and decides to auction the asset off, both currently present bidders learn about all current indications of interest (either from a single bidder or from both bidders).

Consider the response of the seller upon receiving message $m_{i,t} = 1$. We conjecture existence of a responsive equilibrium and derive parameter restrictions for its existence. If the seller plays the equilibrium strategy of auctioning the asset off upon receiving $m_{i,t} = 1$, he obtains the expected payoff of $R_B(\hat{s})$. If the seller deviates and does not put the auction up for sale, then, because the deviation is not observed by the rival bidder, the rival bidder's communication strategy is unaffected: she sends message $m_{-i,t} = 0$ if and only if her signal is below \hat{s} . Because the communication strategy of the rival bidder remains the same and the bidder expects the seller to continue playing a Markov strategy of auctioning the asset off upon receiving message 1 from any of the bidders, she finds it optimal to follow the same strategy of sending message $m_{i,t} = 1$ if and only if her signal exceeds \hat{s} . Thus, if the seller deviates to not putting the auction up for sale upon receiving $m_{i,t} = 1$ at time t , both bidders continue to play the same strategy.

It follows that the seller does not benefit from the deviation if and only if $R_B(\hat{s})$ exceeds his expected payoff from waiting until he receives $m_{1,t} = m_{2,t} = 1$ for the first time and auctioning the asset off then. Because upon the start of the sale process, all current indications of interests are learned by both bidders prior to submitting the bids, in such deviation the seller would have an auction with symmetric bidders whose signals are distributed over $[\hat{s}, 1]$. By the same proof as in Lemma 6, the equilibrium bidding strategy of each bidder would be

$$a_D(s) = \mathbb{E}[v(x)|x \in [\hat{s}, s]] \text{ for } s \in [\hat{s}, 1]. \quad (40)$$

The expected revenue of the seller from such deviation would be

$R_D(\hat{s}) = \mathbb{E}[v(x)|x \in [\hat{s}, \max\{s_1, s_2\}]]$, $s_1 \in [\hat{s}, 1]$, $s_2 \in [\hat{s}, 1]$. Let $\tilde{R}_D(\hat{s})$ denote the discounted expected revenue of the seller from the deviation, and let $\hat{R}_D(\hat{s})$ denote the discounted expected revenue of the seller from holding the auction upon receiving $m_{1,t} = m_{2,t} = 1$, conditional on receiving message 0 from both bidders today. These discounted revenues must satisfy:

$$r\tilde{R}_D(\hat{s}) = \lambda(1-\hat{s})(R_D(\hat{s}) - \tilde{R}_D(\hat{s})) + \lambda((1-\hat{s})\tilde{R}_D(\hat{s}) + \hat{s}\hat{R}_D(\hat{s}) - \tilde{R}_D(\hat{s})) + \nu(R_B(\hat{s}) - \tilde{R}_D(\hat{s})); \quad (41)$$

$$r\hat{R}_D(\hat{s}) = 2\lambda(1-\hat{s})(\tilde{R}_D(\hat{s}) - \hat{R}_D(\hat{s})) + \nu(R_S(\hat{s}) - \hat{R}_D(\hat{s})), \quad (42)$$

Deriving $\hat{R}_D(\hat{s})$ from the second equation and plugging it into the first equation, we obtain that the seller does not benefit from the deviation if and only if

$$R_B(\hat{s}) \geq \tilde{R}_D(\hat{s}) = \omega_1 R_D(\hat{s}) + \omega_2 R_S(\hat{s}) + \omega_3 R_B(\hat{s}), \quad (43)$$

where

$$\begin{aligned}\omega_1 &= \frac{\lambda(1-\hat{s})}{r+\lambda+\nu-\frac{2\lambda^2\hat{s}(1-\hat{s})}{r+2\lambda(1-\hat{s})+\nu}}, \\ \omega_2 &= \frac{\frac{\lambda\hat{s}\nu}{r+2\lambda(1-\hat{s})+\nu}}{r+\lambda+\nu-\frac{2\lambda^2\hat{s}(1-\hat{s})}{r+2\lambda(1-\hat{s})+\nu}}, \\ \omega_3 &= \frac{\nu}{r+\lambda+\nu-\frac{2\lambda^2\hat{s}(1-\hat{s})}{r+2\lambda(1-\hat{s})+\nu}}.\end{aligned}$$

Because $R_B(\hat{s}) > R_S(\hat{s})$, it is always satisfied if \hat{s} is high enough. For any \hat{s} , it is always satisfied if r is high enough.

Proof of Proposition 4. First, consider the decision of the seller to delay the auction. For $\hat{s} = 1$, $\frac{2\lambda(1-\hat{s})}{r+2\lambda(1-\hat{s})}R_B(1) = 0$, while $R_S(1) > 0$. Thus, by Proposition 3, the seller does not benefit from the delay. Second, consider the decision of a bidder to initiate the auction. Note that

$$\Pi_I^*(1, 1) < \Pi_S^*(1, 1) = \lim_{\mu \rightarrow \infty} \frac{\lambda(1-\hat{s})\Pi_N^*(s, \hat{s}) + \mu\Pi_S^*(s, \hat{s})}{r + \lambda(1-\hat{s}) + \lambda + \mu} \quad (44)$$

by Lemma 7, implying that a bidder with signal 1 does not benefit from immediate initiation. Further, from (35), (38), and (39) in Proposition 2 it follows that $\Pi_I^*(s, 1) < \Pi_S^*(s, 1)$, so no bidder with signal $s < 1$ benefits from immediate initiation.

Proof of Proposition 6. Consider the non-initiating bidder with signal s . First, we show that any bid b below $v(s, \hat{s})$ is dominated. Suppose that instead of bidding b , the non-initiating bidder with signal s bids $v(s, \hat{s})$. We compare his payoffs from both strategies for any realization of the rival bidder's bid b_r . If the competing bid is $b_r < b$, the non-initiating bidder wins the auction and pays b_r in both cases. Thus, his payoff is identical in both cases. If the competing bid is $b_r > v(s, \hat{s})$, the non-initiating bidder loses the auction. Thus, his payoff is again identical in both cases. Finally, if the competing bid is $b_r \in (b, v(s, \hat{s}))$, the non-initiating bidder loses the auction and gets zero with bid b , but wins the auction at price b_r and gets $v(s, s_2) - b_r \geq v(s, \hat{s}) - b_r > 0$ with bid $v(s, \hat{s})$. Thus, any bid $b < v(s, \hat{s})$ is dominated by bid $v(s, \hat{s})$. Second, given that the non-initiating bidder with signal s does not bid below $v(s, \hat{s})$, the initiating bidder with signal \hat{s} cannot get a positive payoff for any realization of signal s of the non-initiating bidder. Hence, $\Pi_I^*(\hat{s}, \hat{s}) = 0$. Finally, the proof of the last statement is identical to the proof of Proposition 1, so we omit it for brevity.

Proof of Proposition 7. Consider any hypothetical equilibrium with cut-off \hat{s} . First, consider a bidder-initiated auction. We argue that the non-initiating bidder N never enters the auction. By contradiction, suppose the opposite and let \hat{s}_N denote the lowest type of bidder N that enters the auction. Upon entry, the initiating bidder I re-values the asset to $v(s_I, \hat{s}_N) \geq v(\hat{s}, \hat{s}_N)$, where $s_I \in [\hat{s}, 1]$ is her signal. Then, no bidder pays less than $v(\hat{s}, \hat{s}_N)$ in equilibrium. In turn, bidder N with signal \hat{s}_N wins the auction only when bidder I 's signal is \hat{s} , so that she submits $v(\hat{s}_N, \hat{s})$. Upon

winning, bidder N re-values the asset to $v(\hat{s}_N, \hat{s})$, leaving her without surplus and with negative net surplus once participation costs are accounted for. Such a bidder therefore is better off not entering the auction. This argument holds for any cut-off type \hat{s}_N , so the auction is uncontested. In this case, bidder I bids v_s and obtains the expected net payoff of

$$\Pi_I^*(\hat{s}, \hat{s}) = \mathbb{E}[v(s, x) | x \in [0, \hat{s}]] - v_s - C, \quad (45)$$

while bidder N obtains the payoff of zero. The seller obtains the revenues of v_s .

Second, consider a seller-initiated auction. Let $\hat{s}_{S,ns}$ and $\hat{s}_{S,s}$ be signal cut-offs, above which each bidder enters the auction, if the seller puts the asset up for sale voluntarily and due to the liquidity shock, respectively. Because the cut-off type of the bidder loses the auction with certainty if the rival bidder enters, $\hat{s}_{S,ns}$ and $\hat{s}_{S,s}$ must satisfy:

$$\int_0^{\hat{s}_{S,ns}} (v(\hat{s}_{S,ns}, x) - v_s) \frac{1}{\hat{s}} dx = C, \quad (46)$$

$$\int_0^{\hat{s}_{S,s}} v(\hat{s}_{S,s}, x) \frac{1}{\hat{s}} dx = C. \quad (47)$$

Let s' be defined as the unique solution to $\mathbb{E}[v(s', x) | x \in [0, s']] = v_s + C$. First, we argue that there is no equilibrium with $\hat{s} \in (0, s')$. By contradiction, suppose the opposite. Consider a bidder-initiated auction by type \hat{s} . Because (45) is negative, the expected payoff of this bidder is negative, which contradicts the premise that there is an equilibrium with $\hat{s} \in (0, s')$. Second, consider $\hat{s} = s'$, and a bidder-initiated auction by type s' . Now, the expected payoff of this bidder is zero. In contrast, if type s' waits, her expected payoff will be positive because there is a possibility that the seller will receive a liquidity shock and $\hat{s}_{S,s} < s'$. Second, we argue that there is no equilibrium with $\hat{s} \in (s', 1)$. By contradiction, suppose the opposite. Then, the left-hand side of (46) strictly exceeds C at $\hat{s}_{S,ns} = \hat{s}$. Therefore, $\hat{s}_{S,ns} < \hat{s}$. Thus, if the seller initiates an auction without being receiving a liquidity shock and without being approached by a bidder, his expected payoff strictly exceeds v_s . In contrast, the revenues from a bidder-initiated auction equal v_s . Therefore, if the seller receives an indication of interest from a bidder, he prefers to wait, which contradicts the premise that there is an equilibrium with $\hat{s} \in (s', 1)$.

