HHIF Lecture Series:
Modern Portfolio Theory and Asset Pricing

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Portfolio Management

- Let’s say after performing some analysis you decided to invest in several different securities
- Your combined investments in the securities form a portfolio
- Question: **What are the best weights in this portfolio?**

**Things to consider:**
- Expected Return of the whole portfolio
- Risk of the whole portfolio and how to measure it
- What are your targets for portfolio performance?

To form a portfolio that is expected to meet your targets, need to:
- Model the expected return and risk of securities
- Model the risk of the whole portfolio
- Use the model to determine optimal weights
Modern Portfolio Theory

Assumptions:

- *Standard Deviation* of Return should be used to measure Risk
- Asset Returns are *jointly normally distributed*
- Investors are *risk-averse*: Between two assets with same return, the one with lower risk is chosen

Benefits of *Diversification* - portfolio risk can be reduced by holding uncorrelated assets

Given assets in a portfolio, can plot all possible risk-return combinations of portfolios combined of those assets

- *Min. Variance portfolio* - lowest level of risk possible
- *Efficient Frontier* - highest return for given level of risk

Investor Goal is to maximize Sharpe Ratio:

\[ S = \frac{E(R - R_f)}{\sigma(R - R_f)} \]

- Sometimes assume risk-free rate \((R_f)\) is constant
Modern Portfolio Theory - Optimization

- We consider two common optimization problems in portfolio theory
  - Consider a portfolio $P$ of $n$ assets $A_1, A_2, \ldots, A_n$
  - Asset $A_i$ provides return $R_i$ and has weight $w_i$ in portfolio
  - Let $w$ be the vector of weights, $w = (w_1, w_2, \ldots, w_n)$
  - Let $\Sigma$ be the covariance matrix of $r = (R_1, R_2, \ldots, R_n)$; $\Sigma_{ij} = \sigma_{ij}$

Which weights produce the portfolio with minimum variance?

$$\text{Minimize: } \sigma^2(P) = \sigma^2(w_1R_1 + w_2R_2 + \ldots + w_nR_n)$$
$$= \sum_{i=1}^{n} w_i^2 \sigma^2(R_i) + \sum_{i<j} 2w_i w_j \sigma_{ij} = w^t \Sigma w$$
subject to: $w_1 + w_2 + \ldots + w_n = w^t(1 \ 1 \ldots \ 1) = 1$

Which weights produce the portfolio with maximum Sharpe Ratio?

- We assume for simplicity $R_f$ is constant

$$\text{Maximize: } S(P) = \frac{E(R) - R_f}{\sigma(R)} = \frac{w^t r - R_f}{\sqrt{w^t \Sigma w}}$$
subject to: $w_1 + w_2 + \ldots + w_n = w^t(1 \ 1 \ldots \ 1) = 1$
Modern Portfolio Theory - Optimization

- These problems can be explicitly solved in terms of \( r, \Sigma \)
- They can also be solved using software (e.g. Excel, Matlab, R, etc.)
- We can consider more complicated problems related to portfolio management
  - The portfolio is expected to achieve a specific return
  - There are constraints on portfolio weights
  - Adjusting weights periodically
  - Many others, depending on portfolio objective

What is the big picture here?

- Diversification decreases risk (but also usually expected return)
- Assumption: investors hold portfolios to maximize Sharpe Ratio
- For a given set of assets, the optimal risky portfolio (with maximum Sharpe Ratio) is unique
  - Regardless of risk characteristics of the investor
  - Invest in optimal risky portfolio and risk-free asset depending on risk aversion to get optimal complete portfolio
Modern Portfolio Theory - Graph

![Graph showing the Capital Market Line and the Efficient Frontier, with points for the Optimal Risky Portfolio and the Min Variance Portfolio.](#)
Capital Asset Pricing Model

- Assumptions:
  - MPT: risk aversion, care about mean and variance only
  - Single period investment horizon
  - Investors have homogenous expectations
  - Perfect Market
    - Individual investor cannot affect stock price
    - No taxes or transaction costs; everyone is equally informed
    - Lending and borrowing at risk-free rate

- Using these assumptions we can price assets (e.g. stocks)

- **Market Portfolio**: portfolio of all assets on the market
  - Exists only in theory, impossible to construct
  - However, can approximate, e.g. S&P 500 ($\approx 75\%$ of US equity)

- Total Supply $= \text{Total Demand}$
  - Net supply of risk-free asset is zero
  - Therefore: combine investor portfolios to get market portfolio!
  - Optimal Risky portfolio for every investor is the same!
  - Hence optimal risky portfolio is the market portfolio
Consider an asset $A$ with expected return $R_a$, standard deviation $\sigma_a$, correlation with market $\sigma_{am}$.

Adding $A$ to market portfolio should give same risk-adjusted return.

