Sales-Based Aggregate Rebate Design

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Working Draft

Abstract

Firms offering network products and services can enjoy a free gain in their profit, since they can charge a higher price when the product reaches certain level of popularity in the market. Motivated by this observation, we pose the question as to whether it is beneficial to a firm selling a product with no (or weak) network effect to artificially inject externality? We consider a firm selling an indivisible product to a mass of consumers with uncertain heterogeneous valuations. To artificially induce externality, we propose and analyze an aggregate reward program where the reward paid to the consumers is a function of the aggregate size of the buyers. This induces a global game among the consumers where purchase decisions are conditioned on the offered price and reward program, private valuations, and a common public belief on the uncertainty in the valuations. By analyzing the equilibria of the game, we show that introducing positive externality via an increasing aggregate reward program is indeed harmful to the profit. On the other hand, for any decreasing nonconstant reward program, the seller can adjust the price such that the joint price-reward program is profitable. In particular, the amount of reward (or rebate) for the optimal program can be as high as a full refund if the product fails to reach certain level of popularity. The intriguing power of an aggregate reward program lies in its capacity to discriminate the expected net price over consumer valuations. A carefully designed aggregate reward program can hence charge the buyers with higher valuations at a higher rate, unlike a fixed posted price.

I. INTRODUCTION

Firms selling network goods can enjoy a marginal gain in their profit due to (positive) network effect. For such products, the benefit a user derives from consuming the product is increasing with the number of other users consuming the same product. Early buyers improve the social

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value of the product and firm can charge a higher price when the product reaches certain level of popularity. New technologies and innovations, smartphone applications (e.g., Viber, WhatsApp), online games (e.g., Warcraft), social network websites (e.g., Facebook, Twitter), and online dating services (e.g., Zoosk, Match.com, OkCupid) are among many examples of products with positive network effect or externality. The economic theory of network externalities is pioneered by Farrell and Saloner (1985); Katz and Shapiro (1986), followed by a more recent economics literature on network games with strategic complements.\(^1\) (Ballester et al. (2006); Sundararajan (2007); Galeotti et al. (2010)). Built upon these, Candogan et al. (2012); Bloch and Quérou (2013); Cohen and Harsha (2017); Hu and Wang (2017) study the effect of network externalities on optimal pricing and revenue management over networks in a monopoly setting.

Motivated by the operational value of the network effect discussed above, we pose the question as to whether it is beneficial to a firm selling a product with no (or weak) network effect to artificially inject externality? We consider a firm offering a product in a continuum of agents with uncertain valuations. Firm induces externality by appending what we call an aggregate reward program to the offered price. Buyers pay the price when making a purchase and are paid back a reward at the end of the sales period, where the paid reward is a function of the ex-post aggregate size of the buyers (sales volume). Firm and consumers share a common public belief on the uncertainty in valuations. Upon announcing the price and the reward program, valuations are realized and are privately observed by each agent. The agents then make purchase decisions (buy or not) maximizing their expected payoffs given the offered price and reward program, the public belief on the uncertainty in valuations, and their private valuations.

This setup induces a global game (Morris and Shin (1998, 2003); Carlsson and van Damme (1993)) among the consumers, where heterogeneity of private valuations will result in heterogeneous beliefs on the sales volume at equilibrium. This enables the firm to induce different net prices at different valuations using a non-constant reward program. As one of our main findings, we show that introducing positive externality via an increasing reward function is indeed harmful to the seller’s profit. On the other hand, for any decreasing non-constant reward function, the price can be adjusted in such a way that the joint price-reward program is profitable. Especially, we show that the optimal program can pay back a rebate as high as a full refund if the product

\(^1\)Games of strategic complementarities are those in which the best response of each player is increasing in actions of others Vives (2005).
fails to reach certain level of popularity. The main idea here is that a carefully designed reward program can induce higher net prices (in expectation) at higher valuations, unlike the fixed price case which charges all the buyers the same rate.

A. Literature Review

Our work is closely related to the literature on group-buying and quantity discounts. Our proposed reward program, however, somewhat contrasts the strategy suggested by group-buying schemes. In a group-buying scheme, buyers can enjoy a discounted price if they can achieve certain group size and synchronize their purchase time. In 2010, Groupon, a major player in group-buying industry, was named the fastest-growing company in the history of web by Forbes (Steiner (2010)). Despite the stunning early rise, the industry has experienced a downfall during the recent years: LivingSocial (Groupon’s main competitor), once valued at $6 billion, was recently acquired by Groupon for $0 (Knowledge@Wharton (2017)). Groupon’s stock value has dropped from a high of more than $28 in 2011 to around $4 per share today. Amazon Local, another Groupon’s competitor offering similar packages, has been discontinued in 2015 (Soper (2015)). Given the ups and downs of the group-buying industry and cons and pros of their business model, the future of group-buying sites is yet to be seen.