Finally, immediate initiation is an equilibrium, because the seller only loses from waiting, as his payoff from a bidder-initiated auction equals to v_s and his expected payoff if it receives a liquidity shock is below v_s by the argument in the preceding paragraph. The cut-off \hat{s}_S , above which each bidder enters the auction, is such that type \hat{s}_S 's expected payoff from the auction equals C , yielding (17). Condition $C < \mathbb{E}[v(1, x)] - v_s$ simply means that equation (17) has a solution $\hat{s}_S < 1$.

Model with shareholder activism. We extend our base model by adding an activist investor, and show how it can be beneficial in facilitating auctions of companies. Suppose that there is an activist that arrives to the market (e.g., by discovering the target) with intensity $\lambda_A > 0$. After it arrives, the activist can submit an order to buy fraction α of the target. In addition, there is a liquidity trader that arrives with intensity $\lambda_L > 0$ and submits an order to buy fraction α of

the target.²² If an activist buys fraction α , it can undertake an activism campaign at cost $A > 0$, which results in putting the target up for sale. Assume that a competitive market-maker prices the block at its expected value conditional on receiving the order. In this setup, the activist cannot create value outside of the campaign, so it can only be optimal to buy fraction α to subsequently undertake the campaign. Suppose that upon discovering the target, the activist buys fraction α with probability q . Then, the price of fraction α must be²³

$$\alpha P(q) = \alpha R_S(1) \frac{\lambda_A q}{\lambda_A q + \lambda_L} \left(1 + \frac{\lambda_L}{r + \lambda_A q} \right). \quad (48)$$

Intuitively, the activist invests with intensity $\lambda_A q$. Hence, an order of fraction α comes from it with probability $\frac{\lambda_A q}{\lambda_A q + \lambda_L}$ and from the liquidity trader with probability $\frac{\lambda_L}{\lambda_A q + \lambda_L}$. In the former case, the target value is $R_S(1)$, the expected seller revenue in the absence of bidder initiation. In the latter case, the activist has not arrived yet, so the target value is a discounted value of $R_S(1)$.

The activist's payoff from undertaking a campaign, net of cost of acquiring fraction α , is

$$\alpha R_S(1) - \alpha P(q) - A = \alpha R_S(1) \frac{r \lambda_L}{(q \lambda_A + \lambda_L)(r + q \lambda_A)} - A. \quad (49)$$

Equation (49) is strictly decreasing in q . Hence, if (49) is non-negative at $q = 1$, then the activist buys fraction α with probability one and undertakes the campaign. If (49) is negative at $q = 1$, then the activist buys fraction α with probability $q \in (0, 1)$ at which (49) equals zero, and then undertakes the campaign. The next proposition summarizes the equilibrium:

Proposition 8. *In equilibrium, upon discovering the target, the activist acquires the block with probability q , given by: (a) $q = 1$ if $A \leq \underline{A} \equiv \frac{r \lambda_L \alpha R_S(1)}{(\lambda_A + \lambda_L)(r + \lambda_A)}$; (b) $q = \frac{\sqrt{(r + \lambda_L)^2 + 4 \lambda_L r \frac{\alpha R_S(1) - A}{\lambda_A}} - (r + \lambda_L)}{2 \lambda_A}$ if $A \in (\underline{A}, \bar{A})$; (c) $q = 0$ if $A \geq \bar{A} \equiv \alpha R_S(1)$. The implied price of the block is given by (48).*

Proof of Proposition 8. We compute the implied price of the target, $P(q)$, conditional on the market-maker receiving the order from either the activist or the liquidity trader. If the activist acquires fraction α with probability q upon discovering the target, the Bayes' rule implies that the probability that the order comes from the activist, conditional on receiving the order, is $\frac{\lambda_A q}{\lambda_A q + \lambda_L}$. Therefore, the price of the target equals

$$P(q) = \frac{\lambda_A q}{\lambda_A q + \lambda_L} R_S(1) + \frac{\lambda_L}{\lambda_A q + \lambda_L} \int_0^\infty e^{-rt} R_S(1) q \lambda_A e^{-q \lambda_A t} dt = R_S(1) \frac{\lambda_A q}{\lambda_A q + \lambda_L} \left(1 + \frac{\lambda_L}{r + \lambda_A q} \right). \quad (50)$$

Hence, the activist's expected payoff from acquiring fraction α and undertaking the campaign is $\alpha R_S(1) - \alpha P(q) - A$, yielding (49). If (49) is above zero, the activist is better off acquiring the block; if (49) is below zero, it is better not acquiring the block; and if (49) is equal to zero, it is indifferent. Because (49) is strictly decreasing in $q \geq 0$, the statement of Proposition 6 follows,

²²Here, α is an exogenous parameter but it can be endogenized in a richer model.

²³See the proof of Proposition 8 for the derivation.

with q , in the case of $A \in (\underline{A}, \bar{A})$, given by the quadratic equation

$$\frac{r\lambda_L\alpha R_S(1)}{A} = (q\lambda_A + \lambda_L)(r + q\lambda_A) \Rightarrow q_{1,2} = \frac{-(r + \lambda_L) \pm \sqrt{(r - \lambda_L)^2 + 4\lambda_L r \frac{\alpha R_S(1)}{A}}}{2}. \quad (51)$$

Because $q \in [0, 1]$ by definition and the lower root is negative, the upper root is the relevant one.

Model with toeholds. Consider a common-value framework. Suppose that a bidder who considers approaching the target can submit an order to acquire fraction α of the target without disclosing her intent. In addition, there is a liquidity trader that arrives with intensity $\lambda_L > 0$ and submits an order to buy fraction α of the target. A competitive market-maker prices the block at its expected value conditional on receiving the order. This setup captures, in reduced form, the U.S. practice that a bidder may acquire up to 5% of a company's outstanding shares secretly, beyond which she is required to publicly file Schedule 13(D). For simplicity, assume that the holding cost of the target's shares is infinite, i.e., a bidder never acquires a toehold unless she approaches the target immediately after. Suppose that a bidder that considers approaching the target acquires fraction α with probability q . By analogy with (48), the Bayes' rule implies that the probability that the order comes from one of the two bidders, conditional on receiving the order, is $\frac{2\lambda(1-\hat{s})q}{2\lambda(1-\hat{s})q + \lambda_L}$. Therefore, the price of the target equals:

$$\begin{aligned} P(q) &= \frac{2q\lambda(1-\hat{s})}{2q\lambda(1-\hat{s}) + \lambda_L} R_B(\hat{s}) + \frac{\lambda_L}{2q\lambda(1-\hat{s}) + \lambda_L} \\ &\times \int_0^\infty e^{-rt} \left(\frac{2q\lambda(1-\hat{s})}{2q\lambda(1-\hat{s}) + \mu} R_B(\hat{s}) + \frac{\mu}{2q\lambda(1-\hat{s}) + \mu} R_S(\hat{s}) \right) (2q\lambda(1-\hat{s}) + \mu) e^{-(2q\lambda(1-\hat{s}) + \mu)t} dt \\ &= \frac{2q\lambda(1-\hat{s})}{2q\lambda(1-\hat{s}) + \lambda_L} R_B(\hat{s}) + \frac{\lambda_L}{2q\lambda(1-\hat{s}) + \lambda_L} \frac{2q\lambda(1-\hat{s}) R_B(\hat{s}) + \mu R_S(\hat{s})}{r + 2q\lambda(1-\hat{s}) + \mu}. \end{aligned} \quad (52)$$

Rearranging the terms, we obtain

$$P(q) = \frac{2q\lambda(1-\hat{s})}{2q\lambda(1-\hat{s}) + \lambda_L} \left(1 + \frac{\lambda_L}{r + 2q\lambda(1-\hat{s}) + \mu} \right) R_B(\hat{s}) + \frac{\mu\lambda_L}{(2q\lambda(1-\hat{s}) + \lambda_L)(r + 2q\lambda(1-\hat{s}) + \mu)} R_S(\hat{s}). \quad (53)$$

To simplify the exposition, we focus on the case in which the seller does not voluntarily initiate the auction and is not hit by a liquidity shock, and $q = 1$. When $q = 1$, (53) simplifies to

$$P = \frac{2\lambda(1-\hat{s})}{2\lambda(1-\hat{s}) + \lambda_L} \left(1 + \frac{\lambda_L}{r + 2\lambda(1-\hat{s}) + \mu} \right) R_B(\hat{s}) + \frac{\mu\lambda_L}{(2\lambda(1-\hat{s}) + \lambda_L)(r + 2\lambda(1-\hat{s}) + \mu)} R_S(\hat{s}). \quad (54)$$

When, additionally, $\mu = 0$, the expression further simplifies to

$$\alpha P = \alpha R_B(\hat{s}) \frac{2\lambda(1-\hat{s})}{2\lambda(1-\hat{s}) + \lambda_L} \left(1 + \frac{\lambda_L}{r + 2\lambda(1-\hat{s})} \right). \quad (55)$$

Intuitively, a bidder initiates the auction with intensity $\lambda(1-\hat{s})$. Hence, an order of fraction α comes from one of the bidders with probability $\frac{2\lambda(1-\hat{s})}{2\lambda(1-\hat{s}) + \lambda_L}$ and from the liquidity trader with probability $\frac{\lambda_L}{2\lambda(1-\hat{s}) + \lambda_L}$. In the former case, the target value is $R_B(\hat{s})$, the expected seller revenue from the bidder-initiated auction in the presence of the toehold. In the latter case, the interested bidder has not arrived yet, so the target value is a discounted value of $R_B(\hat{s})$.