Using basic algebraic manipulations, get following result:

$$E(R_a) = R_f + \beta_a (E(R_m) - R_f),$$

where $R_m$ is market return, $\beta_a = \frac{\sigma_{am}}{\sigma_m^2}$ is beta.

This can also be written as $E(R_a) - R_f = \beta_a (E(R_m) - R_f)$

- $E(R_a) - R_f$ is risk premium
- $\beta$ is risk
- $E(R_m) - R_f$ is market risk premium, here "price of risk"

Usually estimate beta by using historical stock and benchmark prices

- Calculate historical returns (daily/monthly/yearly/etc.)
- Find $\sigma_{am}$ and $\sigma_m^2$, then $\beta_a = \frac{\sigma_{am}}{\sigma_m^2}$
- Or run regression: $R_{a,t} - R_f = \beta_0 + \beta_a (R_{m,t} - R_f) + \epsilon_{a,t}$
CAPM - Systematic and Unsystematic Risk

- The CAPM can be rewritten to include an error term $\epsilon_a$:
  \[ R_a = R_f + \beta_a(R_m - R_f) + \epsilon_a \]

  - Assume $E(\epsilon_a)$ and error is normal (from MPT)
  - The error is specific to asset, results in unique risk
  - The $\beta_a(R_m - R_f)$ term results in systematic (market) risk

- By definition of unique risk $R_m$ and $\epsilon_a$ uncorrelated, so:
  \[ \sigma_a^2 = \beta_a^2 \sigma_m^2 + \text{Var}(\epsilon_a) \]

  - Total risk can be decomposed into market and unique risk

- Consider $n$ assets $A_1, A_2, \ldots A_n$ in a portfolio $P$
  - Each asset has weight $w_i$, beta $\beta_i$ and unique risk $\epsilon_i$
  - We can determine unsystematic and unique risk of $P$

  \[ \beta_p = \frac{\sigma_{pm}}{\sigma_m^2} = \frac{\text{Cov}(w_1R_1 + \ldots + w_nR_n, r_m)}{\sigma_m^2} = \sum_{i=1}^{n} w_i \frac{\text{Cov}(R_i, r_m)}{\sigma_m^2} = \sum_{i=1}^{n} w_i \beta_i \]

  Portfolio beta is weighted sum of asset betas
Consider a portfolio $P$ of three stocks $A$, $B$, $C$

- Assume $\beta_A = 0.5, w_A = 0.1$
- $\beta_B = 1.1, w_B = 0.4$
- $\beta_C = 1.8, w_C = 0.4$
- Remaining part - cash, $\beta = 0$

Assume $R_f = 1\%$, market risk premium $E(R_m) - R_f = 9\%$

We can then calculate expected return on whole portfolio

- $\beta_P = w_a\beta_A + w_b\beta_B + w_c\beta_C = 1.21$
- Then expected return of portfolio is $1 + 1.21 \cdot 9 = 11.89\%$
Systematic and Unique Risk for a Portfolio

- Consider a portfolio $P$ of $n$ assets with same notation as before.
- Systematic risk is not hard to compute:
  - $\text{systematic risk} = \beta_p^2 \sigma_m^2 = (w_1 \beta_1 + \ldots + w_n \beta_n)^2 \sigma_m^2$
  - Holding low beta stocks results in lower risk.
  - Almost always $\beta_i > 0$ so $\beta_p > 0$; very hard to eliminate market risk.
- Let us look at unique risk:
  - Recall $R = w_1 R_1 + w_2 R_2 + \ldots + w_n R_n$.
  - According to CAPM $R_i = R_f + \beta_i (R_m - R_f) + \epsilon_i$.
  - So $R = R_f + \beta_p (R_m - R_f) + \epsilon_p$ and $\epsilon_p = w_1 \epsilon_1 + \ldots + w_n \epsilon_n$.
  - We assume unique risks are uncorrelated, so $\text{cov}(\epsilon_i, \epsilon_j) = 0$; then:
    \[
    \text{Var}(\epsilon_p) = w_1^2 \text{Var}(\epsilon_1) + w_2^2 \text{Var}(\epsilon_2) + \ldots + w_n^2 \text{Var}(\epsilon_n)
    \]
  - Assume $w_i = \frac{1}{n}$, $\text{Var}(\epsilon_i) = \sigma^2$.
    - Then $\text{Var}(\epsilon_p) = \frac{1}{n} \sigma^2$. This decreases as $n$ (number of assets) increases.
    - For sufficiently large $n$, $\text{Var}(\epsilon_p)$ (unique risk) becomes very small.
    - Note we cannot have $n \to \infty$, this is real life.
Recap of CAPM and Advantages

- Optimal risky portfolio is market portfolio, therefore:
  \[ \bar{R}_a = \bar{R}_f + \beta_a(\bar{R}_m - \bar{R}_f) + \epsilon_a \]