The benefit of group-buying schemes is mainly attributed to its reliance on the “economies of networking” and “economies of scale” in business press (Mourdoukoutas (2012)). Along the same line, Jing and Xie (2011) suggest that the key advantage of group-buying lies in fostering word-of-mouth: it incentivizes the expert to act as “sales agents” and promote the product to novice customers through interpersonal influences. This makes such strategies to be more suitable for relatively unknown firms, as also shown in Edelman et al. (2016). In a related work, Zhang et al. (2016) study the group-buying mechanisms by explicitly accounting for both the utility from shopping with one’s social circle as well as the inconvenience cost due to the wait time, and show that the former usually outweighs the latter. Kauffman and Wang (2001) find evidence of the positive externality effect on customer bids using customer data from MobShop.com. Selling in large groups is also advantageous in situations involving scale economies (e.g., in restaurant’s industry) as large quantities reduce the marginal cost (Monahan (1984); Kohli and

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2See Mangleburg et al. (2004) and the references therein for the influence of the peers on spending more on shopping.
Park (1989)). Our scope of interest in this work is products with fixed market size and marginal cost. This enables us to single out the operational value of reward programs through their direct effect on the utilities of the firm and consumers in absence of scale economies, and setting aside second order effects such as market expansion via word of mouth and interpersonal influences. In this setting, our results suggest that discounting the price with the size of the buyers is not a profitable strategy.

Another approach to group-buying and threshold discounting is to view them as means using which firm can deal with demand uncertainty. Anand and Aron (2003); Chen and Zhang (2015) use threshold discounting to find the operative demand regime in a scenario where the seller is uncertain about the demand. Unlike our setting, however, demand parameters are assumed to be fully known to buyers. Treating the entire market as a single player with unknown type, where the type determines the operative demand regime, their results can be closely related to Maskin and Riley (1984) which studies optimal quantity discounting of a seller in face of a buyer with uncertain type. A closer work to ours is Marinesi et al. (2016), where demand uncertainty is present at both ends. A seller with capacity constraint uses threshold discounting to both signal the market size to buyers and to condition offering the product during the “slow” season on the market size, hence reducing the supply-demand mismatch. As noted by the authors, however, this strategy can potentially reduce the profit if the seller has no capacity constraint, as assumed in our work.

Another related body of literature which benefits the firm through motivating word of mouth and social interactions is that of referral reward programs, where the seller uses monetary rewards to motivate existing buyers to spread product information thus expanding the market (Biayalogorsky et al. (2001); Aral and Walker (2011); Lobel et al. (2016); Leduc et al. (2017)). Although very similar in nature, group-buying has the advantage of stimulating a larger scale of social interaction as it requires information sharing before any transaction takes place (see Jing and Xie (2011) for a detailed comparison of group-buying and referral reward programs).

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3There are still substantial differences in the assumptions on the type distribution in these works. As a result, while quantity discounts in Maskin and Riley (1984) are everywhere optimal, threshold discounting strategies may not always outperform posted fixed prices as noted by the authors in Anand and Aron (2003); Chen and Zhang (2015).

4Cachon (2004) also uses quantity discounts to encourage early season purchases to reduce the risk due to demand uncertainty.

5Other word of mouth marketing strategies include creating buzz using promotions and frequent zero-pricing (see Campbell et al. (2013), and Achorlou et al. (2016) and references therein).
As stated before, by considering a fixed market size we set aside the second order beneficial marketing effects of the reward programs, including the market expansion via social interactions, in our analysis and fully elaborate on the operational value of such programs resulted from their direct effect on the utilities of the seller and buyers.

To summarize, while firms can enjoy a free boost in their profit for products with an inherent positive network effect, artificially inducing such an effect using an aggregate reward program may be harmful. In particular, for sellers with a established customer base (e.g., Groupon, Amazon, Facebook), with no capacity constraints and a fixed marginal cost (e.g., many digital goods and services), and uncertain average valuation of the product in the market, inducing positive externality (e.g., via group discounting) is harmful to the profit of the firm. Indeed, we show that the optimal aggregate reward strategy works by introducing negative externality, yielding an effective price which is increasing with the average valuation in the market. Nevertheless, incentive programs such as group-buying and referral rewards can be still beneficial due to their effectiveness in fostering word of mouth and social influence, scale economies, and reducing supply-demand mismatch under capacity constraint in situations discussed in the literature of group-buying and referral reward programs.

Along with their analytical complexity and operational challenges, technology-driven markets bring a handful set of features that were not previously available. This urges the need to revisit/improve some of the traditional revenue management strategies as they may no longer be efficient in the larger space of implementable strategies to date. This work uses a simple stylized model to support a theory for a new generation of rebate programs that takes advantage of these new features.  

II. Model

A firm is selling an indivisible product to a mass 1 continuum of consumers indexed by \( i \in [0, 1] \). Consumers have private valuations for the product, normally distributed around an uncertain true value \( v \) which reflects the expected or average valuation of the product in the market. More precisely, we assume that the private valuation of consumer \( i \) is of the form \( v_i = \)

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6The closest implementation that we have found is that of “Guaranteed prize pool” poker tournaments in online poker (e.g., PokerStars and FullTilt), where the platform (also known as the house) guarantees a certain number of participants for the tourney. Each player pays a fee to register for the tourney which goes to the prize pool. If the prize pool falls short of the promised size, then the rest is on the house.
$v + \epsilon_i$, where $\epsilon_i \sim N(0, \sigma^2_\epsilon)$ represents the heterogeneous idiosyncratic tastes of the consumers. Firm and consumers share a common prior $v \sim N(\theta, \sigma^2_\theta)$ on the true value of the product.

The firm announces a price $p$ and a reward program $r : [0, 1] \rightarrow \mathbb{R}^+ \cup \{0\}$ when launching the product. Consumers pay price $p$ to the firm when they buy the product and receive a reward valued at $r(\bar{a})$ at the end of the purchase period, where $\bar{a}$ is the ex-post aggregate size of the buyers or sales volume.