Consider a bidder-initiated auction when bidder I has toehold α . The expected payoffs of bidders I and N with signal s and bid b are

$$\Pi_j(b, s, \hat{s}) = \int_{\underline{s}_k}^{\phi_k(b, \hat{s})} (v(s, x) - (1 - \alpha_j)b) \frac{1}{\bar{s}_k - \underline{s}_k} dx + \int_{\phi_k(b, \hat{s})}^{\bar{s}_k} \alpha_j a_k(x, \hat{s}) \frac{1}{\bar{s}_k - \underline{s}_k} dx, \quad (56)$$

where $j \neq k \in \{I, K\}$ and $(\alpha_I, \alpha_N) = (\alpha, 0)$. For intuition, consider bidder I . The first term in (56) captures the payoff of bidder I upon winning. In this case, bidder I acquires the remaining $1 - \alpha$ shares at bid b , and obtains asset value $v(s, x)$. The second term in (56) captures the payoff of bidder I upon losing. In this case, bidder I sells fraction α to the rival at her bid $a_N(s, \hat{s})$. Bidder N only obtains the payoff if she wins. Taking the first-order conditions of (56), we obtain

$$\frac{\partial \phi_j(b, \hat{s})}{\partial b} (v(s, \phi_j(b, \hat{s})) - (1 - \alpha_k)b - \alpha_k a_j(\phi_j(b, \hat{s}), \hat{s})) - (1 - \alpha_k)(\phi_j(b, \hat{s}) - \underline{s}_j) = 0 \quad (57)$$

for $j \neq k \in \{I, N\}$. Compared to (2) in the model without toeholds, (57) includes an additional term. In equilibrium, $b = a_j(s, \hat{s})$ must satisfy (57) for $j \in \{I, N\}$, implying $s = \phi_j(b, \hat{s})$. Also, note that $a_j(\phi_j(b, \hat{s}), \hat{s}) = b$. Plugging in and rearranging the terms, we obtain

$$\frac{\partial \phi_j(b, \hat{s})}{\partial b} = \frac{(1 - \alpha_k)(\phi_j(b, \hat{s}) - \underline{s}_j)}{v(\phi_k(b, \hat{s}), \phi_j(b, \hat{s})) - b}. \quad (58)$$

The system of equations (58) is solved subject to the following boundary conditions: $a_j(\bar{s}_j, \hat{s}) \equiv \bar{a}(\hat{s})$ and $a_j(\underline{s}_j, \hat{s}) \equiv \underline{a}(\hat{s})$, so that the support of possible equilibrium bids for both bidders is $[\underline{a}(\hat{s}), \bar{a}(\hat{s})]$. In turn, the equilibrium inverse bidding functions solve (58) subject to boundary conditions $1 = \phi_I(\bar{a}(\hat{s}), \hat{s})$, $\hat{s} = \phi_N(\bar{a}(\hat{s}), \hat{s})$, $\hat{s} = \phi_I(v(\hat{s}, 0), \hat{s})$, $0 = \phi_N(v(\hat{s}, 0), \hat{s})$.

The payoff of bidder I with signal \hat{s} from the auction, net of the cost of acquiring toehold, is

$$\Pi_I^*(\hat{s}, \hat{s}) - \alpha P = \alpha \left(\int_0^{\hat{s}} a_N(x, \hat{s}) \frac{1}{\hat{s}} dx - R_B(\hat{s}) \frac{2\lambda(1 - \hat{s})}{2\lambda(1 - \hat{s}) + \lambda_L} \left(1 + \frac{\lambda_L}{r + 2\lambda(1 - \hat{s})} \right) \right), \quad (59)$$

where $\Pi_I^*(\hat{s}, \hat{s}) = \Pi_I(\underline{a}(\hat{s}), \hat{s}, \hat{s})$ and $R_B(\hat{s}) = \mathbb{E}[\max\{a_I(s_1, \hat{s}), a_N(s_2, \hat{s})\}]$. A necessary condition for a bidder to acquire a toehold and initiate the auction is that (59) is positive. Condition (59) is never satisfied if \hat{s} is low enough and always satisfied if \hat{s} is high enough. Intuitively, if too many types are expected to initiate, the price of a toehold is high, so it is too expensive for the bidder with signal \hat{s} . Thus, factors that increase the price of a toehold also limit bidder initiation.

For an equilibrium, a bidder with signal \hat{s} must be indifferent between approaching the seller and waiting. The cut-off signal is determined from the indifference condition (13) computed at $\mu = 0$. As in Proposition 3, the seller never initiates the auction if $\frac{2\lambda(1 - \hat{s})}{r + 2\lambda(1 - \hat{s})} R_B(\hat{s}) \geq R_S(\hat{s})$.

We complete the extension by briefly discussing the general case of $q \in [0, 1]$ and $\mu \geq 0$. In this case, the payoff to bidder I with signal \hat{s} from the auction, net of the cost of acquiring toehold, is

$$\Pi_I^*(\hat{s}, \hat{s}) - \alpha P = \alpha \left(\int_0^{\hat{s}} a_N(x, \hat{s}) \frac{1}{\hat{s}} dx - \frac{2q\lambda(1 - \hat{s}) R_B(\hat{s})}{2q\lambda(1 - \hat{s}) + \lambda_L} \left(1 + \frac{\lambda_L}{r + 2q\lambda(1 - \hat{s}) + \mu} \right) - \frac{\mu \lambda_L R_S(\hat{s})}{(2q\lambda(1 - \hat{s}) + \lambda_L)(r + 2q\lambda(1 - \hat{s}) + \mu)} \right). \quad (60)$$

For an equilibrium, first, a bidder with signal \hat{s} must be indifferent between approaching the seller

and waiting. The cut-off signal is determined from the indifference condition (13). Second, to have seller initiation in equilibrium, for some $\mu > 0$ the seller must be indifferent between initiating the auction and waiting: $\frac{2\lambda(1-\hat{s})}{r+2\lambda(1-\hat{s})}R_B(\hat{s}) = R_S(\hat{s})$.

Example 4 in the appendix provides a quasi-closed form solution for the case $v(s_i, s_{-i}) = \frac{1}{2}(s_i + s_{-i})$. For $r = 0.05$, $\lambda = 0.5$, and $\lambda_L = 0.5$, the equilibrium with no seller initiation is $\hat{s} = 0.934$, the equilibrium with both seller and bidder initiation is $\hat{s} = 0.96$ and $\mu = 0.16$, and the equilibrium with immediate seller initiation is $\hat{s} = 1$.

7.2 Examples

Example 1: auction stage, the common-value framework. Suppose that $v(s_i, s_{-i}) = \frac{1}{2}(s_i + s_{-i})$. To find the equilibrium, let $\phi_j(b, \hat{s}) = \alpha_j + \beta_j b$, $j \in \{I, N\}$. Then, the first two boundary conditions in (4) become:

$$1 = \alpha_I + \beta_I \bar{a}(\hat{s}), \quad \hat{s} = \alpha_N + \beta_N \bar{a}(\hat{s}) \quad \Rightarrow \quad 1 + \hat{s} = \alpha_I + \alpha_N + (\beta_I + \beta_N) \bar{a}(\hat{s}). \quad (61)$$

The second two boundary conditions become:

$$\hat{s} = \alpha_I + \beta_I \frac{\hat{s}}{2}, \quad 0 = \alpha_N + \beta_N \frac{\hat{s}}{2} \quad \Rightarrow \quad \hat{s} = \alpha_I + \alpha_N + (\beta_I + \beta_N) \frac{\hat{s}}{2}. \quad (62)$$

The difference between (61) and (62) yields $\beta_I + \beta_N = \frac{1}{\bar{a}(\hat{s}) - \hat{s}/2}$. Next, plugging $\phi_j(b, \hat{s})$ into differential equations (3) yields:

$$\beta_j = \frac{\alpha_j + \beta_j b - \underline{s}_j}{\frac{1}{2}(\alpha_I + \alpha_N + (\beta_I + \beta_N)b) - b}. \quad (63)$$

Adding the two equations up at $b = \bar{a}(\hat{s})$ results in $\beta_I + \beta_N = \frac{1}{(1+\hat{s})/2 - \bar{a}(\hat{s})}$. Combining the two equations for $\beta_I + \beta_N$ yields $\bar{a}(\hat{s}) = \frac{1+2\hat{s}}{4}$ and the range of bids $b \in [\frac{\hat{s}}{2}, \frac{1+2\hat{s}}{4}]$ for both bidders. With $\bar{a}(\hat{s})$ known, the coefficients in $\phi_j(b, \hat{s})$ can be found from boundary conditions. The resulting inverses of bidding strategies are:

$$\phi_I(b, \hat{s}) = \hat{s}(2\hat{s} - 1) + 4(1 - \hat{s})b, \quad \phi_N(b, \hat{s}) = -2\hat{s}^2 + 4\hat{s}b. \quad (64)$$

The bidding strategies given signals are, in turn, inverses of $\phi_j(b, \hat{s})$. They are linear in own signals:

$$a_I(s, \hat{s}) = \frac{s + \hat{s}(1 - 2\hat{s})}{4(1 - \hat{s})}, \quad a_N(s, \hat{s}) = \frac{s + 2\hat{s}^2}{4\hat{s}}. \quad (65)$$

Example 2: auction stage, the private-value framework. Suppose that $v(s_i) = s_i$. The solution to the system of differential equations (8) subject to boundary conditions (10) and the

minimum serious bid $\frac{\hat{s}}{2}$ obtained from (9), derived by Kaplan and Zamir (2012), is

$$\phi_I(b) = \hat{s} + \frac{\hat{s}^2}{(\hat{s} - 2b) c_I e^{-\frac{\hat{s}}{\hat{s}-2b}} - 4b}, \phi_N(b) = \frac{\hat{s}^2}{(\hat{s} - 2b) c_N e^{\frac{\hat{s}}{\hat{s}-2b}} + 4(\hat{s} - b)}, \quad (66)$$

where constants c_N and c_I are determined from boundary conditions $\phi_I\left(\hat{s} - \frac{\hat{s}^2}{4}\right) = 1$ and $\phi_N\left(\hat{s} - \frac{\hat{s}^2}{4}\right) = \hat{s}$. The range of bids is $b \in \left[\frac{\hat{s}}{2}, \hat{s} - \frac{\hat{s}^2}{4}\right]$ for both bidders.