- Portfolio beta is weighted sum of asset betas
- Using regression can estimate systematic and unique risk
  - Market risk usually cannot be eliminated
  - Unique risk can be greatly reduced with many assets

Advantages

- A simple way to estimate asset and portfolio return and risk
- Helps explain the benefits of diversification and decomposition of risk
- Allows for measurement of portfolio performance
  - For a specific period, find average return \( \bar{R} \) on portfolio
  - Calculate \( \hat{\beta} \), \( \bar{R}_f \), \( \bar{R}_m \) based on prices during period
  - Perform regression: \( \bar{R} - \bar{R}_f = \alpha + \hat{\beta}(\bar{R}_m - \bar{R}) \)
  - \( \alpha \) called Jensen’s alpha; if positive - beat the market
Drawbacks of CAPM

• CAPM on average explains only 80% of portfolio returns
  ▶ Some other models do much better (e.g. 90% for Fama-French)
• The market portfolio should include everything
  ▶ Stocks, bonds, commodities, real estate, human capital, etc.
  ▶ Evidence that using a more complete portfolio explains more of returns
• Very strong assumptions about distribution of returns
  ▶ Stock returns are often not normally distributed
  ▶ Standard Deviation, Covariance are not constant
• Fama and French have shown that small cap value stocks exhibit abnormal positive returns
• Not clear that investors have "homogenous" expectations
  ▶ Although many argue that outlying expectations "cancel out"
• Various constraints on portfolio selection and trading
  ▶ Institutional Level, e.g. investment grade, margin requirements
  ▶ Individual Investor level - risk aversion, irrational behavior?
Arbitrage Pricing Theory

- Assumptions:
  - No taxes or transaction costs
  - Some agents are smart and can form desired portfolios
  - Securities have finite expected values and standard deviations

Assumptions allow a smart investor to execute any trading strategy

- **Arbitrage**: "free lunch"; there exists a strategy which:
  1. Starts with portfolio $P$ containing nothing.
  2. At the end is guaranteed to produce positive money.

  (*Self-financing* condition - cannot invest external funds in portfolio)

- Example: two 1-yr zero-coupon bonds with 3% and 4% yields

- APT: whenever there is arbitrage, some smart investor(s) will immediately exploit it \( \Rightarrow \) Therefore there is no arbitrage!

- Note: in real life can have arbitrage
  - Maybe not worth to exploit due to transaction costs, trading rules, etc.
Multifactor Model

- Asset returns are driven by \( k \) factors
  - e.g. inflation, unemployment, manufacturing, market, etc.
- Return \( R_i \) on Asset \( i \) is given by:
  \[
  R_i = E(R_i) + \beta_{i,1}\tilde{f}_1 + \beta_{i,2}\tilde{f}_2 + \cdots + \beta_{i,k}\tilde{f}_k + \epsilon_k
  \]
  - Assume for \( 1 \leq j \leq k \), Factor \( j \) has return \( F_j \)
  - Define \( \tilde{f}_j = F_j - E(F_j) \) so we see \( E(\tilde{f}_j) = 0 \)
  - \( \beta_{i,j} \) is factor loading/beta - how return \( R_i \) responds to change in \( \tilde{f}_j \)
  - \( \epsilon_k \) is unique risk (not explained by factors); assume \( E(\epsilon_k) = 0 \)

- Consider a stock of a copper mining company with a factor loading of 1.5 on a US manufacturing index (e.g. Purchasing Managers Index) and a factor loading of 0.6 on inflation.
  If the manufacturing index increases by 4\% and inflation increases by 5\%, we expect the return on the stock to increase by:
  \[ 1.5 \cdot 4 + 0.6 \cdot 5 = 9\% . \]
APT Model

- Assuming the factor model and no arbitrage, we must have:

\[
E(R_i) = \lambda_0 + \beta_{i,1}\lambda_1 + \beta_{i,2}\lambda_2 + \cdots + \beta_{i,k}\lambda_k
\]

(1)

- Here \( \lambda_j \) is risk premium on factor \( j \); the extra return you expect to get for an extra unit of exposure to factor \( j \).
- Note that risk premium can be negative.
- \( \lambda_0 \) corresponds to the return on an asset with no factor risks, i.e. risk-free asset.