Upon observing the price and reward program and given the private valuations and the common public prior on true value, agents simultaneously decide as to whether make a purchase or not. The payoff of not adopting ($a_i = 0$) is normalized to zero. The payoff of adoption ($a_i = 1$) is given by a quasi-linear function of the form

$$u_i = v_i + r(\bar{a}) - p,$$

where $\bar{a} = \int_0^1 a_j\,dj$ is the ex-post sales volume. Consumers take actions maximizing their expected payoffs, and hence,

$$a_i = 1\{E[u_i|v_i, p, r(\cdot)] > 0\}.$$

The utility of the firm offering price $p$ and reward program $r(\cdot)$ is given by

$$\Pi(p, r(\cdot)) = (p - r(\bar{a}))\bar{a},$$

where $\bar{a}$ is the sales volume resulted from the strategies of the consumers and the realization of the true value $v$.

We call a joint price-reward program profitable if it yields an expected profit higher than that of an optimally chosen price with no reward. A natural question then is whether there exists a profitable reward program at all. This can be thought of as a bet between the firm and each consumer whose outcome depends on the realization of the true value. Therefore, a profitable reward program may seem out of reach, given the information edge of the consumers over the firm.\footnote{The marginal cost is normalized to zero.}

\footnote{Besides the common prior on the true value $v$, each consumer’s valuation provides her with a noisy private observation of $v$.}
III. Profitability of Aggregate Reward Programs

Let us start with the case where the true value of the product $v$ is perfectly known, that is $\sigma_\theta = 0$. It is easy to verify that a reward program in this case works merely as a shift in the offered price. To see this, note that given $v$ consumers can correctly foresee the sales volume $\bar{a}$ at equilibrium and subsequently adjust the price and make a purchase if and only if $v_i > c = p - r(\bar{a})$. Therefore, the offered pair of price-reward program $(p, r(\cdot))$ works the same as $(c = p - r(\bar{a}), 0)$, that is offering a price $p - r(\bar{a})$ with no reward.

For an uncertain true value ($\sigma_\theta \neq 0$), given the dependence of the paid reward to the aggregate size of the buyers, the offered pair of $(p, r(\cdot))$ induces a global game among the consumers; To make a purchase decision, each consumer needs to form a belief on the valuations of other consumers, as well as a belief on the beliefs of other consumers on the valuations of others, and so on (see Morris and Shin (1998, 2003); Carlsson and van Damme (1993) for the theory of global games). Therefore, the first step in analyzing the profitability of aggregate reward programs is to characterize the equilibria of the subgame among the consumers.

The negligible effect of individual consumers on the aggregate action in continuum models makes the Bayes Nash equilibria of the game symmetric. We turn our attention here to equilibria in the class of monotone or threshold strategies. A symmetric monotone strategy with threshold $c$ is of the form $a_i = 1\{v_i > c\}$, that is, a consumer makes a purchase if and only if her private valuation is above $c$. We first derive conditions ensuring the existence of such equilibria for the consumers’ subgame.

For a monotone strategy $a_i = 1\{v_i > c\}$, the aggregate size of the buyers for a realization $v$ of the true value is $\bar{a} = \Phi(\frac{v - c}{\sigma_\epsilon})$. Given the private valuation $v_i = v + \epsilon_i$, consumer $i$ updates her belief on the true value as $v|v_i \sim N(\tau v_i + (1 - \tau)\theta, \sigma_v^2)$, where $\tau = \frac{\sigma_\theta^2}{\sigma_\epsilon^2 + \sigma_\theta^2}$ and $\sigma_v^2 = \frac{\sigma_\theta^2 \sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_\theta^2}$. The expected payoff of adoption is

$$E_v[u_i|v_i] = v_i - p + E_v[r(\Phi(\frac{v - c}{\sigma_\epsilon}))].$$ (1)

We next present conditions under which $\frac{\partial}{\partial v_i}E_v[u_i|v_i] \geq 0$, that is, the expected payoff from a purchase is higher for a consumer with higher valuation.

**Lemma 1.** Let $r_{\text{min}}$ and $r_{\text{max}}$ denote the minimum and maximum reward paid to a buyer under the reward program $r(\cdot)$. Then, the expected payoff of adoption, assuming a monotone symmetric
strategy for the consumers, is increasing with their valuations if

\[ r_{\text{max}} - r_{\text{min}} \leq \sqrt{2\pi\sigma_\epsilon} \sqrt{1 + \left(\frac{\sigma_\epsilon}{\sigma_\theta}\right)^2}. \]

**Proof.** See the appendix. \[\blacksquare\]

**Assumption 1.** \( r_{\text{max}} - r_{\text{min}} \leq r_M, \text{ where } r_M = \sqrt{2\pi\sigma_\epsilon} \sqrt{1 + \left(\frac{\sigma_\epsilon}{\sigma_\theta}\right)^2}. \)

Making this assumption on the reward spread, the strategy \( a_i = 1\{v_i > c\} \) is an equilibrium if and only if the cutoff agent is indifferent between making a purchase or not, that is \( \mathbb{E}_v[u_i|v_i = c] = 0 \), or

\[ c - p + \mathbb{E}_{v|c}[r(\Phi(\frac{v - c}{\sigma_\epsilon}))] = 0. \tag{2} \]

From (1), it is easy to see that \( \lim_{c \to -\infty} \mathbb{E}_v[u_i|v_i = c] = -\infty \) and \( \lim_{c \to +\infty} \mathbb{E}_v[u_i|v_i = c] = +\infty \), proving the existence of a monotone equilibrium strategy for the subgame induced among consumers given any pair \((p, r(\cdot))\) satisfying Assumption 1.