Example 3: initiation stage, the private-value framework. We consider one set of parameters that gives a single equilibrium with seller initiation only, and one set of parameters that gives an additional equilibrium with both bidder and seller initiation. Let $\nu \rightarrow 0$. First, suppose that $v(s_i) = s_i$. Using (6) and bidding strategies in Example 2, $\Pi_I^*(\hat{s}, \hat{s}) = \frac{\hat{s}}{4}$ and $\Pi_N^*(\hat{s}, \hat{s}) = \frac{\hat{s}^2}{4}$. By Lemma 6, the equilibrium in the seller-initiated auction is $a_S(s) = \frac{s}{2}$, implying $\Pi_S^*(\hat{s}, \hat{s}) = \frac{\hat{s}}{2}$.

Consider the initiation game. The indifference condition for \hat{s} , (13), simplifies to a cubic equation

$$(r + \lambda) \frac{\hat{s}}{4} + \lambda(1 - \hat{s}) \left(\frac{\hat{s}}{4} - \frac{\hat{s}^2}{4} \right) = \mu \left(\frac{\hat{s}}{2} - \frac{\hat{s}}{4} \right). \quad (67)$$

This equation has closed-form solutions $\hat{s}_1 = 0$, which, by Proposition 1, is not an equilibrium, and $\hat{s}_{2,3} = 1 \pm \left(-\frac{r+\lambda-\mu}{\lambda} \right)^{1/2}$ if condition $\mu > r + \lambda$ is satisfied. Note however that this condition contradicts restriction $\mu < \hat{\mu}(\hat{s})$, which in this case simplifies to $\mu < r + \lambda\hat{s}$.²⁴ Intuitively, while a bidder with the cut-off signal is indifferent between sending indication of interest $m_{i,t}$ equal to 0 or 1, a bidder with signal immediately above (or below) the cut-off has incentives to switch her $m_{i,t}$ from 1 to 0 (or from 0 to 1). By Proposition 4, $\hat{s} = 1$ and $\mu = \infty$ is the only equilibrium; in it, all auctions are seller-initiated.

Second, suppose that $v(s_i) = s_i^2$. Repeating all computations, $\Pi_I^*(\hat{s}, \hat{s}) = \frac{2}{3\sqrt{3}}\hat{s}^2$, $\Pi_N^*(\hat{s}, \hat{s}) = \hat{s}^2 - \bar{a}(\hat{s})$, and $\Pi_S^*(\hat{s}, \hat{s}) = \frac{2}{3}\hat{s}^2$, where $\bar{a}(\hat{s})$ is found numerically. The indifference condition (13) simplifies to

$$(r + \lambda) \frac{2}{3\sqrt{3}}\hat{s}^2 + \lambda(1 - \hat{s}) \left(\left(\frac{2}{3\sqrt{3}} - 1 \right) \hat{s}^2 + \bar{a}(\hat{s}) \right) = \mu \left(\left(\frac{2}{3} - \frac{2}{3\sqrt{3}} \right) \hat{s}^2 \right). \quad (68)$$

The seller's best response is characterized in Proposition 3 and is computed numerically. For the case $r = 0.05$ and $\lambda = 0.5$, Figure 3 illustrates the bidders' and seller's best responses, and two equilibria, with seller initiation only ($\hat{s} = 1$ and $\mu = \infty$) and with both bidder and seller initiation ($\hat{s} = 0.977$ and $\mu = 0.752$). It is easy to check that the second equilibrium satisfies restriction $\mu < \hat{\mu}(\hat{s})$, which in this case is equal to $\mu < \frac{1}{\sqrt{2}-1}r + \sqrt{2}\lambda + \lambda\hat{s} = 1.316$. Additionally, if $\nu \in (0.752, \frac{1}{\sqrt{2}-1}r + \sqrt{2}\lambda + \lambda\hat{s})$ and \hat{s} solves (13) for $\mu = \nu$, there is an additional equilibrium, in which the seller does not voluntarily initiate an auction, \hat{s} and $\mu = \nu$. For example, if $\nu = 1$ then $\hat{s} = 0.43$ and $\mu = 1$ is such an equilibrium.

²⁴This can be seen from $\hat{\mu}(\hat{s}) = \frac{a(\hat{s})}{\hat{s}-a(\hat{s})} (r + \lambda) - \lambda(1 - \hat{s}) = r + \lambda\hat{s}$.

Example 4: the common-value framework with toeholds. Suppose that $v(s_i, s_{-i}) = \frac{1}{2}(s_i + s_{-i})$. The general solution to the system of differential equations (58) is given by

$$\phi_I(b, \hat{s}) = \hat{s} + c_1 ((1 - \alpha)\phi_N(b, \hat{s}))^{\frac{1}{1-\alpha}}, \quad (69)$$

$$b = \frac{\phi_N(b, \hat{s})}{2(2 - \alpha)} + c_2 \phi_N(b, \hat{s})^{-\frac{1}{1-\alpha}} + \frac{\hat{s}}{2} + \frac{c_1}{4} ((1 - \alpha)\phi_N(b, \hat{s}))^{\frac{1}{1-\alpha}}. \quad (70)$$

First, use boundary conditions $\hat{s} = \phi_I(v(\hat{s}, 0), \hat{s})$ and $0 = \phi_N(v(\hat{s}, 0), \hat{s})$ to show that (70) can only be satisfied with equality if $c_2 = 0$. Second, plug in $1 = \phi_I(\bar{a}(\hat{s}), \hat{s})$ and $\hat{s} = \phi_N(\bar{a}(\hat{s}), \hat{s})$ in (69) to obtain $c_1 = \frac{1 - \hat{s}}{((1 - \alpha)\hat{s})^{\frac{1}{1-\alpha}}}$.

Plugging $\hat{s} = \phi_N(\bar{a}(\hat{s}), \hat{s})$ and expressions for c_1 and c_2 into (70), we obtain $\bar{a}(\hat{s})$:

$$\bar{a}(\hat{s}) = \frac{1 + \hat{s}}{4} + \frac{\hat{s}}{2(2 - \alpha)}. \quad (71)$$

The upper boundary on bids is increasing in α , consistent with the intuition that toeholds result in a more aggressive bidding. Additionally, (70) becomes

$$b = \frac{\phi_N(b, \hat{s})}{2(2 - \alpha)} + \frac{\hat{s}}{2} + \frac{1 - \hat{s}}{4} \left(\frac{\phi_N(b, \hat{s})}{\hat{s}} \right)^{\frac{1}{1-\alpha}}. \quad (72)$$

Equivalently, the equilibrium bid of bidder N with signal s is

$$a_N(s, \hat{s}) = \frac{s}{2(2 - \alpha)} + \frac{\hat{s}}{2} + \frac{1 - \hat{s}}{4} \left(\frac{s}{\hat{s}} \right)^{\frac{1}{1-\alpha}}. \quad (73)$$

Plugging the expression for c_1 into (69), we obtain

$$\phi_I(b, \hat{s}) = \hat{s} + (1 - \hat{s}) \left(\frac{\phi_N(b, \hat{s})}{\hat{s}} \right)^{\frac{1}{1-\alpha}}. \quad (74)$$

Substituting the non-linear term in (74) using (72), we obtain

$$\phi_I(b, \hat{s}) = 4b - \hat{s} - \frac{2\phi_N(b, \hat{s})}{2 - \alpha}, \quad (75)$$

Equivalently, the equilibrium bid of bidder I with signal s is

$$a_I(s, \hat{s}) = \frac{s + \hat{s}}{4} + \frac{\phi_N(a_I(s, \hat{s}), \hat{s})}{2(2 - \alpha)}, \quad (76)$$

where $\phi_N(a_I(s, \hat{s}), \hat{s})$ is the signal of bidder N corresponding to bid $a_I(s, \hat{s})$ made by bidder I with signal s . For the case $\alpha = 0$ (no toeholds), $\phi_N(a_I(s, \hat{s}), \hat{s}) = 4\hat{s}a_I(s, \hat{s}) - 2\hat{s}^2$, and we have the same solution as in Example 1. For the case $\alpha > 0$, a numerical inversion is needed.

