

Technical Notes and Correspondence

On the NP-Hardness of Checking Matrix Polytope Stability and Continuous-Time Switching Stability

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Abstract—Motivated by questions in robust control and switched linear dynamical systems, we consider the problem checking whether all convex combinations of k matrices in $R^{n \times n}$ are stable. In particular, we are interested whether there exist algorithms which can solve this problem in time polynomial in n and k . We show that if $k = \lceil n^d \rceil$ for any fixed real $d > 0$, then the problem is NP-hard, meaning that no polynomial-time algorithm in n exists provided that $P \neq NP$, a widely believed conjecture in computer science. On the other hand, when k is a constant independent of n , then it is known that the problem may be solved in polynomial time in n . Using these results and the method of measurable switching rules, we prove our main statement: verifying the absolute asymptotic stability of a continuous-time switched linear system with more than n^d matrices $A_i \in R^{n \times n}$ satisfying $0 \succeq A_i + A_i^T$ is NP-hard.

Index Terms—Robust control, switched systems, uncertain systems.

I. INTRODUCTION

Let $\{A_1, \dots, A_k\}$ be a finite set of real $n \times n$ matrices. We define the corresponding matrix polytope \mathcal{A} as those matrices which can be written as $\sum_{i=1}^k \alpha_i A_i$ for some nonnegative real numbers $\alpha_1, \dots, \alpha_k$ adding up to 1. We are concerned with the following decision problem, which we will call the (k, n) -POLYTOPE-STABILITY problem: given k rational $n \times n$ matrices $\{A_1, \dots, A_k\}$ to decide whether every matrix in \mathcal{A} is Hurwitz stable, i.e., has eigenvalues with negative real parts.

The (k, n) -POLYTOPE-STABILITY problem, and some of its natural generalizations, have been considered before in the control theory literature [1]–[4], [6]–[11], [16], [18], [21], [22], [24], [25], [27], [29]. Notable results include the solution of the $k = 2$ case in [1], [11], [24], also known as the stability testing of affine representations; a Lyapunov-search algorithm in [10]; and a recent approach based on LMI relaxations [16], [9], [27].

Our first result is that when there are $\lceil n^d \rceil$ different $n \times n$ rational matrices A_i , where d is any positive real number (in the above notation, the $(\lceil n^d \rceil, n)$ -POLYTOPE-STABILITY problem) the problem of deciding whether there exists an unstable matrix in \mathcal{A} is NP-hard.

On the other hand, in many circumstances where the (k, n) -POLYTOPE-STABILITY problem appears, k is constant that does not vary with n . Consider, for example, testing the stability of an $n \times n$ interval matrix where all but r of the entries are known precisely. It is easy to see that this is a special case of the polytope stability problem with $k = 2^r$. Moreover, if r is fixed, but n is allowed to vary, we end up with a polytope stability problem with fixed k .

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We note that the (k, n) -POLYTOPE-STABILITY problem can be solved in $n^{O(k)}$ elementary operations by reducing the problem to deciding whether a certain multivariate polynomial does not have real roots, which can in turn be solved using quantifier elimination algorithms – see [15] for a detailed writeup (the reduction to a root localization problem for a multivariate polynomial was first done in [16]). If the number k is fixed, this gives a deterministic polynomial-time algorithm.

The second and main subject of our technical note is the (k, n) – Continuous Time Absolute Switching Stability Problem ((k, n) – CTASS problem): given k rational matrices A_1, \dots, A_k in $R^{n \times n}$ to decide whether there exists a norm $\|\cdot\|$ in R^n and $a > 0$ such that the induced operator norms satisfy the inequalities

$$\|\exp(A_i t)\| \leq e^{-at} : a > 0, t \geq 0, 1 \leq i \leq k. \quad (1)$$

We consider matrices that satisfy the further restriction of being non-strict contractions in the two-norm

$$\|\exp(A_i t)\|_2 \leq 1, t \geq 0, 1 \leq i \leq k$$

which is equivalent to requiring

$$0 \succeq A_i + A_i^*.$$

We show that, even in this case, checking the condition of (1) with more than n^d matrices $A_i \in R^{n \times n}$, where d is any positive real number, is also NP-hard. As far as we know, this is the first hardness result for continuous-time switched linear systems.

We note that it is well known that the stability of the polytope \mathcal{A} is necessary, but not in general sufficient, for the absolute switching stability condition of (1) to hold. Luckily, these two conditions are equivalent for the “gadgets” used in our proof of NP-HARDNESS of polytope stability.