**Lemma 2.** For any pair of price-reward programs \((p, r(\cdot))\) satisfying Assumption 1, the game induced among the consumers admits a symmetric monotone Bayes Nash equilibrium.

**Proof.** See the appendix. \[\blacksquare\]

We can similarly come up with conditions ensuring the uniqueness of such an equilibrium. However, we postpone this to a later time because, as we will see in the sequel, the uniqueness condition will be trivial for the reward programs of our interest.

For a product of true value \( v \), the strategy \( a_i = 1\{v_i > c\} \) results in a sales volume \( \bar{a} = \Phi(\frac{v - c}{\sigma_\epsilon}) \), charging each buyer a “net price” of \( p - r(\Phi(\frac{v - c}{\sigma_\epsilon})) \). We can hence write the expected utility of the firm as

\[ \mathbb{E}_v[\Pi(p, r(\cdot))] = \mathbb{E}_v[(p - r(\Phi(\frac{v - c}{\sigma_\epsilon})))\Phi(\frac{v - c}{\sigma_\epsilon})], \]

where from the indifference equation in (2) we have \( p = c + r_c \), where \( r_c = \mathbb{E}_{v|c}[r(\Phi(\frac{v - c}{\sigma_\epsilon}))] \) is the reward expected at the cutoff. Another useful representation is to write the expected utility of the firm as the sum (integral) of the expected profit made at each valuation \( v_i \) weighted by its density.
among all the realizations of $v$ and $\epsilon_i$ resulting in $v_i$. From $v_i | v \sim N(v, \sigma^2)$ and $v \sim N(\theta, \sigma^2_{\theta})$ we get $v_i \sim N(\theta, \sigma^2_{\epsilon} + \sigma^2_{\theta})$. The net price expected at valuation $v_i$ is $p - \mathbb{E}_{v_i}[r(\Phi(v - c \sigma_{\epsilon}))]$, as already established in deriving (1). Observing that the net price expected at valuation $v_i$ is the same as the expected net price charged by the seller at valuation $v_i$, we obtain

$$
\mathbb{E}_{v_i}[\Pi(p, r(\cdot))] = \int_c^{\infty} (p - \mathbb{E}_{v_i}[r(\Phi(v - c \sigma_{\epsilon}))]) \frac{\phi\left(\frac{v_i - v}{\sigma_{\epsilon}}\right)}{\sqrt{\sigma^2_{\epsilon} + \sigma^2_{\theta}}} dv_i
$$

$$
= \int_c^{\infty} (c + \mathbb{E}_{v_i}[r(\Phi(v - c \sigma_{\epsilon}))]) - \mathbb{E}_{v_i}[r(\Phi(v - c \sigma_{\epsilon}))] \frac{\phi\left(\frac{v_i - v}{\sigma_{\epsilon}}\right)}{\sqrt{\sigma^2_{\epsilon} + \sigma^2_{\theta}}} dv_i
$$

$$
= \mathbb{E}_{v}[\Pi(c, 0)] + \int_c^{\infty} \mathbb{E}_{v_i}[r(\Phi(v - c \sigma_{\epsilon}))] - \mathbb{E}_{v_i}[r(\Phi(v - c \sigma_{\epsilon}))] \frac{\phi\left(\frac{v_i - v}{\sigma_{\epsilon}}\right)}{\sqrt{\sigma^2_{\epsilon} + \sigma^2_{\theta}}} dv_i \quad (3)
$$

where $\mathbb{E}_{v}[\Pi(c, 0)]$ is the expected utility of the firm offering a price $c$ with no reward. The following result is immediate.

**Lemma 3.** Suppose the consumers follow an equilibrium strategy of the form $a_i = 1\{v_i > c\}$, given the joint price-reward program $(p, r(\cdot))$. Then, $\mathbb{E}_{v_i}[\Pi(p, r(\cdot))] > \mathbb{E}_{v_i}[\Pi(c, 0)]$ if and only if the ex-ante expected reward paid per purchase is less than the reward expected at the cutoff, that is,

$$
\int_c^{\infty} \mathbb{E}_{v_i}[r(\Phi(v - c \sigma_{\epsilon}))] - \mathbb{E}_{v_i}[r(\Phi(v - c \sigma_{\epsilon}))] \frac{\phi\left(\frac{v_i - v}{\sigma_{\epsilon}}\right)}{\sqrt{\sigma^2_{\epsilon} + \sigma^2_{\theta}}} dv_i \quad (4)
$$

**Proof.** See the appendix. \[\square\]

If the condition in (4) fails, then offering the price $c$ with no reward would yield a profit no less than that of the joint price-reward program. We next use this condition to verify the profitability of monotone (both increasing and decreasing) reward functions. We first show the following monotonicity property for the expected rewards for this class of reward programs.

**Lemma 4.** Suppose the consumers follow a threshold strategy of the form $a_i = 1\{v_i > c\}$. Then, for any decreasing non-constant reward program $r(\cdot)$ the expected rewards of the consumers
are strictly decreasing with their valuations. In other words,

\[ v_1 > v_2 \Rightarrow \mathbb{E}_{v|v_1}[r(\Phi(\frac{v - c}{\sigma_e}))] < \mathbb{E}_{v|v_2}[r(\Phi(\frac{v - c}{\sigma_e}))]. \]

Similarly, the expected rewards are increasing with consumers’ valuations for any increasing reward program.

