A Class of Near-Optimal Local Minima for Witsenhausen’s Problem

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Witsenhausen’s counterexample:

- A simple two-stage LQG decentralized control problem.
- Optimal controller is **nonlinear** in regime $k^2\sigma_0^2 = 1$ for large $\sigma_0$.
- Asymmetry of information can lead to nonlinear solutions in multi-stage decentralized decision-making.
- **Optimal controller still unknown**, conjectured to be near piecewise-linear.

\[
\min \{ k^2\mathbb{E}[u_1^2] + \mathbb{E}[x_2^2] \} 
\]
Motivation

A cherry-pick of related work:\(^1\)

- Nonlinear designs largely outperform linear strategies (Mitter and Sahai (99)).

- Bounds on proximity to optimality using information theoretic techniques (Grover and Sahai (2013)).

- Heuristic techniques to approximate optimal solution (Li et al. (2009), Baglietto et al. (2001), Lee et al. (2001)).

- Little known on topological properties of optimal solution (continuous or not?, number of fixed points, etc.), or even local optima.

- Wu and Verdú (2011) using optimal transport theory: optimal controller is strictly increasing with a **real analytic left inverse**.

- **This does not imply continuity** of the optimal solution.

\(^1\) There is plenty of elegant related work which could not be covered in this short talk.
1. Pose the problem as a leader-follower coordination game:
   - $\theta \sim N(0, \sigma^2)$ observable to leader
   - Payoffs:
     
     $u_L = -r_L(\theta - a_L)^2 - (1 - r_L)(a_F - a_L)^2$
     
     $u_F = -(a_L - a_F)^2$

   - Follower observes $s = a_L + \delta$, where $\delta \sim N(0, 1)$.

2. Construct a class of **near-piecewise-linear strategies** for the leader, **invariant** under best response for large $\sigma$.

3. Show existence of **equilibria** with leader’s strategy in this set.

4. Show existence of a collection of **local minima** of the same order as of the global optimal solution.
Invariant Set of Near-Piecewise-Linear Strategies

Perfect Bayesian equilibria of the game:

\[ a^*_F(s) = \mathbb{E}_{\nu^*}[a_L^*|s] = \int_{\mathbb{R}} a_L \nu^*(a_L|s) da_L, \]

\[ a^*_L(\theta) = \text{argmax}_{a_L} -r_L(\theta - a_L)^2 - (1 - r_L) \int_{\mathbb{R}} (a^*_F(s) - a_L)^2 \phi(s - a_L) ds. \]

\( \nu^*(\cdot|s) \): the follower’s belief on leader’s action given \( s \).

Partition \( N(0, \sigma^2) \) into \( 2m + 1 \) segments:

- \( B^0_k = [b^0_k, b^0_{k+1}) \) for \( k \in \mathbb{N}_m \).
  \( B^0_0 = (b^0_{-1}, b^0_1), B^0_{-k} = (b^0_{-k-1}, b^0_{-k}] \).

- Symmetric: \( b^0_{-k} = -b^0_k \). Unbounded tails: \( b^0_{m+1} = -b^0_{-m-1} = +\infty \).
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- Symmetric: \( b_{-k}^0 = -b_k^0 \). Unbounded tails: \( b_{m+1}^0 = -b_{-m-1}^0 = +\infty \).

- Let centroid \( c_k^0 = \mathbb{E}_{N(0, \sigma^2)}[\theta|\theta \in B_k^0] \).
Invariant Set of Near-Piecewise-Linear Strategies

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- Symmetric: \( b_{-k}^0 = -b_k^0 \). Unbounded tails: \( b_{m+1}^0 = -b_{-m-1}^0 = +\infty \).

- Let centroid \( c_k^0 = \mathbb{E}_{N(0, \sigma^2)}[\theta|\theta \in B_k^0] \). Then, \( b_k^0 = \frac{c_{k-1}^0 + c_k^0}{2} \) (interval endpoints are equidistant from adjacent centroids).

Such a partition exists and is unique.
An Invariant Set of Near-Piecewise-Linear Strategies

$A^m_L(r_L, \sigma)$: a set of $(2m + 1)$-segmented increasing odd functions.

We consider strategies with the following properties:

- segments close to $B^0_k$
- a fixed point in each segment in a certain vicinity of $c^0_k$
- almost linear with a slope close to $r_L$ over each segment
An Invariant Set of Near-Piecewise-Linear Strategies

$A^m_L(r_L, \sigma)$: a set of \((2m + 1)\)-segmented increasing odd functions.