We stress that our result says nothing about the solvability of concrete, finite-size instances for the problem – for instance, this technical note has no new implications for testing the stability of all convex combinations of 3 matrices in $R^{8 \times 8}$. Rather our result implies that if $P \neq NP$ then any algorithm for solving this problem with n^d matrices $n \times n$ matrices, for any $d > 0$, will have a worst-case operations count which grows faster than any polynomial in n .

This provides an explanation why many approaches to this problem fail. Despite an extensive literature devoted to this problem cited above, no polynomial time algorithms are known. See, in particular, [7], for proofs that various intuitive approaches fail. Our result implies that in fact any polynomial time algorithm for this problem would immediately disprove the $P \neq NP$ conjecture.

II. NP-HARDNESS OF STABILITY TESTING OF MATRIX POLYTOPES

In this section, we consider the computational complexity of deciding whether every matrix in the set \mathcal{A} is stable. We will show that this problem is NP-hard through a reduction from the maximum clique problem, which is known to be NP-complete [17]. The details of this reduction are described below.

Theorem 1: $(n, 2n + 2)$ -POLYTOPE-STABILITY is NP-hard.

Notice that the well-known interval stability problem¹ corresponds to (k, m) -POLYTOPE-STABILITY with k exponential in m . In other words NP-hardness of interval stability, shown in [20] and [23], does not imply NP-hardness of $(n, 2n + 2)$ -POLYTOPE-STABILITY.

We first give a series of definition and lemmas before proving Theorem 1. Given k rational matrices A_1, \dots, A_k in $R^{n \times n}$, we will refer to the problem of deciding whether there exists a singular matrix in the associated polytope \mathcal{A} as the (k, n) -POLYTOPE-NONSINGULARITY problem.

Lemma 2: There is a polynomial-time reduction from the (k, n) -POLYTOPE-NONSINGULARITY problem to the $(k, 2n)$ -POLYTOPE-STABILITY problem

Proof: Given a square matrix $A \in R^{n \times n}$ define B as

$$B = \begin{pmatrix} 0_{n \times n} & A^T \\ -A & -I_{n \times n} \end{pmatrix}.$$

We claim B is Hurwitz if and only if A is nonsingular. This follows because spectrum of B can be written as $\sigma(B) = \cup_{1 \leq i \leq n} \{(1/2)(-1 \pm \sqrt{1 - (4s_i(A))^2})\}$, where $s_1(A) \leq \dots \leq s_n(A)$ are singular values of the matrix A .

Suppose we are given k $n \times n$ matrices A_i , and we want to decide whether the set \mathcal{A} contains a nonsingular matrix. Define

$$B_i = \begin{pmatrix} 0_{n \times n} & A_i^T \\ -A_i & -I_{n \times n} \end{pmatrix}.$$

Since

$$\sum_i \alpha_i B_i = \begin{pmatrix} 0_{n \times n} & (\sum_i \alpha_i A_i)^T \\ -\sum_i \alpha_i A_i & -I_{n \times n} \end{pmatrix},$$

when $\sum_i \alpha_i = 1$, it follows by the previous item that testing POLYTOPE-NONSINGULARITY with the set \mathcal{A} is the same as testing POLYTOPE-STABILITY on the set

$$\mathcal{B} = \left\{ B \mid B = \sum_i \alpha_i B_i, \sum_i \alpha_i = 1, \alpha_i \geq 0 \text{ for all } i \right\}.$$

However, note that the construction has doubled the dimension, since the matrices B_i belong to $R^{2n \times 2n}$. This concludes the proof that (k, n) -POLYTOPE-NONSINGULARITY can be reduced to $(k, 2n)$ -POLYTOPE-STABILITY. ■

Consider the problem of deciding whether there exists a nonnegative vector p in R^n whose components sum to 1 such that $p^T M p = 1$ for an arbitrary invertible matrix M . We will consider M^{-1} to be the input to this problem. We will refer to the problem as the n -QUADRATIC-THRESHOLD problem.

Lemma 3: There is a polynomial-time reduction from the n -QUADRATIC-THRESHOLD problem to the $(n, n + 1)$ -POLYTOPE-NONSINGULARITY problem.