**Proof.** See the appendix.

Based on the above result, any increasing reward function clearly violates the condition in (3), and hence cannot be profitable. On the other hand, this condition clearly holds for any decreasing non-constant reward function \( r(\cdot) \) and any threshold \( c \). This thus implies that offering a price \( p = c + \mathbb{E}_{v|c}[r(\Phi(\frac{v - c}{\sigma_e}))] \) together with the reward program \( r(\cdot) \) will yield an expected profit higher than offering a price \( c \) with no reward. Choosing \( c \) to be the optimal price for the no reward case, we have just found a profitable price-reward program pair.

**Theorem 1.** Every decreasing non-constant reward function \( r(\cdot) \) satisfying Assumption 1 can be joined by an appropriate price \( p \) such that the joint price-reward program \( (p, r(\cdot)) \) is profitable. On the other hand, there exists no profitable increasing reward program.

**Proof.** See the appendix.

**IV. Popularity-Guaranteed Rebate Program**

Next, we aim to find the optimal joint price-reward program. We do this in two steps: choosing the target consumers (controlled by threshold \( c \)), and the reward program extracting the maximum profit from them. We can then find the corresponding optimal price using the relation \( p = c + \mathbb{E}_{v|c}[r(\Phi(\frac{v - c}{\sigma_e}))] \).

Proceeding backward, we start by finding the optimal reward program given the equilibrium strategy \( a_i = 1\{v_i > c\} \) for the consumers. Using (3), we can write the expected utility of the firm as

\[ \mathbb{E}_{\nu}[\Pi(p, r(\cdot))] = \mathbb{E}_{\nu}[\Pi(c, 0)] + \Delta \Pi(c, r(\cdot)), \]

where \( \mathbb{E}_{\nu}[\Pi(c, 0)] \) is the expected utility of the firm offering price \( c \) with no reward, and

\[ \Delta \Pi(c, r(\cdot)) = \int_{c}^{\infty} (\mathbb{E}_{v|c}[r(\Phi(\frac{v - c}{\sigma_e}))] - \mathbb{E}_{v|v_i}[r(\Phi(\frac{v - c}{\sigma_e}))]) \frac{\phi(\frac{\theta - v_i}{\sqrt{\sigma^2 + \sigma^2_\theta}})}{\sqrt{\sigma^2 + \sigma^2_\theta}} dv_i, \]  

(5)
is the expected gain/loss resulted from the reward program. Denote with \( r^*_c(\cdot) \) the reward program maximizing the above gain, that is \( r^*_c(\cdot) = \arg \max_{r(\cdot)} \Delta \Pi(c, r(\cdot)) \), satisfying Assumption 1. We find \( r^*_c(\cdot) \) by squeezing the search space in a few steps. Rearranging the terms in (11), we can find

\[
\Delta \Pi(c, r(\cdot)) = \int_{-\infty}^{\infty} r(\Phi(\frac{v - c}{\sigma_\epsilon})) \int_{c}^{\infty} \left( \frac{\phi(\frac{v - \mu_v}{\sigma_v})}{\sigma_v} - \frac{\phi(\frac{v - \mu_i}{\sigma_v})}{\sigma_v} \right) \frac{\phi\left(\frac{\theta - v_i}{\sqrt{\sigma_v^2 + \sigma_\theta^2}}\right)}{\sqrt{\sigma_v^2 + \sigma_\theta^2}} dv_i dv, \tag{6}
\]

where we recall that \( v_i | v \sim N(\mu_i, \sigma_v^2) \) with \( \mu_i = \tau v_i + (1 - \tau) \theta \) and \( \sigma_v^2 = \frac{\sigma^2 \sigma_v^2}{\sigma_v^2 + \sigma_\theta^2} \). Let \( q(v|v_i) = \frac{\phi\left(\frac{v - \mu_i}{\sigma_v}\right)}{\sigma_v} \) denote the likelihood of the true value \( v \) observing a valuation \( v_i \). We can write (12) as

\[
\Delta \Pi(c, r(\cdot)) = \int_{-\infty}^{\infty} r(\Phi(\frac{v - c}{\sigma_\epsilon})) \Phi\left(\frac{\theta - c}{\sqrt{\sigma_v^2 + \sigma_\theta^2}}\right)(q(v|c) - q(v|v_i \geq c)) dv,
\]

where \( q(v|c) \) and \( q(v|v_i \geq c) \) are the likelihood of the true value \( v \) from the cutoff and its expected (or average) likelihood among the buyers. From the monotone likelihood ratio property (MLRP) of the normal distribution (see Milgrom (1981)), it follows that there exists a unique value \( v_c \) at which

\[
q(v_c|c) = q(v_c|v_i \geq c), \tag{7}
\]

that is, the true value whose likelihood at the cutoff is the same as its average likelihood among all buyers. Focusing on the buyers, true values \( v > v_c \) have a less than average likelihood from the cutoff while values \( v < v_c \) are more likely at cutoff. As a result, firm gains from the reward program for \( v < v_c \) and loses for \( v > v_c \). Let \( a_c = \Phi\left(\frac{v_c - c}{\sigma_\epsilon}\right) \) be the corresponding sales volume for true value \( v_c \). Then, the optimal reward program is of the form \( r^*_c(\bar{a}) = r_M \) for \( \bar{a} \leq a_c \) and \( r^*_c(\bar{a}) = 0 \) for \( \bar{a} > a_c \), where \( r_M \) is defined in Assumption 1. This means that the optimal reward program pays a fixed rebate back if the product fails to reach certain level of popularity.