Property 1: For every $a_L(\theta) \in A^m_L(r_L, \sigma)$, there exist $2m + 1$ segments $B_k = [b_k, b_{k+1})$, for $k \in \mathbb{N}_m$, $B_0 = (-b_1, b_1)$, and $B_{-k} = (b_{-k-1}, b_{-k}]$, with $b_{m+1} = -b_{-m-1} = +\infty$ such that:

- $a_L(\theta)$ is increasing and odd, and is smooth over each interval.
- $a_L(\theta)$ has a unique fixed point ($a_L(c_k) = c_k$) in each segment.
$A^m_L(r_L, \sigma)$: a set of $(2m + 1)$-segmented increasing odd functions.

Property 2: For every $k \in \mathbb{N}_m, |b_k - \frac{c_{k-1} + c_k}{2}| \leq 0.1r_L$. Moreover, $|c_k - c^0_k| \leq 2.9$.

Let $\bar{x}_k := x^0_k + 3$ and $\underline{x}_k := x^0_k - 3$, where quantizer's half-steps $x^0_k = \frac{c^0_k - c^0_{k-1}}{2}$. Then $\bar{x}_k$ and $\underline{x}_k$ bound lengths of half-intervals $[c_{k-1}, b_k]$ and $[b_k, c_{k+1}]$. 
An Invariant Set of Near-Piecewise-Linear Strategies

$A_L^m(r_L, \sigma)$: a set of $(2m + 1)$-segmented increasing odd functions.

Property 3: Over inner intervals $B_k$ ($k \neq -m, m$), $r \leq \frac{d}{d\theta} a_L(\theta) \leq \bar{r}$, where $r = r_L(1 - 0.5r_L^2\sigma^2)$ and $\bar{r} = r_L(1 + 0.5r_L^2\sigma^2)$. Same bound over the tail if $b_m < \theta < c_m + \sqrt{e}\sigma\bar{x}_m$. For $\theta > c_m + \sqrt{e}\sigma\bar{x}_m$ we have $a_L(\theta) \leq c_m + 3r_L(\theta - c_m)$. 
A_L^m(r_L, \sigma): Invariance under best response operator
An Invariant Set of Near-Piecewise-Linear Strategies

$A_L^m(r_L, \sigma)$: Invariance under best response operator

$$a_L(\theta) \xrightarrow{\text{BR}} a_F(s)$$
$A^m_L(\sigma, r_L)$: Invariance under best response operator

$$a_L(\theta) \xrightarrow{BR} a_F(s) \xrightarrow{BR} \tilde{a}_L(\theta)$$
An Invariant Set of Near-Piecewise-Linear Strategies

$A^m_L(r_L, \sigma)$: Invariance under best response operator

\[ a_L(\theta) \xrightarrow{\text{BR}} a_F(s) \xrightarrow{\text{BR}} \tilde{a}_L(\theta) \]

$A^m_L(r_L, \sigma)$ invariant for $m \in M(\sigma) = \{ m \in \mathbb{N} | 2 \sqrt{2 \ln \sigma} + 4 < x_1^0 < 4 \sqrt{\ln \sigma} \}$ and sufficiently large $\sigma$ in the regime $\frac{1}{2} \leq r_L \sigma^2 \leq 1$.

- choice of $m \in M(\sigma) \Rightarrow$ efficient strategies.
- uses results from asymptotic quantization theory.
The set of \((2m + 1)\)-segmented strategies \(A^m_L(r_L, \sigma)\) for the leader, characterized by Property 1-3, is invariant under the best response for any
\[
m \in M(\sigma) = \{m \in \mathbb{N}| 2 \sqrt{2 \ln \sigma + 4} < x_1^0 < 4 \sqrt{\ln \sigma}\},
\]
and sufficiently large \(\sigma\) in the regime \(\frac{1}{2} \leq r_L \sigma^2 \leq 1\). Moreover, the game has an equilibrium for which:

i) \(a^*_L(\theta, r_L, \sigma) \in A^m_L(r_L, \sigma), \) and

ii) \((a^*_L(\theta, r_L, \sigma), a^*_F(s) = \mathbb{E}_\delta[a^*_L|s])\) maximizes the expected payoff of the leader over all pair of strategies \((a_L(\theta, r_L, \sigma), a_F(s) = \mathbb{E}_\delta[a_L|s])\) where \(a_L(\theta, r_L, \sigma) \in A^m_L(r_L, \sigma)\).