Proof: Define

$$X_i^{(n)} = \begin{pmatrix} M^{-1} & e_i \\ e_i^T & 1 \end{pmatrix}$$

where e_i is the column vector with 1 in the i 'th entry and zeros elsewhere. Define $\mathcal{S}_n = \{p \in R^n \mid \sum_i p_i = 1, p_i \geq 0 \forall i\}$ and let

$$\mathcal{X} = \left\{ X \mid X = \sum_i p_i X_i, p \in \mathcal{S}_n \right\}.$$

¹The interval stability problem is to determine, given numbers $\{\underline{a}_{ij}, \bar{a}_{ij}\}_{i,j=1..m}$, whether every matrix A satisfying $A_{ij} \in [\underline{a}_{ij}, \bar{a}_{ij}]$ is stable.

In other words, \mathcal{X} is the set of matrices of the form

$$\begin{pmatrix} M^{-1} & p^T \\ p & 1 \end{pmatrix}$$

with $p \in \mathcal{S}_n$. By the Schur complement formula such a matrix is singular if and only if $p^T M p = 1$. Thus given an invertible matrix M , we can solve the n -QUADRATIC-THRESHOLD problem by solving an instance of the $n + 1$ -POLYTOPE-NONSINGULARITY problem with the polytope \mathcal{X} . ■

Let $G = (V, E)$ be an undirected graph. A subset of the vertices C is called a clique if $v_1, v_2 \in C$ implies $(v_1, v_2) \in E$. The n -MAX-CLIQUE problem is the problem of determining the size of the largest clique, denoted by $\omega(G)$, in an undirected graph G on n vertices.

Lemma 4: There is a polynomial-time reduction from the n -MAX-CLIQUE problem to the n -QUADRATIC-THRESHOLD problem

Proof:

1) It is known that [19]

$$1 - \frac{1}{\omega(G)} = \max_{p \in \mathcal{S}_n} p^T M p \quad (2)$$

where M is the adjacency matrix of the graph G .

2) Because the QUADRATIC-THRESHOLD problem is defined only for nonsingular matrices, for our reduction to work we will need to modify M to insure its nonsingularity. To this end, we consider the matrices $M_i = M + (1/(n^2 + i))I$ for $i = 1, \dots, n + 1$. At least one M_i must be nonsingular, because M cannot have $n + 1$ eigenvalues. We find a nonsingular M_i (this can be done in polynomial time with Gaussian elimination for each $i = 1, \dots, n + 1$). Let us denote this nonsingular M_i by M_{i^*} .

In the proof below, we will threshold the form $p^T M_{i^*} p$; recall from Lemma 3 that this requires the computation of $M_{i^*}^{-1}$. This involves a polynomial number of computations in n , and the bit-sizes remain polynomial as well; for a proof see Corollary 3.2a of [26].

3) We have that for $p \in \mathcal{S}_n$

$$p^T M_{i^*} p = p^T M p + \frac{1}{n^2 + i^*} \sum_i p_i^2.$$

Because $p_i \in [0, 1]$ and $\sum_i p_i = 1$ imply that $\sum_i p_i^2 \leq 1$, we have

$$p^T M p \leq p^T M_{i^*} p \leq p^T M p + \frac{1}{n^2 + i^*}. \quad (3)$$

It follows that:

$$1 - \frac{1}{\omega(G)} \leq \max_{p \in \mathcal{S}_n} p^T M_{i^*} p \leq 1 - \frac{1}{\omega(G)} + \frac{1}{n^2 + i^*}. \quad (4)$$

4) Because the optimal solution of (2) is $1 - 1/\omega(G)$, and $\omega(G)$ is an integer between 1 and n , this optimal solution must be in the set $S = \{0, 1/2, 2/3, 3/4, \dots, 1 - 1/n\}$. Because the gap between the elements of S is less than $1/n^2$, and consequently less than $1/(n^2 + i^*)$, it follows from (4) that the largest element of S smaller than $\max_{p \in \mathcal{S}_n} p^T M_{i^*} p$ must be $1 - 1/\omega(G)$, the solution of the MAX-CLIQUE problem. Let this element be called k^* ; then, it follows that:

$$\frac{1}{k^*} p^T M_{i^*} p \geq 1 \quad (5)$$

for some $p \in \mathcal{S}_n$, and k^* is the largest element of S with this property.