To solve for \hat{s} using the indifference condition (13), we compute the equilibrium payoff from

the auction of bidder I with signal \hat{s} :

$$\Pi_I^*(\hat{s}, \hat{s}) = \alpha \int_0^{\hat{s}} a_N(s, \hat{s}) \frac{1}{\hat{s}} ds = \alpha \left(\frac{1 - \alpha(1 - \hat{s})}{4(2 - \alpha)} + \frac{\hat{s}}{2} \right). \quad (77)$$

Equilibrium payoffs of bidders N and both bidders in a seller-initiated auction with signals \hat{s} are

$$\Pi_N^*(\hat{s}, \hat{s}) = \int_{\hat{s}}^1 \left(\frac{\hat{s} + x}{2} - \bar{a}(\hat{s}) \right) \frac{1}{1 - \hat{s}} dx = \frac{\hat{s}}{2} \frac{1 - \alpha}{2 - \alpha}; \quad (78)$$

$$\Pi_S^*(\hat{s}, \hat{s}) = \int_0^{\hat{s}} \left(\frac{\hat{s} + x}{2} - a_S(\hat{s}) \right) \frac{1}{\hat{s}} dx = \frac{\hat{s}}{4}. \quad (79)$$

Next, we need to calculate the equilibrium price of fraction α of the seller. Let $H_B(b, \hat{s})$ denote the c.d.f. of the winning bid b in the bidder-initiated auction:

$$\begin{aligned} H_B(b, \hat{s}) &= \Pr(a_I(s_1, \hat{s}) \leq b, a_N(s_2, \hat{s}) \leq b) = \Pr(s_1 \leq \phi_I(b, \hat{s}), s_2 \leq \phi_N(b, \hat{s})) \\ &= \frac{\phi_I(b, \hat{s}) - \hat{s} \phi_N(b, \hat{s})}{1 - \hat{s}} \frac{1}{\hat{s}} = \frac{4b - 2\hat{s} - \frac{2\phi_N(b, \hat{s})}{2 - \alpha} \phi_N(b, \hat{s})}{1 - \hat{s}} \frac{1}{\hat{s}}, \end{aligned} \quad (80)$$

so that the corresponding p.d.f. is

$$\begin{aligned} h_B(b, \hat{s}) &= \frac{4 - \frac{2}{2 - \alpha} \phi'_{N,1}(b, \hat{s}) \phi_N(b, \hat{s})}{1 - \hat{s}} \frac{1}{\hat{s}} + \frac{4b - 2\hat{s} - \frac{2}{2 - \alpha} \phi_N(b, \hat{s}) \phi'_{N,1}(b, \hat{s})}{1 - \hat{s}} \frac{1}{\hat{s}} \\ &= \frac{4\phi_N(b, \hat{s}) + 4b\phi'_{N,1}(b, \hat{s}) - 2\hat{s}\phi'_{N,1}(b, \hat{s}) - \frac{4}{2 - \alpha} \phi_N(b, \hat{s}) \phi'_{N,1}(b, \hat{s})}{\hat{s}(1 - \hat{s})}. \end{aligned} \quad (81)$$

Here, $\phi_N(b, \hat{s})$ is the numerical solution of (72). Applying the implicit function theorem to (72),

$$\phi'_{N,1}(b, \hat{s}) = \frac{1}{\frac{1}{2(2 - \alpha)} + \frac{1 - \hat{s}}{4} \frac{1}{1 - \alpha} \phi_N(b, \hat{s})^{\frac{1}{1 - \alpha} - 1} \hat{s}^{-\frac{1}{1 - \alpha}}}. \quad (82)$$

P.d.f. $h_B(b, \hat{s})$ can be calculated numerically. Finally, using (59), the necessary condition for a bidder to acquire a toehold and initiate the auction when the seller does not initiate auctions ($\mu = 0$) is

$$\alpha \left(\frac{1 - \alpha(1 - \hat{s})}{4(2 - \alpha)} + \frac{1}{2} \hat{s} \right) - \frac{2\lambda(1 - \hat{s})}{2\lambda(1 - \hat{s}) + \lambda_L} \left(1 + \frac{\lambda_L}{r + 2\lambda(1 - \hat{s})} \right) \int_{\underline{a}(\hat{s})}^{\bar{a}(\hat{s})} b h_b(b, \hat{s}) db. \quad (83)$$

7.3 For online publication: Complete dynamics with private values

We focus on the case of private values. Assume that at date $t = 0$, each potential bidder randomly draws a private signal from the uniform distribution $[0, 1]$. Thus, in this extension, the distribution of initial and subsequent bidders' signals is the same. However, to the extent that there exists a stationary cut-off signal $\hat{s} < 1$, to which the game eventually converges, the game becomes

non-stationary in the meantime. To simplify the analysis, we consider non-stationary equilibria, in which the seller does not voluntarily initiate the auction but can be hit by a liquidity shock ($\mu = \nu > 0$). We conjecture and later confirm that the cut-off \hat{s}_t is a decreasing function of time: $\hat{s}'_t < 0$.

In a conjectured non-stationary equilibrium, at any time a bidder can initiate the auction for two reasons. First, a new bidder with signal $s \geq \hat{s}_t$ replaces an old bidder. Second, over time the decreasing cut-off \hat{s}_t reaches one of the currently present bidders' signal. Upon the start of the auction, the non-initiating bidder learns about the rival bidder's indication of interest but cannot distinguish between these two events.

7.3.1 Equilibria in bidder- and seller-initiated auctions

First, we solve for the equilibrium at the auction stage. For the remainder of this subsection, to make the notation simpler we omit the subscript for time in \hat{s}_t .

7.3.1.1 A bidder-initiated auction. Because cut-off \hat{s} , by conjecture, is decreasing over time, the probability over a short interval of time dt that bidder I is a new bidder with signal at or above \hat{s} is $\lambda dt(1 - \hat{s})$, while the probability that bidder I is an old bidder whose signal was reached by decreasing cut-off \hat{s} is $-\hat{s}'dt/\hat{s}$. The Bayes' rule implies that conditional on initiation, bidder N believes the first event occurs with probability $p = \frac{-\hat{s}'/\hat{s}}{\lambda(1-\hat{s}) - \hat{s}'/\hat{s}}$, while the second event occurs with probability $1 - p$.

We define equilibrium bidding strategies through cumulative distribution functions. Specifically, let $F_j(b, \hat{s}, p)$, $j \in \{I, N\}$ denote the probability that bid b by bidder I (or bidder N) wins against a randomly drawn, from her perspective, bid of bidder N (or bidder I). We conjecture that the equilibrium takes the following form. The set of "serious" bids is $[\underline{a}(\hat{s}, p), \bar{a}(\hat{s}, p)]$. Bidder N with signal s bids $a_N(s, \hat{s}, p) \in [\underline{a}(\hat{s}, p), \bar{a}(\hat{s}, p)]$, $s_i \geq \underline{a}(\hat{s}, p)$. Bidder I with signal \hat{s} plays the mixed strategy of bidding over interval $[\underline{a}(\hat{s}, p), \hat{a}(\hat{s}, p)]$ for some $\hat{a}(\hat{s}, p) \in [\underline{a}(\hat{s}, p), \bar{a}(\hat{s}, p)]$. Bidder I with signal $s_i > \hat{s}$ bids $a_I(s, \hat{s}, p) \in [\hat{a}(\hat{s}, p), \bar{a}(\hat{s}, p)]$. Note that $F_I(\hat{a}(\hat{s}, p)) = p$.

The expected payoff of bidder N with signal s and bid b is

$$\Pi_N(b, s, \hat{s}, p) = \mathbb{E}[v(s) - b | x = \hat{s} \text{ or } x \in [\hat{s}, 1]] = (v(s) - b) F_I(b, \hat{s}, p). \quad (84)$$

Intuitively, bidder N 's bid exceeds the bid of her rival with probability $F_I(b, \hat{s}, p)$. In equilibrium, $b = a_N(s, \hat{s}, p)$, implying $s = \phi_N(b, \hat{s}, p)$ such that $\phi_N(b, \hat{s}, p)/\hat{s} = F_N(b, \hat{s}, p)$. Therefore, taking the first-order condition of (84) and using the equilibrium condition, for any $b \in (\underline{a}(\hat{s}, p), \bar{a}(\hat{s}, p))$,

$$\frac{\partial F_I(b, \hat{s}, p)}{\partial b} (v(F_N(b, \hat{s}, p) \hat{s}) - b) = F_I(b, \hat{s}, p). \quad (85)$$

Next, consider bidder I with signal \hat{s} . Randomization among bids $b \in [\underline{a}(\hat{s}, p), \hat{a}(\hat{s}, p)]$ requires that for any such b and some constant C , her expected payoff is

$$\Pi_N(b, \hat{s}, \hat{s}, p) = \mathbb{E}[v(\hat{s}) - b | x \in [0, \hat{s}]] = (v(\hat{s}) - b) F_N(b, \hat{s}, p) = C. \quad (86)$$

If this condition does not hold, bidder I would deviate to the most profitable set of bids.

Finally, the expected payoff of bidder N with signal $s > \hat{s}$ and bid b is

$$\Pi_N(b, s, \hat{s}, p) = \mathbb{E}[v(s) - b | x \in [0, \hat{s}]] = (v(s) - b) F_N(b, \hat{s}, p). \quad (87)$$

In equilibrium, $b = a_I(s, \hat{s}, p)$ for $s > \hat{s}$, implying $s = \phi_I(b, \hat{s}, p)$ such that $p + (1 - p) \frac{\phi_I(b, \hat{s}, p) - \hat{s}}{1 - \hat{s}} = F_I(b, \hat{s}, p)$. Intuitively, the probability for bidder N to win with bid $b \geq \hat{a}(\hat{s}, p)$ against bidder I is $p \Pr[s < \phi_I(b, \hat{s}, p) | s = \hat{s}] + (1 - p) \Pr[s < \phi_I(b, \hat{s}, p) | s \in [\hat{s}, 1]] = p + (1 - p) \frac{\phi_I(b, \hat{s}, p) - \hat{s}}{1 - \hat{s}}$. Therefore, taking the first-order condition of (87) and using the equilibrium condition, for any $b \in [\hat{a}(\hat{s}, p), \bar{a}(\hat{s}, p)]$,

$$\frac{\partial F_N(b, \hat{s}, p)}{\partial b} \left(v \left(\hat{s} + \frac{F_I(b, \hat{s}, p) - p}{1 - p} (1 - \hat{s}) \right) - b \right) = F_N(b, \hat{s}, p). \quad (88)$$

The system of three equations, (85), (86), and (88), is solved subject to boundary conditions

$$1 = F_j(\bar{a}(\hat{s}, p), \hat{s}, p), \quad 0 = F_I(\underline{a}(\hat{s}, p), \hat{s}, p), \quad \frac{v^{-1}(\underline{a}(\hat{s}, p))}{\hat{s}} = F_N(\underline{a}(\hat{s}, p), \hat{s}, p), \quad p = F_I(\hat{a}(\hat{s}, p), \hat{s}, p) \quad (89)$$

for $j \in \{I, N\}$. These conditions are similar to (10) but also account for randomization by bidder I with signal \hat{s} . From the third boundary condition and (86), for any $b \in [\underline{a}(\hat{s}, p), \hat{a}(\hat{s}, p)]$,

$$F_N(b, \hat{s}, p) = \frac{v^{-1}(\underline{a}(\hat{s}, p))}{\hat{s}} \frac{v(\hat{s}) - \underline{a}(\hat{s}, p)}{v(\hat{s}) - b}. \quad (90)$$