For this reward program, the reward expected at valuation \( v_i \) is

\[
\mathbb{E}_{v|v_i}[r^*_c(\Phi(\frac{v - c}{\sigma_\epsilon}))] = r_M \mathbb{E}_{v|v_i}[1\{v < v_c\}] = r_M \Phi\left(\frac{v_c - \mu_i}{\sigma_v}\right).
\]

Incorporating this into (3), we can write the expected utility of the firm under the optimal reward
program as

$$
\mathbb{E}_v[\Pi(p^*_c, r^*_c(\cdot))] = \mathbb{E}_v[\Pi(c, 0)] + r_M \int_{c}^{\infty} \left( \Phi\left( \frac{v_c - \mu_c}{\sigma_v} \right) - \Phi\left( \frac{v_c - \mu_i}{\sigma_v} \right) \right) \frac{\phi\left( \frac{\theta - v_i}{\sqrt{\sigma_v^2 + \sigma^2} \theta} \right)}{\sqrt{\sigma_v^2 + \sigma^2}} dv_i. 
$$

(8)

The optimal joint price-reward program can then be obtained by targeting the subset of consumers (controlled by $c$) maximizing the above utility. The optimal joint price-reward program is fully characterized in the next theorem.

**Theorem 2.** The optimal joint price-reward program $(p^*, r^*(\cdot))$ satisfying Assumption 1 is of the form $r^*(\bar{a}) = r_M \{\bar{a} \leq a_c\}$ and $p^* = c + r_M \Phi\left( \frac{v_c - \mu_v}{\sigma_v} \right)$, where $\mu_c = \tau c + (1 - \tau) \theta$, $\sigma_v = \frac{\sigma_c \sigma_v}{\sigma_c^2 + \sigma^2}$, $\tau = \frac{\sigma^2}{\sigma_c^2 + \sigma^2}$, $a_c = \Phi\left( \frac{\theta - c}{\sigma_c} \right)$, $r_M = \sqrt{2\pi} \sigma_c \sqrt{1 + (\frac{2}{\sigma_v})^2}$, and $(c, v_c)$ is the solution of

$$
\frac{\phi\left( \frac{v_c - \mu_v}{\sigma_v} \right)}{\Phi\left( \frac{\theta - c}{\sigma_c} \right)} = \frac{\phi\left( \frac{\theta - c}{\sigma_c} \right)}{\Phi\left( \frac{\theta - c}{\sigma_c^2 + \sigma^2} \right)} = 1 - \sqrt{2\pi} \phi\left( \frac{v_c - \mu_v}{\sigma_v} \right). 
$$

(9)

**Proof.** See the appendix.

### V. Discussions and Comparison with the Optimal No-Reward Pricing

The optimal joint price-reward program characterized in Theorem 2 and the corresponding statistics, as well as comparison with the optimal no reward case, are described in Figure 1-2. The intriguing power of an aggregate reward program lies in its capacity to discriminate the expected net price (i.e., the paid price less the reward to-be-paid) over valuations, as can be seen in Figure 2. In what follows, we lay down a sequence of key observations in order to spell out this idea.

By conditioning the reward on the size of the buyers, the seller can effectively discriminate the net price on the realization of the true value $v$. A key ingredient here is the heterogeneity of consumers’ beliefs on $v$, which follows from the heterogeneity in valuations. Discriminating the price over the true value $v$ together with the heterogeneity of consumers’ beliefs on $v$ results in heterogeneity in the net price expected by the consumers. This enables the seller to indirectly induce different expected prices at different valuations. Having had identical beliefs, a reward program would merely work as a price shift which could be fully overseen and accounted for by the consumers. Returning to Figure 2, we can see that a properly designed aggregate reward...
Fig. 1. Ex-post profit and net prices for different realizations of the true value $v$ for both the optimal joint price-reward program characterized in Theorem 2 and the optimal no reward case, for a sample case with $\theta = 5$, $\sigma_x = \sigma_y = 1$. For the optimal joint program, the implied net price is $p^* = 5.85$ for realizations $v > v_c = 4.61$, and $p^* - r_M = 2.31$ for $v < 4.61$, resulting in an average profit of 4.41. For the optimal no reward case, the fixed offered price is $p^*_0 = 3.86$ with an average profit of 3.05.

Program can charge higher net prices (in expectation) at higher valuations, while still below the willingness-to-pay of the consumers with valuations above the cutoff. This is the source of the profit in excess of the no-reward fixed-price profit which charges every consumer the same price.

Our last result concerns the effect of the proposed popularity-guaranteed rebate program on the sales volume and its welfare implications. To show an increase in the sales volume compared to the optimal no reward case, we need to show that the threshold $c$ for the optimal joint price-reward program falls below the optimal no reward price. This easily follows from two observations: Setting the reward program optimally, for the cutoff value $c$ maximizing the expected profit given by (14) the marginal gain/loss resulted from the reward program has to be the same as the marginal loss/gain in the expected profit resulted from offering price $c$ with no reward. On the other hand, the utility of an optimally set reward program is decreasing with the cutoff value $c$. 