For each $k \in S$, the existence of a $p \in \mathcal{S}_n$ satisfying (5) can, due to the invertibility of M_i^* , be decided by evaluating $p^T M_i^* p$ at an arbitrary $p \in \mathcal{S}_n$, followed up with a call to the QUADRATIC-THRESHOLD problem. This is the reduction from MAX-CLIQUE to QUADRATIC-THRESHOLD. ■

Proof of Theorem 1: Lemmas 2, 3, 4 provide a reduction from the MAX-CLIQUE problem to the POLYTOPE-STABILITY problem. The size of the problem goes from n in the QUADRATIC-THRESHOLD problem to $(n, n + 1)$ after the reduction to POLYTOPE-NONSINGULARITY; and from $(n, n + 1)$ to $(n, 2(n + 1))$ in the reduction from POLYTOPE-NONSINGULARITY to POLYTOPE-STABILITY. Since MAX-CLIQUE is known to be NP-complete [17], it follows that $(n, 2n + 2)$ -POLYTOPE-STABILITY is NP-hard. ■

Remark: Note that the matrices A_i created in our reduction have entries whose bit-size are polynomial in n .

Remark:

- 1) We note that our results easily imply the NP-HARDNESS of POLYTOPE-STABILITY with $\lceil n^d \rceil$ extreme points, for any real $d > 0$ (here $\lceil x \rceil$ refers to the smallest integer which is at least x). Indeed, the $\lceil n^d \rceil$ -MAX-CLIQUE problem remains NP-COMPLETE, and therefore the $(\lceil n^d \rceil, 2\lceil n^d \rceil + 2)$ -POLYTOPE-STABILITY remains NP-Hard. However, we clearly do not make the problem any easier by increasing the dimension, so that the $(\lceil n^d \rceil, 2n + 2)$ problem is NP-Hard.
- 2) We have remarked that the (k, n) -POLYTOPE STABILITY problem may be solved in polynomial time if k is upper bounded by a constant. We may ask about the reverse question: what happens when n is upper bounded by a constant? Is the problem solvable in polynomial time as a function of k ? The answer is yes. Caratheodory's theorem implies that any matrix in \mathcal{A} may be expressed as a convex combination of $n^2 + 1$ matrices. Thus we reduce the problem of checking (k, n) polytope stability to the problem of checking $\binom{k}{n^2+1}$ different $(n^2 + 1, n)$ polytope stability problems. When n is upper bounded, checking $(n^2 + 1, n)$ -POLYTOPE-STABILITY (say by computing the determinant explicitly and using quantifier elimination) takes a constant number of operations, so the number of operations grows as $\binom{k}{n^2+1}$, which, when n is upper bounded, is polynomial in k .
- 3) Essentially, our construction boils down to the following determinantal representation

$$Q(p_1, \dots, p_n) = \det(A_0 + \sum_{1 \leq i \leq n} p_i A_i). \quad (6)$$

Such representations exists for any polynomial $Q(p_1, \dots, p_n)$ [28]. In our case $Q(p_1, \dots, p_n) = \langle Mp, p \rangle - 1$.

III. NP-HARDNESS OF CHECKING CONTINUOUS-TIME ABSOLUTE SWITCHING STABILITY

In this section, we will show that checking the absolute switching stability of a class of continuous-time linear switched systems is NP-hard. Recall that the (k, n) continuous time switching stability problem is: given k rational matrices A_1, \dots, A_k in $R^{n \times n}$, to decide whether there exists a norm $\|\cdot\|$ in R^n and $a > 0$ such that the induced operator norms satisfy the inequalities

$$\|\exp(A_i t)\| \leq e^{-at} : t \geq 0, 1 \leq i \leq k. \quad (7)$$

We consider a subcase of the problem where the matrices A_i satisfy the (nonstrict) Lyapunov inequalities $0 \succeq A_i + A_i^T, 1 \leq i \leq k$. In assuming this, we only make the problem easier, as we assume that the matrices A_i have a nice geometrical structure; indeed, the previous requirement corresponds to requiring that solutions of the equation

$\dot{x}(t) = A_i x(t), i = 1, \dots, k$ with measurable switching rules, are nonincreasing in the 2-norm.

Nevertheless, we will show that testing continuous time stability is NP-hard already in this case. The following lemma – together with Theorem 1 – proves this. As far as we know it is the first hardness result in the area of continuous time absolute switching stability.