The minimum bid $\underline{a}(\hat{s}, p)$ must be optimal for bidder I with signal \hat{s} . Therefore,

$$\frac{v^{-1}(\underline{a}(\hat{s}, p))}{\hat{s}} (v(\hat{s}) - \underline{a}(\hat{s}, p)) \geq F_N(b, \hat{s}, p) (v(\hat{s}) - b), \quad \forall b. \quad (91)$$

In equilibrium, bidder N never bids above her valuation: $a_N(s, \hat{s}, p) \leq v(s)$. Therefore, $\phi_N(b, \hat{s}, p) \geq v^{-1}(b)$, which implies $F_N(b) \geq \frac{v^{-1}(b)}{\hat{s}}$. Hence,

$$\frac{v^{-1}(\underline{a}(\hat{s}, p))}{\hat{s}} (v(\hat{s}) - \underline{a}(\hat{s}, p)) \geq \frac{v^{-1}(b)}{\hat{s}} (v(\hat{s}) - b), \quad \forall b. \quad (92)$$

Therefore,

$$\underline{a}(\hat{s}, p) = \arg \max_b \frac{v^{-1}(b)}{\hat{s}} (v(\hat{s}) - b) \quad \Rightarrow \quad \text{F.O.C.:} \quad \frac{v(\hat{s}) - \underline{a}(\hat{s}, p)}{v'(v^{-1}(\underline{a}(\hat{s}, p)))} = v^{-1}(\underline{a}(\hat{s}, p)). \quad (93)$$

Note that in equilibrium, $\underline{a}(\hat{s}, p) \equiv \underline{a}(\hat{s})$. The following lemma summarizes the equilibrium in the bidder-initiated first-price auction in the non-stationary setting:

Lemma 8 (equilibrium in the bidder-initiated PV auction, the non-stationary setting). *The equilibrium is unique (up to the non-serious bids of types $s < v^{-1}(\underline{a}(\hat{s}, p))$ of non-initiating bidders). The equilibrium probabilities of the initiating and non-initiating bidders to win with bid b , $F_j(b, \hat{s}, p)$, $j \in \{I, N\}$, satisfy (85), (86), and (88), with boundary conditions (89) and*

the lowest serious bid given by (93).

Example 6 derived below provides the closed form solution for the case $v(s_i) = s_i$. Figure 4 illustrates the equilibrium bids for the case $\hat{s} = 0.5$ and $p = 0.5$.

7.3.1.2 A seller-initiated auction. If the auction is initiated by the seller, all parties believe that each bidder's signal is distributed uniformly over $[0, \hat{s}]$. As a result, auction outcomes are identical to those derived in Lemma 6. The following lemma summarizes the equilibrium:

Lemma 9 (equilibrium in the seller-initiated PV auction, the non-stationary setting). *The equilibrium is unique. The symmetric equilibrium bidding strategies of the non-initiating bidders, $a_S(s)$, are increasing functions that are independent of \hat{s} and p and solve (23) with boundary condition $a_S(0) = v(0)$. Specifically, $a_S(s)$ is given by (11).*

7.3.2 The initiation game

We solve for symmetric Markov Perfect Bayesian equilibria, in which the seller does not voluntarily initiate the auction ($\mu = \nu$). We also derive the conditions, for which this strategy of the seller is optimal.

Suppose that a bidder believes that a rival bidder approaches the seller either because it replaced the previous rival and received a signal in excess of \hat{s}_t , which occurs with probability $\lambda(1 - \hat{s}_t)dt$, or because the current rival's signal is reached by decreasing cut-off \hat{s}_t , which occurs with probability $-\hat{s}'_t dt / \hat{s}_t$. Denote the expected continuation value of this bidder by $V_B(s, t, \nu) \equiv V_B(s, \hat{s}_t, p_t, t, \mu = \nu)$. This value satisfies

$$\begin{aligned} V_B(s, t, \nu) &= \max \left\{ \Pi_I^*(s, \hat{s}_t, p_t), \frac{V'_{B,t}(s, t, \nu) + (\lambda(1 - \hat{s}_t) - \hat{s}'_t / \hat{s}_t) \Pi_N^*(s, \hat{s}_t, p_t) + \nu \Pi_S^*(s, \hat{s}_t)}{r + \lambda(1 - \hat{s}_t) - \hat{s}'_t / \hat{s}_t + \lambda + \nu} \right\} \\ &= \max \left\{ \Pi_I^*(s, \hat{s}_t, p_t), \frac{V'_{B,t}(s, t, \nu) + \lambda \frac{1 - \hat{s}_t}{1 - p_t} \Pi_N^*(s, \hat{s}_t, p_t) + \nu \Pi_S^*(s, \hat{s}_t)}{r + \lambda \frac{1 - \hat{s}_t}{1 - p_t} + \lambda + \nu} \right\}, \end{aligned} \quad (94)$$

The intuition behind (94) is as follows. The continuation value is the maximum of the expected payoff from immediate initiation, which yields $\Pi_I^*(s, \hat{s}_t, p_t)$, and waiting, which yields the second term of (94). The change in the continuation value with time in a non-stationary setting is captured by $V'_{B,t}(s, t, \nu)$. Four independent events can occur if the bidder chooses to wait. First, with intensity $\lambda(1 - \hat{s}_t)$, a new rival bidder with signal above \hat{s}_t indicates the interest to the seller, and the seller reacts by putting the asset up for sale. The non-initiating bidder obtains $\Pi_N^*(s, \hat{s}_t, p_t)$ in this case. Second, with intensity $-\hat{s}'_t / \hat{s}_t = \frac{p_t}{1 - p_t} \lambda(1 - \hat{s}_t)$, a current rival bidder's signal is reached by decreasing cut-off \hat{s}_t , so that she indicates the interest to the seller, thereby triggering a sale. The non-initiating bidder, again, obtains $\Pi_N^*(s, \hat{s}_t, p_t)$. Third, with intensity ν , the seller puts the asset up for sale, and the bidder obtains $\Pi_S^*(s, \hat{s}_t)$. Fourth, with intensity λ , the bidder experiences a shock and leaves the market.

By continuity of $\Pi_I^*(\cdot)$, $\Pi_N^*(\cdot)$, and $\Pi_S(s, \hat{s}_t)$ in s , the cut-off signal must satisfy

$$\begin{aligned} \Pi_I^*(\hat{s}_t, \hat{s}_t, p_t) &= \frac{V'_{B,t}(\hat{s}_t, t, \nu) + \lambda \frac{1-\hat{s}_t}{1-p_t} \Pi_N^*(\hat{s}_t, \hat{s}_t, p_t) + \nu \Pi_S^*(s, \hat{s}_t)}{r + \lambda \frac{1-\hat{s}_t}{1-p_t} + \lambda + \nu} \Rightarrow \\ (r + \lambda) \Pi_I^*(\hat{s}_t, \hat{s}_t, p_t) + \lambda \frac{1-\hat{s}_t}{1-p_t} (\Pi_I^*(\hat{s}_t, \hat{s}_t, p_t) - \Pi_N^*(\hat{s}_t, \hat{s}_t, p_t)) &= V'_{B,t}(\hat{s}_t, t, \nu) + \mu(\Pi_S^*(\hat{s}, \hat{s}) - \Pi_I^*(\hat{s}_t, \hat{s}_t, p_t)). \end{aligned} \quad (95)$$

In addition, the smooth-pasting condition at \hat{s}_t must be satisfied:

$$V'_{B,t}(s, t, \nu) = \Pi_{I,\hat{s}_t}^{*'}(\hat{s}_t, \hat{s}_t, p_t) \hat{s}_t' + \Pi_{I,p_t}^{*'}(\hat{s}_t, \hat{s}_t, p_t) p_t'. \quad (96)$$

Lemma 10 (Smooth-pasting condition for the value function, the non-stationary setting). *The smooth-pasting condition for a bidder with signal \hat{s}_t is*

$$V'_{B,t}(s, t, \nu) = -\frac{p_t}{1-p_t} \lambda \hat{s}_t (1-\hat{s}_t) \frac{\partial F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)}{\partial \hat{s}_t} (v(\hat{s}_t) - \hat{a}(\hat{s}_t, p_t)), \quad (97)$$

where $F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t) = \frac{v^{-1}(a(\hat{s}_t, p_t))}{\hat{s}_t} \left(\frac{v(\hat{s}_t) - a(\hat{s}_t, p_t)}{v(\hat{s}_t) - \hat{a}(\hat{s}_t, p_t)} \right)$.