Proof based on careful best response analysis, very involved!
Local Minima and Performance Guarantees

Expected disutility of the leader:

\[ U(a_L, a_F) = r_L \int_{\mathbb{R}} (\theta - a_L(\theta))^2 \frac{\phi(\frac{\theta}{\sigma})}{\sigma} d\theta \]

\[ + (1 - r_L) \int \int_{\mathbb{R}^2} (a_F(s) - a_L(\theta))^2 \phi(s - a_L(\theta)) \frac{\phi(\frac{\theta}{\sigma})}{\sigma} ds d\theta. \] (1)

**Lemma 1**

Any pair of equilibrium strategies \((a^*_L, a^*_F)\) characterized by Theorem 1, where \(a^*_L \in A^m_L(r_L, \sigma)\) and \(a^*_F(s) = \mathbb{E}_\delta[a^*_L|s]\) is a local minimum of \(U\).

- \(U\) maps to the original Witsenhausen’s cost.
- \((a^*_L, a^*_F)\) equilibrium \(\Rightarrow\) cost **cannot be improved** by changing (only) one of the strategies.
- Not ruling out a lower cost by simultaneously changing both.
- Idea of the proof:
  - The best response image of an infinitesimal variation in \(a^*_L\) also lies in \(A^m_L\).
  - \((a^*_L, a^*_F)\) is the **minimizer** of \(U\) over all pairs \((a_L, a_F)\) with \(a_L \in A^m_L\) (Theorem 1).
Analytically proved the existence of local minima for Witsenhausen’s problem with a near-piecewise-linear strategy for the leader (or first controller).

Next: evaluate their performance w.r.t optimal cost.

Consider pair of strategies \((a^0_L, a^0_F)\):

- \(a^0_L\): piecewise-linear strategy with segments \(B^0_k\), constant slope \(r^L\), and centroids \(c^0_k\) as fixed points.
- \(a^0_F\): optimal \((2m + 1)\)-level MSE quantizer (i.e., constant value of \(c^0_k\) over segment \(B^0_k\)).

Any pair \((a_L, a_F)\) with \(a_L \in A^L_m\) can be used to upper-bound \(U(a^*_L, a^*_F)\) from Theorem 1:

\[
a^0_L \in A^L_m \Rightarrow U(a^*_L, a^*_F) \leq U(a^0_L, a^0_F).
\]

\(U(a^0_L, a^0_F)\) is easier to play with ...

Exact asymptotics are known for expected loss of optimal \((2m + 1)\)-level MSE quantizer.
Performance Guarantees (cont’d)

\[ U(a^0_L, a^0_F) = r_L (1 - r_L)^2 D^0_L + (1 - r_L) D^0_F, \]

\[ D^0_L = \int_{\mathbb{R}} (\theta - a^0_F(\theta))^2 \frac{\phi(\frac{\theta}{\sigma})}{\sigma} d\theta, \]

\[ D^0_F = \int_{\mathbb{R}} \int_{\mathbb{R}} (a^0_F(s) - a^0_L(\theta))^2 \phi(s - a^0_L(\theta)) \frac{\phi(\frac{\theta}{\sigma})}{\sigma} ds d\theta. \]

- \( D^0_F \) can be upper-bounded as:
  \[ D^0_F \leq 4 \sqrt{\frac{2}{e}} \frac{(2 - r_L)^2}{(1 - r_L)^2} \phi(\frac{x^0_1}{\sqrt{2}}) + r_L^2 D^0_L. \]

- \( D^0_L \) is the expected loss of an optimal \((2m + 1)\)-level MSE quantizer for a source \( \theta \sim N(0, \sigma^2) \).

- Exact asymptotics for \( D^0_L \) from asymptotic quantization theory.

- for large \( m \), \( D^0_L \approx \frac{c_\infty}{(2m+1)^2} \), where \( c_\infty \) is the Panter-Dite constant of a normal source:
  \[ c_\infty = \frac{1}{12} \left( \int_{\mathbb{R}} \left( \frac{\phi(\frac{\theta}{\sigma})}{\sigma} \right)^{\frac{1}{3}} d\theta \right)^3 = \frac{\sqrt{3}\pi}{2} \frac{\sigma^2}{x^0_1}. \]

- Another exact asymptotic:
  \[ (2m + 1) \frac{x^0_1}{\sigma} \approx \frac{\sqrt{6}\pi}{2} \Rightarrow D^0_L \approx \frac{(x^0_1)^2}{\sqrt{3}}. \]
Performance Guarantees (cont’d)

\[ U(a^0_L, a^0_F) = r_L(1 - r_L)^2 D^0_L + (1 - r_L) D^0_F, \]

\[ D^0_L = \int_{\mathbb{R}} (\theta - a^0_F(\theta))^2 \frac{\phi(\frac{\theta}{\sigma})}{\sigma} d\theta, \]

\[ D^0_F = \int \int_{\mathbb{R}} (a^0_F(s) - a^0_L(\theta))^2 \phi(s - a^0_L(\theta)) \frac{\phi(\frac{\theta}{\sigma})}{\sigma} ds d\theta. \]