Lemma 5: Consider the following $2n \times 2n$ matrices $B_i, 1 \leq i \leq k < \infty$:

$$B_i = \begin{pmatrix} 0_{n \times n} & A_i^T \\ -A_i & -I_{n \times n} \end{pmatrix}.$$

Then there exists a norm $\|\cdot\|$ in R^{2n} and $a > 0$ such that the induced operator norms satisfy the following inequalities:

$$\|\exp(B_i t)\| \leq e^{-at} : t \geq 0, 1 \leq i \leq k \quad (8)$$

if and only if all matrices in the convex hull \mathcal{A} are nonsingular.

Before proving the lemma, we need the following auxiliary claim. Consider the following family of “differential” equations²:

$$\dot{x}(t) = \left[\sum_{i=1}^k p_i(t) B_i \right] x(t)$$

with initial condition satisfying $\|x(0)\|_2 = 1$. Here $(p_1(t) \dots p_k(t))$ is a Lebesgue-measurable vector function whose range is a subset of \mathcal{S}_k . Since the $\sum_i p_i B_i$ lies in a bounded set of matrices, the above equation has a unique Lipschitz solution. Since $0 \succeq B_i + B_i^T$, it follows that $\|x(t)\|_2 \leq 1$ for $t \geq 0$.

1) *Claim:* Assume that there is no induced norm satisfying (7). Then, there exists a measurable vector function $(p_1(t) \dots p_k(t))$ whose range is a subset of \mathcal{S}_k , and a vector $x(0)$ with $\|x(0)\|_2 = 1$ such that the solution of

$$\dot{x}(t) = \left[\sum_{i=1}^k p_i(t) B_i \right] x(t) \quad (9)$$

with initial condition $x(0)$ satisfies $\|x(1)\|_2 = 1$.

Proof of Claim: We will prove the contrapositive of the claim. It can be seen that the set of all possible values of $x(t)$ at time $t = 1$ produced by choices of $p(t), x(0)$ which satisfy our assumptions is compact; see Theorem 4.7 in [13] for details of the proof. So, suppose the conclusion is not true, by compactness this means there exists $\delta > 0$ such that for every $p, x(0)$ satisfying our assumptions, we have that $\|x(1)\|_2 < 1 - \delta$. Thus there exists an $\epsilon > 0$ such that the system

$$\dot{x}(t) = \left[\sum_{i=1}^k p_i(t) (B_i + \epsilon I) \right] x(t) \quad (10)$$

has the same property (i.e., there exists $\hat{\delta}$ with $\|x(1)\|_2 < 1 - \hat{\delta}$ for all suitable choices of $x(0), p(t)$).

We will define a norm such that $\|e^{B_i t}\| \leq e^{-at}$ for some $a > 0$ and all $t \geq 0$, thus violating (8). Define a norm on R^n as follows. For $q \in R^n$

$$\|q\|_n = \sup_{u \geq 1} \sup_{\text{measurable } p(z), z \in [0, u] \text{ with range in } \mathcal{S}_k} \|x(u)\|_2$$

where $x(\cdot)$ is a solution to (10) with $x(0) = q$.

²Strictly speaking, they ought to be viewed as integral equations

$$x(t) = x(0) + \int_0^t \left(\sum_{i=1}^k p_i(\tau) B_i \right) x(\tau) d\tau.$$

This norm induces a norm on $R^{n \times n}$

$$\|Q\|_{n \times n} = \sup_{x \in R^n, \|x\|_n=1} \|Qx\|_n.$$

However

$$\|e^{(B_i + \epsilon I)t} x\|_n \leq \|x\|_n$$

since premultiplication by $e^{B_i + \epsilon I}$ corresponds to simply requiring that $p(z) = e_i$ for the first t time units. Therefore

$$\|e^{B_i t}\|_{n \times n} \leq e^{-\epsilon t}$$

which proves the claim.

Proof of Lemma 5: We remark that the argument is very similar to Theorem 4.7 in [13] and Corollary 2.8 in [14].