Proof. First, take a partial derivative of $\Pi_I^*(s, \hat{s}_t, p_t) = \Pi_I(a_I(s, \hat{s}, p), s, \hat{s}_t, p_t)$ with respect to \hat{s}_t and compute it at point $s \downarrow \hat{s}_t$, using that for bidder I , $\lim_{s \downarrow \hat{s}_t} a_I(s, \hat{s}_t, p_t) = \hat{a}(\hat{s}_t, p_t)$:

$$\begin{aligned} \Pi_{I,\hat{s}_t}^{*'}(\hat{s}_t, \hat{s}_t, p_t) &= \left(\frac{\partial F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)}{\partial a_I} \frac{\partial \hat{a}(\hat{s}_t, p_t)}{\partial \hat{s}_t} + \frac{\partial F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)}{\partial \hat{s}_t} \right) (v(\hat{s}_t) - a_I(\hat{s}_t, p_t)) \\ &\quad - F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t) \frac{\partial \hat{a}(\hat{s}_t, p_t)}{\partial \hat{s}_t}. \end{aligned} \quad (98)$$

The first-order condition for bidder I , (88), computed at $s \downarrow \hat{s}_t$, reduces to

$$\frac{\partial F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)}{\partial a_I} (v(\hat{s}_t) - a_I(\hat{s}_t, p_t)) = F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t). \quad (99)$$

Plugging (99) into (98), we obtain

$$\Pi_{I,\hat{s}_t}^{*'}(\hat{s}_t, \hat{s}_t, p_t) = \frac{\partial F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)}{\partial \hat{s}_t} (v(\hat{s}_t) - a_I(\hat{s}_t, p_t)). \quad (100)$$

From (90), $F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t) = \frac{v^{-1}(a(\hat{s}_t, p_t))}{\hat{s}_t} \left(\frac{v(\hat{s}_t) - a(\hat{s}_t, p_t)}{v(\hat{s}_t) - \hat{a}(\hat{s}_t, p_t)} \right)$ immediately follows. Second, take a partial derivative of $\Pi_I^*(s, \hat{s}_t, p_t)$ with respect to p_t and compute it at point $s \downarrow \hat{s}_t$, using that for bidder I , $\lim_{s \downarrow \hat{s}_t} a_I(s, \hat{s}_t, p_t) = \hat{a}(\hat{s}_t, p_t)$:

$$\begin{aligned} \Pi_{I,p_t}^{*'}(\hat{s}_t, \hat{s}_t, p_t) &= \left(\frac{\partial F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)}{\partial a_I} \frac{\partial \hat{a}(\hat{s}_t, p_t)}{\partial p_t} + \frac{\partial F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)}{\partial p_t} \right) (v(\hat{s}_t) - a_I(\hat{s}_t, p_t)) \\ &\quad - F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t) \frac{\partial \hat{a}(\hat{s}_t, p_t)}{\partial p_t}. \end{aligned} \quad (101)$$

Again, using the first-order condition for bidder I at $s \downarrow \hat{s}(t)$,

$$\Pi_{I,p_t}^{*'}(\hat{s}_t, \hat{s}_t, p_t) = \frac{\partial F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)}{\partial p_t} (v(\hat{s}_t) - a_I(\hat{s}_t, p_t)). \quad (102)$$

Note that $F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)$ does not depend on p_t other than through its first argument. Hence,

$$\Pi_{I,p}^{*'}(\hat{s}_t, \hat{s}_t, p_t) = 0. \quad (103)$$

Finally, by definition of p_t , p_t and s_t are linked through differential equation $\hat{s}'_t = -\frac{p_t}{1-p_t} \lambda \hat{s}_t (1 - \hat{s}_t)$. This concludes the proof.

Equation (95) is a differential equation on \hat{s}_t : p_t is a function of \hat{s}'_t . The initial condition is $\hat{s}_0 = 1$. Its solution gives the evolution of \hat{s}_t over time. This equation is intuitive. For the cut-off type \hat{s}_t , the cost of waiting equals the benefit. The cost of waiting (the left-hand side) comes from two sources. The first source, represented by the first term on the left-hand side of (95), is discounting due to a positive discount rate and the possibility that the bidder leaves the market. The second source, represented by the second term on the left-hand side of (95), is the possibility that either a strong rival bidder appears or an existing rival bidder with a sufficiently high signal decides to abandon waiting; such a rival then initiates the auction lowering the expected rents of the bidder. The benefit of waiting (the right-hand side) comes from the possibility that the seller initiates the auction without being approached by a bidder, and from the increase in the continuation value due to the decrease in cut-off \hat{s}_t over time. This decrease increases the profitability of the bidder-initiated auction for the cut-off type \hat{s}_t , because such a bidder faces a marginally weaker rival bidder.

For \hat{s}_t to be the equilibrium cut-off, we need to verify that the seller does not benefit from early voluntary initiation. At each time t , the expected payoff to the seller from strategy $\mu = \nu$, denoted by $V_S(t, \nu)$, needs to satisfy

$$rV_S(t, \nu) = V'_{S,t}(t, \nu) + 2(\lambda(1 - \hat{s}_t) + p_t)R_B(\hat{s}_t, p_t), \quad (104)$$

where $R_B(\hat{s}_t, p_t)$ is the seller's expected revenue in the bidder-initiated auction. The seller does not benefit from early voluntary initiation if and only if $V_S(t, \nu) \geq R_S(\hat{s}_t)$ for any t , where $R_S(\hat{s}_t)$ is his expected revenue in the seller-initiated auction.

The equilibrium admits a quasi-closed form solution and is solved for numerically. Example 7 derived below provides the solution for the case $v(s_i) = s_i$. Figure 5 illustrates the behavior of the inference about the cause of bidder initiation p_t as a function of the current cut-off \hat{s}_t and the speed of convergence to the stationary cut-off $\hat{s} = 0.368$ for the case $r = 0.05$, $\lambda = 0.5$, and $\nu = 0.75$.

Example 6: auction stage, the private-value framework, the non-stationary setting. Suppose that $v(s_i) = s_i$. From (93), $\underline{a}(\hat{s}, p) = \frac{\hat{s}}{2}$. Multiplying (85) by $\frac{1-\hat{s}}{1-p}$, multiplying (88) by \hat{s}

and adding the equations up, we obtain

$$\begin{aligned}
& \frac{1-\hat{s}}{1-p} F'_{I,b}(b, \hat{s}, p) (F_N(b, \hat{s}, p) \hat{s} - b) + \hat{s} F'_{N,b}(b, \hat{s}, p) \left(\hat{s} + \frac{F_I(b, \hat{s}, p) - p}{1-p} (1 - \hat{s}) - b \right) = \frac{1-\hat{s}}{1-p} F_I(b, \hat{s}, p) + \hat{s} F_N(b, \hat{s}, p) \\
& \Rightarrow d \left(\hat{s} F_N(b, \hat{s}, p) \left(\hat{s} + \frac{F_I(b, \hat{s}, p) - p}{1-p} (1 - \hat{s}) \right) \right) = d \left(\frac{1-\hat{s}}{1-p} F_I(b, \hat{s}, p) b + \hat{s} F_N(b, \hat{s}, p) b \right) \\
& \Rightarrow \hat{s} F_N(b, \hat{s}, p) \left(\hat{s} + \frac{F_I(b, \hat{s}, p) - p}{1-p} (1 - \hat{s}) \right) = \frac{1-\hat{s}}{1-p} F_I(b, \hat{s}, p) b + \hat{s} F_N(b, \hat{s}, p) b + c,
\end{aligned} \tag{105}$$

for some constant c . This equation holds for $b \in [\hat{a}(\hat{s}, p), \bar{a}(\hat{s}, p)]$. Evaluating at $\bar{a}(\hat{s}, p)$ gives c :

$$c = \hat{s} (1 - \bar{a}(\hat{s}, p)) - \frac{1-\hat{s}}{1-p} \bar{a}(\hat{s}, p). \tag{106}$$

Evaluating (105) at $\hat{a}(\hat{s}, p)$ gives an alternative expression for c , which we use to obtain $\hat{a}(\hat{s}, p)$ as a function of given $\hat{a}(\hat{s}, p)$:

$$c = \frac{\hat{s}^2}{4} - \frac{1-\hat{s}}{1-p} p \hat{a}(\hat{s}, p) \Rightarrow \bar{a}(\hat{s}, p) = \frac{\hat{s} - \frac{\hat{s}^2}{4} + \frac{1-\hat{s}}{1-p} p \hat{a}(\hat{s}, p)}{\hat{s} + \frac{1-\hat{s}}{1-p}}. \tag{107}$$

In the stationary setting, $p = 0$ so $\bar{a}(\hat{s}, p) = \frac{\hat{s}^2}{4} - \hat{s}$, as before. What remains is to solve for $\hat{a}(\hat{s}, p)$. Plugging c into (105),

$$\hat{s} F_N(b, \hat{s}, p) \left(\hat{s} - b + \frac{F_I(b, \hat{s}, p) - p}{1-p} (1 - \hat{s}) \right) = \frac{1-\hat{s}}{1-p} F_I(b, \hat{s}, p) b + \frac{\hat{s}^2}{4} - \frac{1-\hat{s}}{1-p} p \hat{a}(\hat{s}, p), \tag{108}$$

implying that in the range $b \in [\hat{a}(\hat{s}, p), \bar{a}(\hat{s}, p)]$,

$$\hat{s} F_N(b, \hat{s}, p) = \frac{\frac{1-\hat{s}}{1-p} F_I(b, \hat{s}, p) b + \frac{\hat{s}^2}{4} - \frac{1-\hat{s}}{1-p} p \hat{a}(\hat{s}, p)}{\hat{s} - b + \frac{F_I(b, \hat{s}, p) - p}{1-p} (1 - \hat{s})}. \tag{109}$$

Inserting (109) into $v(\cdot)$ in (85) and transforming,

$$\begin{aligned}
& F'_{I,b}(b, \hat{s}, p) \left(\frac{\frac{1-\hat{s}}{1-p} F_I(b, \hat{s}, p) b + \frac{\hat{s}^2}{4} - \frac{1-\hat{s}}{1-p} p \hat{a}(\hat{s}, p)}{\hat{s} - b + \frac{F_I(b, \hat{s}, p) - p}{1-p} (1 - \hat{s})} - b \right) = F_I(b, \hat{s}, p) \\
& \Rightarrow F'_{I,b}(b, \hat{s}, p) = \frac{\left(\hat{s} - b + \frac{F_I(b, \hat{s}, p) - p}{1-p} (1 - \hat{s}) \right) F_I(b, \hat{s}, p)}{\frac{\hat{s}^2}{4} - \frac{1-\hat{s}}{1-p} p \hat{a}(\hat{s}, p) - b \hat{s} + b^2 + \frac{p}{1-p} (1 - \hat{s}) b}.
\end{aligned} \tag{110}$$

The boundary condition is

$$F_I(\bar{a}(\hat{a}(\hat{s}, p))) = F_I \left(\frac{\hat{s} - \frac{\hat{s}^2}{4} + \frac{1-\hat{s}}{1-p} p \hat{a}(\hat{s}, p)}{\hat{s} + \frac{1-\hat{s}}{1-p}}, \hat{s}, p \right) = 1. \tag{111}$$

This differential equation has a closed-form solution which is omitted for brevity and available from the authors upon request. The remaining value, $\hat{a}(\hat{s}, p)$, is numerically determined from the

second boundary condition:

$$F_I(\hat{a}(\hat{s}, p), \hat{s}, p) = p. \quad (112)$$

It can be shown that $\frac{\hat{s}}{2} \leq \hat{a}(\hat{s}, p) \leq \frac{3\hat{s}}{4}$ and $\hat{a}(\hat{s}, p)$ is strictly increasing in p . In particular, $\hat{a}(\hat{s}, 0) = \frac{\hat{s}}{2}$, so that the solution converges to that of the base model; $\hat{a}(\hat{s}, 1) = \frac{3\hat{s}}{4}$. In addition, $\frac{3\hat{s}}{4} \leq \bar{a}(\hat{s}, p)$ and $\bar{a}(\hat{s}, p)$ is strictly decreasing in p . In particular, $\bar{a}(\hat{s}, 1) = \frac{3\hat{s}}{4}$, so that at $p = 1$, bidder I always randomizes between bids. This is because at $p = 1$, $\hat{s} = 1$, so bidder I always has exactly the cut-off signal.