- \( D^0_F \) can be upper-bounded as: \( D^0_F \leq 4 \sqrt{\frac{2}{e}} \frac{(2 - r_L)^2}{(1 - r_L)^2} \phi(\frac{x_1^0}{\sqrt{2}}) + r_L^2 D^0_L. \)
- \( D^0_L \) is the expected loss of an optimal \((2m + 1)\)-level MSE quantizer for a source \( \theta \sim N(0, \sigma^2) \).
- Exact asymptotics for \( D^0_L \) from asymptotic quantization theory.
- for large \( m \), \( D^0_L \approx \frac{c_\infty}{(2m+1)^2} \), where \( c_\infty \) is the Panter-Dite constant of a normal source:

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- Another exact asymptotic: \((2m + 1) \frac{x_1^0}{\sigma} \approx \frac{\sqrt{6} \pi}{2} \Rightarrow D^0_L \approx \frac{(x_1^0)^2}{\sqrt{3}}. \)
Lemma 2

For the pair of equilibrium strategies \((a^*_L, a^*_F)\) characterized by Theorem 1, where \(a^*_L \in A_L^m(r_L, \sigma)\), \(a^*_F(s) = \mathbb{E}_\delta[a^*_L|s]\), and \(m \in M(\sigma)\) we have

\[
\liminf_{\sigma \to \infty} \frac{r_L(1-r_L)(x_1^0)^2}{\sqrt{3}} + 4 \sqrt{\frac{2}{e}} \frac{(2-r_L)^2}{(1-r_L)} \phi\left(\frac{x_1^0}{\sqrt{2}}\right) \geq 1.
\]

- The above asymptotic upper bound on \(U(a^*_L, a^*_F)\) is minimized when \(x_1^0 \approx 2 \sqrt{2 \ln \sigma}\) for large \(\sigma\), yielding a cost \(\approx \frac{8r_L \ln \sigma}{\sqrt{3}}\).

- \(M(\sigma) = \{m \in \mathbb{N} | 2 \sqrt{2 \ln \sigma} + 4 < x_1^0 < 4 \sqrt{\ln \sigma}\} \Rightarrow\) cost (asymptotically) as low as \(\frac{8r_L \ln \sigma}{\sqrt{3}}\) is achievable for some \(m \in M(\sigma)\).

- To compare with the global optimum, we can use known lower bounds on the optimal cost of Witsenhausen’s problem.

- Next lemma is an immediate result of the bounds derived in Grover et al. (2013).
Lemma 3

Denote with $U^*(\sigma)$ the minimum value of the cost functional $U(a_L, a_F)$ in the regime $r_L \sigma^2 = 1$. Then, $\limsup_{\sigma \to \infty} \frac{\ln \sigma}{6 \sigma^2} U^*(\sigma) \leq 1$.

A quite loose but helpful lower bound!

Theorem 2

Any pair of equilibrium strategies $(a^*_L, a^*_F)$ characterized by Theorem 1, where $a^*_L \in A^m_L(r_L, \sigma)$ and $a^*_F(s) = E_\delta[a^*_L|s]$, is a local minimum of the cost functional $U$ in (1). Moreover, the set $M(\sigma)$ is nonempty for sufficiently large values of $\sigma$ and,

$$
\liminf_{\sigma \to \infty} \frac{8r_L \ln \sigma}{\sqrt{3} \min_{m \in M(\sigma)} U(a^*_L, a^*_F)} \geq 1.
$$

In the regime $r_L \sigma^2 = 1$, all these local minima are within constant factor of the optimal cost, with at least one being less than 27.8 times away from the optimal value as $\sigma \to \infty$. 
Witsenhausen’s seminal counterexample: a simple two-stage LQG decentralized control problem, where optimal solution is nonlinear.

Little known about the structure of global and local optima.

Simulations suggest optimality of near-piecewise-linear strategies in large $\sigma$ regime.

We view the problem as a leader-follower game of incomplete information.

We prove existence of near-piecewise-linear local optima with a cost at most a constant factor away from the optimal one, in large $\sigma$ regime.
Concluding Remarks

- Still many fundamental open questions about the structural properties of local/global optima.

- It is well-known that optimal solution has a real analytic left inverse. Same can be shown for local optima.

- This does not imply continuity. Can the optimal solution be discontinuous?

- When viewed as an LQG leader-follower game, is there a phase transition in the emergence of nonlinear equilibria?

- Can nonlinear equilibria emerge when actions are strategic substitutes?

- The latter relates to the yet open conjecture on uniqueness of equilibrium in classical Kyle model in Finance.
Thank You!