First, we show the “only if” part, that is, assuming (8), all the matrices in the convex hull \mathcal{A} must be nonsingular. Indeed, suppose not; suppose there exists a vector p such that $\sum_{i=1}^k p_i B_i$ is not stable, with $\sum_{i=1}^k p_i = 1$ and all $p_i \geq 0$. Let λ be an eigenvalue of $\sum_{i=1}^k p_i B_i$ with nonnegative real part; then, e^λ will be an eigenvalue of $e^{\sum_{i=1}^k p_i B_i}$. Thus, $e^{\sum_{i=1}^k p_i B_i}$ has an eigenvalue of magnitude at least 1, so that

$$\left\| e^{\sum_{i=1}^k p_i B_i} \right\| \geq 1 \quad (11)$$

where $\|\cdot\|$ is the same norm as in (8). However, by the Baker-Campbell-Hausdorff formula, for all $\epsilon > 0$, there exists m large enough so that

$$\left\| e^{\sum_{i=1}^k p_i B_i} \right\| \leq (1 + \epsilon) \left\| \left(\prod_{i=1}^k e^{(1/m)p_i B_i} \right)^m \right\|.$$

Since $\|e^{(1/m)p_i B_i}\|^m \leq e^{-a p_i}$ by (8), we have that large enough m

$$\left\| e^{\sum_{i=1}^k p_i B_i} \right\| \leq (1 + \epsilon) e^{-a \sum_{i=1}^k p_i} = (1 + \epsilon) e^{-a} < 1 \quad (12)$$

where we pick ϵ small enough for the last inequality to hold. Equations (11) and (12) are in contradiction. We conclude that all the matrices in \mathcal{B} are indeed Hurwitz. By Lemma 2, this implies all matrices in \mathcal{A} are nonsingular. This proves the “only if” part.

Next, we show the “if” part. Let $x(0), p(t)$ be such that the conclusion of the above claim is satisfied. The corresponding curve $x(t) : 0 \leq t \leq 1$, satisfying (9) is Lipschitz; $\|x(t)\|_2 = 1$ for all $0 \leq t \leq 1$. It follows from the structure of matrices B_i that if $C \in \mathcal{B}$ then for all non-zero vectors x the inner product $\langle Cx, x \rangle \leq 0$ and $\langle Cx, x \rangle = 0$ if and only if $x \in R^n \oplus 0_n$. Here $R^n \oplus 0_n$ is the subspace spanned by $\{e_1, \dots, e_n\}$. If for some $\tau \in [0, 1]$ the vector $x(\tau)$ does not belong to $R^n \oplus 0_n$ then it also holds by continuity of the curve $x(t)$ in a sufficiently small neighbourhood $[\tau, \tau + \epsilon]$. This implies that $\langle x(\tau), Cx(\tau + \delta) \rangle < -b < 0$ for some $b > 0$ and all $0 \leq \delta \leq \epsilon$ and $C \in \mathcal{B}$. Define the vector $v(\epsilon) = \int_\tau^{\tau+\epsilon} (\sum_{i=1}^k p_i(s) B_i) x(s) ds$. It is clear that $\|v(\epsilon)\|_2 \leq K\epsilon$ for some constant K and that $\langle x(\tau), v(\epsilon) \rangle \leq -b\epsilon$. Therefore

$$\begin{aligned} \|x(\tau + \epsilon)\|_2^2 &= \|x(\tau)\|_2^2 + 2\langle x(\tau), v(\epsilon) \rangle + \|v(\epsilon)\|_2^2 \\ &\leq 1 - 2b\epsilon + K^2\epsilon^2. \end{aligned}$$

We get for small enough ϵ the inequality $\|x(\tau + \epsilon)\|_2 < 1$. We conclude that $x(t) \in R^n \oplus 0_n : 0 \leq t \leq 1$.

This gives that $\int_0^t (\sum_{1 \leq i \leq k} p_i(\tau) A_i) x_1(\tau) d\tau = 0$ for all $0 \leq t \leq 1$, where $x_1(\tau)$ is the vector formed by the first n components of $x(\tau)$. As the Lebesgue measurable vector function on $[0, 1](p_1(t) \dots p_k(t)) \in \mathcal{S}_k$ is bounded thus it follows that $(\sum_{1 \leq i \leq k} p_i(\tau) A_i) x_1(\tau) = 0$ up to measure zero. Since the last n components of $x(\tau)$ are zero, and $\|x(t)\|_2 = 1$, we have $\|x_1(t)\|_2 = 1, 0 \leq t \leq 1$. Therefore there must exist a singular matrix in \mathcal{A} . ■

Remark: In the discrete time case, it is known that given two $n \times n$ rational matrices A, B it is NP-hard to check if there exists a norm $\|\cdot\|$ in R^n such that the induced norms $\|A\|, \|B\| < 1$ [5]. On the other hand, it had been observed in [12], that a slight modification of a construction in [5] gives a direct proof of the following statement: given two $n \times n$ rational matrices A, B with nonnegative entries and $\|A\|_{l_1}, \|B\|_{l_1} \leq 1$ it is NP-HARD to check if there exists a norm $\|\cdot\|$ in R^n such that the induced norms $\|A\|, \|B\| < 1$. However, the continuous-time counterpart of this last problem is “easy” (see Theorem 2.1 in [14]). Based on this, it is unclear whether it is possible to modify the constructions from [5], [12] to handle the continuous time case.