- Where does Arbitrage Pricing Come in?
  - Two portfolios with the same exposure to each factor should have the same price.
  - For assets 1, 2, ..., \( n \) in real life, there exist unique numbers \( \lambda_0, \lambda_1, \ldots, \lambda_k \) such that equation (1) holds for every asset.
  - Otherwise there is arbitrage!
Consider stocks of two companies: Ebay and Amazon

- Assume each company will buy back all its shares in one year
- Assume you can buy the stock now and cannot sell until the buyback
- Neither company pays dividends so capital gains is all you will get

Consider two cases for the online retail market in one year

- Measure "online retail market index" \( I \); now \( I = 100 \)
- Massive Expansion: \( I = 120 \) (20% return); probability 70%
- Minor Expansion: \( I = 105 \) (5% return); probability 30%
APT Example

- Estimate prices for each stock in one year:

<table>
<thead>
<tr>
<th></th>
<th>EBAY</th>
<th>AMZN</th>
</tr>
</thead>
<tbody>
<tr>
<td>I=120</td>
<td>$50.0</td>
<td>$300</td>
</tr>
<tr>
<td>I=105</td>
<td>$31.0</td>
<td>$160</td>
</tr>
<tr>
<td>E(Price)</td>
<td>$44.0</td>
<td>$258</td>
</tr>
<tr>
<td>Price Today</td>
<td>$34.7</td>
<td>$187.8</td>
</tr>
<tr>
<td>Return</td>
<td>27.7%</td>
<td>37.4%</td>
</tr>
</tbody>
</table>

- Estimate factor loadings $\beta_{EBAY,I}, \beta_{AMZN,I}$

  - For EBAY stock: $R_{EBAY} = E(R_{EBAY}) + \beta_{EBAY,I} \cdot \tilde{f}_I + \epsilon_{EBAY}$
  
  - Recall $\tilde{f}_I = F_I - E(F_I)$, NOT the factor return!
  
  Major Expansion: $44.1 = E(R_{EBAY}) + \beta_{EBAY,I} \cdot 4.5$
  Minor Expansion: $-10.6 = E(R_{EBAY}) + \beta_{EBAY,I} \cdot (-10.5)$

  - Solving get: $E(R_{EBAY}) = 27.7\%$ (already know this) and $\beta_{EBAY,I} = 3.65$

  - Do the same thing for AMZN, get: $E(R_{AMZN}) = 37.4\%$ (already know this) and $\beta_{AMZN,I} = 4.97$
APT Example

- Now that we know factor loadings, can find risk premia
- \( E(R_{EBAY}) = \lambda_0 + \beta_{EBAY,I} \cdot \lambda_I \)

For EBAY: \( 27.7 = \lambda_0 + 3.65\lambda_I \)
For AMZN: \( 37.4 = \lambda_0 + 4.97\lambda_I \)

- Solving, get \( \lambda_I = 7.35\% \), \( \lambda_0 = 0.87\% \)
- \( \lambda_I \) - risk premium on the "Internet Retail" factor
- \( \lambda_0 \) - return on asset not exposed to factor
  - If there are no other factors, this is risk-free rate
Applying APT

- The previous approach can be done for more than 2 factors
- Can apply APT to historical data to price assets

Two general approaches:

- **Principal Component Analysis**
  - Statistical procedure to extract factors from historical returns; estimates $\beta_{ij}$s and $\lambda_j$s together
  - However factors have no real life significance

- **Come up with factors yourself**
  - Could use macroeconomic factors (e.g. inflation, GDP, manufacturing index. etc.)
    - See Chen, Roll, Ross 1986 paper
  - Could use Fama-French factors - based on size and P/B ratio
    - See Fama-French three-factor model
  - Once you have identified factors, get factor returns, and estimate $\tilde{f}_j$s
  - Perform regression of $R_i$ on $\tilde{f}_j$s to estimate $\beta_{ij}$s
  - Perform cross-sectional regression of $E(R_i)$s on $\beta_{ij}$s to estimate $\lambda_j$s
Concluding Remarks

- Asset allocation is crucial when it comes to investing
- Finding optimal portfolio weights is a very hard problem
  - Depends on portfolio objectives and constraints
- Sharpe Ratio is often used to measure performance

- CAPM: a quick way to estimate portfolio return and risk
  - Makes a lot of assumptions that don’t hold true in real life
  - But does illustrate the benefits of diversification
- APT: also used to estimate return and risk, but harder to do
  - Identifying the factors is a major challenge
    - How many factors do we really need?
  - Need to be careful about which statistical tools and which data is used to estimate betas and risk premia

- These are just models!
  - Should be used as guidelines, not just blindly followed
  - Good idea to think about why a model outputs certain results
Any Questions?

Upcoming Events:

- HHIF Quarterly Meeting
  - Friday, March 4, 6:00-8:00 p.m., South Dining Room, Hart House
References

1. prof. Jiang Wang (MIT) - Arbitrage Pricing Theory Lecture

2. prof. Giovanna Nicodano (University of Toledo) - Multi-Factor Models and the APT Lecture
   http://web.econ.unito.it/nicodano/handout_bkm1_apt.pdf

3. Investopedia: Financial Concepts
   http://www.investopedia.com/university/concepts/default.asp