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Fig. 2. Net price expected at valuation $v_i$ for the optimal joint price-reward program characterized in Theorem 2 for a sample case with $\theta = 5$, $\sigma_v = \sigma_\theta = 1$, resulting in $v_i \sim N(5, 2)$. Expected payoff of purchase for valuations above the threshold $c = 2.97$ is positive, hence making a purchase.

More precisely, it follows from (14) that

$$
\frac{\partial}{\partial c} \mathbb{E}[\Pi(c, 0)] = \sqrt{2\pi} \phi \left( \frac{v_c - \mu_c}{\sigma} \right) \Phi \left( \frac{\theta - c}{\sqrt{\sigma^{2\epsilon} + \sigma^2_\theta}} \right) > 0, \tag{10}
$$

where $\mathbb{E}[\Pi(c, 0)] = c \Phi \left( \frac{\theta - c}{\sqrt{\sigma^{2\epsilon} + \sigma^2_\theta}} \right)$ is the expected profit of offering price $c$ with no reward. Denote with $p_0^*$ the optimal price for the no reward case. $\mathbb{E}[\Pi(c, 0)]$ is increasing for $c < p_0^*$ and decreasing for $c > p_0^*$. This together with (10) implies that $c < p_0^*$, that is, the cutoff of the optimal joint price-reward program is less than the optimal no-reward price, hence resulting in a higher sales volume. Given the quasi-linear payoff of the consumers and that prices are a transfer from buyers to the seller, we can show that the aggregate welfare\(^\dagger\) is increasing with the sales volume. The proposed popularity-guaranteed rebate program thus also improves the social welfare.

\(^{10}\)See the proof of Theorem 2 on how to obtain this.

\(^{11}\)That is, the aggregate utility of all the consumers and the seller.
Proposition 1. The optimal popularity-guaranteed rebate program characterized in Theorem 1 improves both the sales volume and the expected social welfare compared to the optimal no-reward pricing.

Proof. See the appendix.

REFERENCES


APPENDIX

Proof of Lemma 1. For \( \frac{\partial}{\partial v_i} \mathbb{E}_v[u_i|v_i] \geq 0 \) to hold, it suffices to have \( |\frac{\partial}{\partial v_i} \mathbb{E}_v[u_i|v_i]| \leq 1 \) for all \( v \) and \( i \). Let \( \mu_i = \tau v_i + (1 - \tau)\theta \). Then,

\[
\frac{\partial}{\partial v_i} \mathbb{E}_v[u_i|v_i] = \mathbb{E}_v[u_i|v_i] \left[ \frac{\partial}{\partial v_i} (\Phi(\frac{v - c}{\sigma_\epsilon})) (\frac{v - \mu_i}{\sigma_v}) \right] = \frac{\tau}{\sigma_v} \mathbb{E}_v[u_i|v_i] \left[ \frac{\partial}{\partial v_i} (\Phi(\frac{v - c}{\sigma_\epsilon})) (\frac{v - \mu_i}{\sigma_v}) \right].
\]

Using this, it is easy to see that

\[
|\frac{\partial}{\partial v_i} \mathbb{E}_v[u_i|v_i]| \leq \frac{\tau}{\sigma_v} \int_{r_{\min}}^{r_{\max}} x \phi(x) dx = \frac{\tau}{\sqrt{2\pi}\sigma_v} (r_{\max} - r_{\min}).
\]

Therefore, if \( r_{\max} - r_{\min} \leq \frac{\sqrt{2\pi}\sigma_v}{\tau} \) then \( \frac{\partial}{\partial v_i} \mathbb{E}_v[u_i|v_i] \geq 0 \), which completes the proof.

\[\Box\]

Proof of Lemma 2. Proof is given in the body of the paper right above the lemma.

\[\Box\]

Proof of Lemma 3. Proof follows directly from (3).

\[\Box\]

Proof of Lemma 4. Recall that \( v|x \sim N(\tau x + (1 - \tau)\theta, \sigma_v^2) \), where \( \tau = \frac{\sigma_\theta^2}{\sigma^2 + \sigma_\theta^2} \) and \( \sigma_v^2 = \frac{\sigma_x^2 \sigma_\theta^2}{\sigma^2 + \sigma_\theta^2} \).

Therefore,

\[
\mathbb{E}_v[u|x] = \int_{-\infty}^{\infty} r(\Phi(\frac{v - c}{\sigma_\epsilon})) \varphi(\frac{v - \tau x - (1 - \tau)\theta}{\sigma_v}) \sigma_v \, dv
\]

from which the lemma immediately follows.

\[\Box\]

Proof of Theorem 1. Most of the proof is already given in the text above the theorem. We here prove the uniqueness of the threshold equilibrium strategy for the consumers’ subgame given a decreasing reward function \( r(\cdot) \) satisfying Assumption 1. To see this, recall the indifference equation

\[
c + \mathbb{E}_v[c|r(\Phi(\frac{v - c}{\sigma_\epsilon}))] = p,
\]

where \( v|c \sim N(\tau c + (1 - \tau)\theta, \sigma_v^2) \). It is easy to see that the LHS is strictly increasing in \( c \) for any decreasing reward function, implying the uniqueness of \( c \). This is important, as in case of multiple equilibria seller cannot predict the purchase behavior of the consumers.

\[\Box\]