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Delay-Independent Stability of Reset Systems

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Abstract—Reset control systems have potential advantages to overcome fundamental limitations of LTI compensation. However, since a reset compensator may destabilize a stable base LTI system, stability needs to be guaranteed in advance for a proper practical application of reset control. This work extends previous stability results of reset control systems to consider the case of LTI plants with time-delays. Stability independent of the delay criteria are developed both by means of LMIs and in the frequency domain by using the KYP Lemma.

Index Terms—Hybrid systems, reset control, time-delay systems.

I. INTRODUCTION

Reset control systems were one of the first attempts to overcome fundamental limitations of LTI control systems. Its development was started fifty years ago with the work of Clegg [7], that introduced a nonlinear integrator based on a reset action. Basically, since the integrator output is set to zero when its input is zero, a faster system response without excessive overshooting may be expected, thus avoiding limitation of its LTI counterpart. Although several other nonlinear compensators were developed, all based on describing function analysis, it is

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in a series works by Horowitz ([10], [12]), where reset control systems were impulsed by introducing the first order reset element (FORE).

The implementation of reset control is very simple, it consists of resetting the state (or part of it) of a feedback LTI compensator (referred to as the base LTI compensator) at every instant in which its input is zero. Usually the design of the reset control is strongly dependent on a proper election of the base LTI control system. A common approach is to design the base LTI system to be stable and to fulfill some performance specifications, and then including reset over some compensator states to improve performance and robustness. However, this method should be carefully applied, since it is well known [3] that the reset action may destabilize a base LTI control system, this being one of the main drawbacks for the practical application of reset control. Thus, reset control may be used to overcome fundamental limitations of LTI control systems, but it may work worse than its base LTI system, being the stability problem particularly important.

This work is focused on the problem of guaranteeing stability of a reset control system based on a stable base LTI system. The goal is to obtain extra conditions over the base LTI system for stability of the reset control system. This stability problem has been addressed in [3], [4] for the case of finite dimensional LTI plants, by using quadratic Lyapunov functions. The result is that stability can be checked by the strictly positive realness of a transfer function matrix H_β (the H_β condition), or alternatively by a pair of LMIs. More recently, a generalization of this latter result has been made in [15], where a pair of LMIs is obtained by using less restrictive Lyapunov functions. On the other hand, reset control systems are a special case of *hybrid systems* [8], [13], [18], and thus they have been also approached from this perspective as a special case of systems with impulse effects. For example, the work [9] addresses the stability problem of this type of systems with the goal of analyzing the stability of switching between LTI controllers.

This work will solve a generalization of the above problem for the case of plants with time-delays. Time-delay systems is a classical area in control engineering and other disciplines, that has received a considerable attention in the last decade, specially with the appearance of new techniques based on linear matrix inequalities (LMI's). A good survey of stability results may be found for example in the monograph [11]. In general, stability analysis can be separated in delay-independent and delay-dependent. In this work, a delay-independent analysis will be followed, the delay-dependent case being approached elsewhere. The content of the work is as follows. In Section II, we introduce some preliminaries related with the definition of the reset system by means of impulsive delay differential equations, including existence of solutions. Then, quadratic stability is defined and characterized via LMIs in Section III. Section IV presents the frequencial interpretation of stability, using a generalized KYP Lemma. Finally, Section V shows a numerical example that illustrates the results.

II. PRELIMINARIES

Consider a time-delay reset system given by the impulsive differential equation

$$\text{IDDE} : \begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h), & x(t) \notin \mathcal{M} \\ x(t^+) = A_R x(t), & x(t) \in \mathcal{M} \\ x(t) = \phi(t), & t \in [-h, 0] \end{cases} \quad (2.1)$$

for arbitrary values of A and A_d , and where the reset matrix A_R takes the form

$$A_R = \text{diag}(I_{n_p}, 0_{n_\rho}) \quad (2.2)$$