Example 7: auction stage, the private-value framework, the non-stationary setting.

Suppose that $v(s_i) = s_i$. Using bidding strategies in Example 6, $\Pi_I^*(\hat{s}, \hat{s}, p) = \frac{a(\hat{s}, p)}{\hat{s}} (\hat{s} - \underline{a}(\hat{s}, p)) = \frac{\hat{s}}{4}$ and $\Pi_N^*(\hat{s}, \hat{s}, p) = \hat{s} - \bar{a}(\hat{s}, p)$. Additionally, from Lemma 9, $\Pi_S^*(\hat{s}, \hat{s}) = \frac{\hat{s}}{2}$.

The indifference condition for \hat{s}_t , (95), simplifies to

$$(r + \lambda) \frac{\hat{s}_t}{4} + \lambda \frac{1 - \hat{s}_t}{1 - p_t} \left(\bar{a}(\hat{s}_t, p_t) - \frac{3\hat{s}_t}{4} \right) = \frac{p_t}{1 - p_t} \lambda \hat{s}_t (1 - \hat{s}_t) \frac{1}{2} \left(\frac{\hat{a}(\hat{s}_t, p_t) - \frac{\hat{s}_t}{2}}{\hat{s}_t - \hat{a}(\hat{s}_t, p_t)} + \frac{1}{2} \right) + \nu \left(\frac{\hat{s}}{2} - \frac{\hat{s}}{4} \right), \quad (113)$$

where $\bar{a}(\hat{s}_t, p_t)$ and $\hat{a}(\hat{s}_t, p_t)$ are numerically determined from (111) and (112). Equation (113) sets up a differential equation for \hat{s}_t . Specifically, first, (113) is used to obtain $p(\hat{s}_t)$; second, differential equation $\hat{s}_t' = -\frac{p(\hat{s}_t)}{1 - p(\hat{s}_t)} \lambda \hat{s}_t (1 - \hat{s}_t)$ with the initial condition $\hat{s}_0 = 1$ is numerically solved.

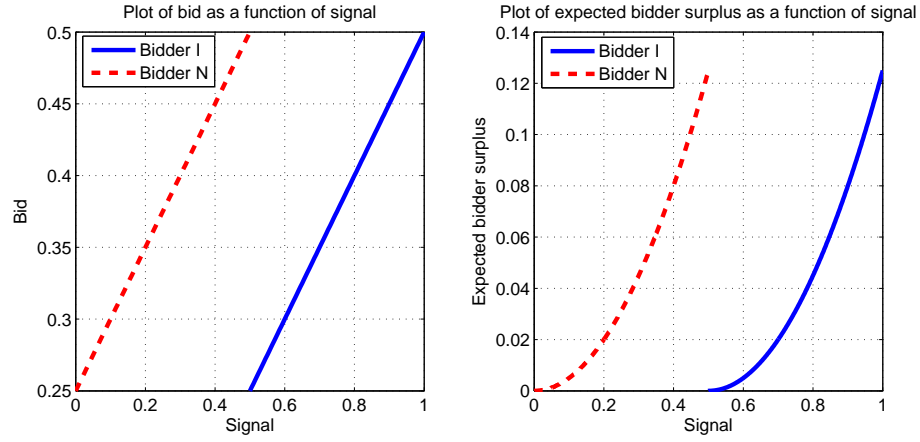


Figure 1: **Equilibrium bids and expected payoffs of bidders in a bidder-initiated common-value auction.** The left panel plots the equilibrium bids as functions of signals for the initiating bidder (the normal line) and the non-initiating bidder (the dashed line). The right panel plots the corresponding expected surpluses of each bidder.

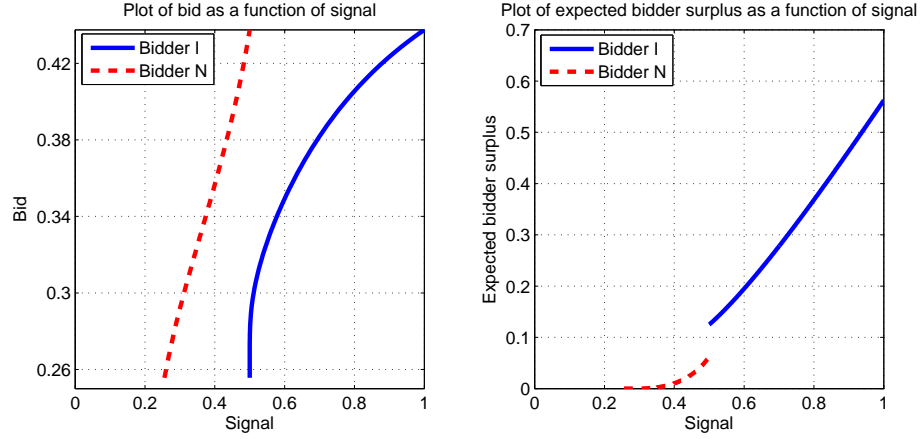


Figure 2: **Equilibrium bids and expected payoffs of bidders in a bidder-initiated private-value auction.** The left panel plots the equilibrium bids as functions of signals for the initiating bidder (the normal line) and the non-initiating bidder (the dashed line). The right panel plots the corresponding expected surpluses of each bidder.

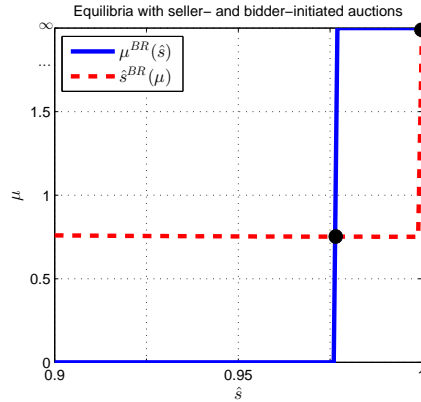


Figure 3: **Equilibria with seller- and bidder-initiated auctions.** For the case of no involuntary seller initiation, $\nu \rightarrow 0$, the figure plots best responses of the seller (the normal line) and each bidder (the dashed line), as well as multiple equilibria (circles). With involuntary seller initiation, $\nu > 0$, there may be an additional equilibrium with \hat{s} determined by the bidder's best response and $\mu = \nu$.

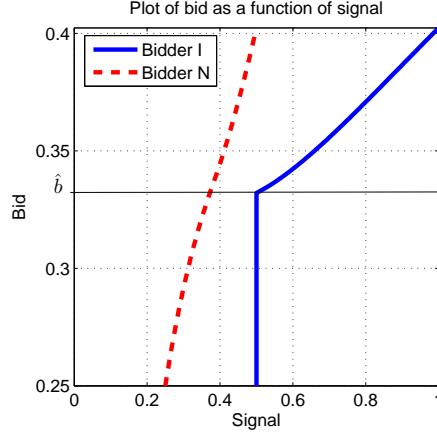


Figure 4: **Equilibrium bids in a bidder-initiated private-value auction in the non-stationary setting.** The figure plots the equilibrium bids as functions of signals for the initiating bidder (the normal line) and the non-initiating bidder (the dashed line).

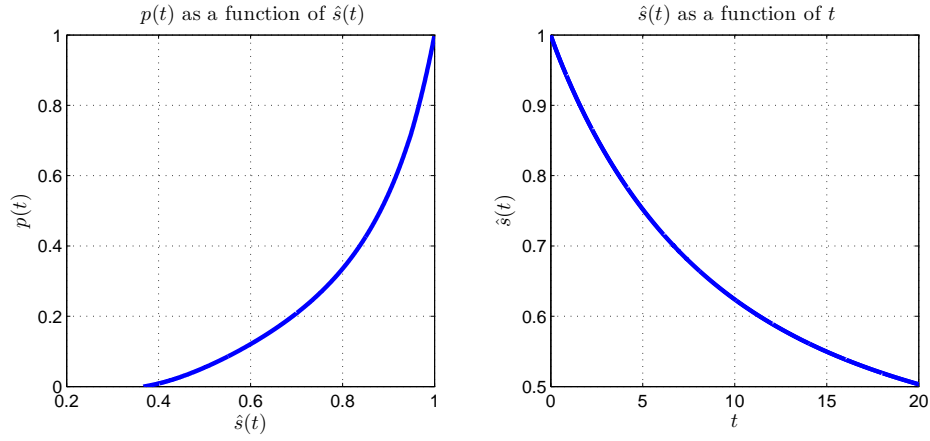


Figure 5: **The dynamics p_t and \hat{s} in the non-stationary setting.** The left panel plots the behavior of the conditional probability that bidder I 's signal is exactly at the cut-off, p_t , as a function of the cut-off, \hat{s}_t . The right panel plots the behavior of the cut-off as a function of time.