Proof of Theorem 2.
We find the optimal joint price-reward program. We do this in two steps: choosing the
target consumers (controlled by threshold $c$), and the reward program extracting the maximum profit from them. We can then find the corresponding optimal price using the relation
$$p = c + \mathbb{E}_{v|c}[r(\Phi(\frac{v - c}{\sigma_e}))].$$

Proceeding backward, we start by finding the optimal reward program given the equilibrium strategy $a_i = 1\{v_i > c\}$ for the consumers. Using (3), we can write the expected utility of the firm as $\mathbb{E}_v[\Pi(p, r(\cdot))] = \mathbb{E}_v[\Pi(c, 0)] + \Delta\Pi(c, r(\cdot))$, where $\mathbb{E}_v[\Pi(c, 0)]$ is the expected utility of the firm offering price $c$ with no reward, and
$$\Delta\Pi(c, r(\cdot)) = \int_{-\infty}^{\infty} (\mathbb{E}_{v|c}[r(\Phi(\frac{v - c}{\sigma_e}))]) - \mathbb{E}_{v|v_i}[r(\Phi(\frac{v - c}{\sigma_e}))]) \frac{\phi(\frac{\theta - v}{\sqrt{\sigma^2 + \sigma^2}})}{\sqrt{\sigma^2 + \sigma^2}} dv,$$
(11)
is the expected gain/loss resulted from the reward program. Denote with $r^*_c(\cdot)$ the reward program maximizing the above gain, that is $r^*_c(\cdot) = \arg\max_{r(\cdot)} \Delta\Pi(c, r(\cdot))$, satisfying Assumption 1. We find $r^*_c(\cdot)$ by squeezing the search space in a few steps. Rearranging the terms in (11), we can find
$$\Delta\Pi(c, r(\cdot)) = \int_{-\infty}^{\infty} r(\Phi(\frac{v - c}{\sigma_e})) \int_{-\infty}^{\infty} \frac{\phi(\frac{v - \mu_c}{\sigma_v})}{\sigma_v} - \frac{\phi(\frac{v - \mu_i}{\sigma_v})}{\sigma_v} \frac{\phi(\frac{\theta - v}{\sqrt{\sigma^2 + \sigma^2}})}{\sqrt{\sigma^2 + \sigma^2}} dv dv,$$
(12)
where we recall that $v|v_i \sim N(\mu_i, \sigma^2_v)$ with $\mu_i = \tau v_i + (1 - \tau)\theta$ and $\sigma^2_v = \frac{\sigma^2_v}{\sigma^2_v + \sigma^2_\theta}$. Let $q(v|v_i) = \frac{\phi(\frac{v - \mu_i}{\sigma_v})}{\sigma_v}$ denote the likelihood of the true value $v$ observing a valuation $v_i$. We can write (12) as
$$\Delta\Pi(c, r(\cdot)) = \int_{-\infty}^{\infty} r(\Phi(\frac{v - c}{\sigma_e})) \Phi(-\frac{\theta - c}{\sqrt{\sigma^2 + \sigma^2}}) (q(v|c) - q(v|v_i \geq c)) dv,$$
where $q(v|c)$ and $q(v|v_i \geq c)$ are the likelihood of the true value $v$ from the cutoff and its expected (or average) likelihood among the buyers. From the monotone likelihood ratio property (MLRP) of the normal distribution (see Milgrom (1981)), it follows that there exists a unique value $v_c$ at which
$$q(v_c|c) = q(v_c|v_i \geq c),$$
(13)
that is, the true value whose likelihood at the cutoff is the same as its average likelihood among

\[12\] For the cutoff, $v|c \sim N(\mu_c, \sigma^2_v)$ with $\mu_c = \tau c + (1 - \tau)\theta$. 
all buyers. Focusing on the buyers, true values \( v > v_c \) have a less than average likelihood from the cutoff while values \( v < v_c \) are more likely at cutoff. As a result, firm gains from the reward program for \( v < v_c \) and loses for \( v > v_c \). Let \( a_c = \Phi\left(\frac{v_c - c}{\sigma_c}\right) \) be the corresponding sales volume for true value \( v_c \). Then, the optimal reward program is of the form \( r^*_c(\bar{a}) = r_M \) for \( \bar{a} \leq a_c \) and \( r^*_c(\bar{a}) = 0 \) for \( \bar{a} > a_c \), where \( r_M \) is defined in Assumption 1. This means that the optimal reward program pays a fixed rebate back if the product fails to reach certain level of popularity.

For this reward program, the reward expected at valuation \( v_i \) is

\[
E_{v_i}[r^*_c(\Phi(\frac{v - c}{\sigma_c}))] = E_{v_i}[\mathbb{1}\{v < v_c\}] = r_M \Phi(\frac{v - \mu_i}{\sigma_v}).
\]

Incorporating this into (3), we can write the expected utility of the firm under the optimal reward program as

\[
E_v[\Pi(p^*_c, r^*_c(\cdot))] = E_v[\Pi(c, 0)] + r_M \int_{\mathbb{R}} \left( \Phi\left(\frac{v_c - \mu_c}{\sigma_v}\right) - \Phi\left(\frac{v_c - \mu_i}{\sigma_v}\right) \right) \frac{\phi\left(\frac{\theta - v_i}{\sqrt{\sigma^2 + \sigma^2}}\right)}{\sqrt{\sigma^2 + \sigma^2}} dv_i. \tag{14}
\]

The optimal joint price-reward program can then be obtained by targeting the subset of consumers (controlled by \( c \)) maximizing the above utility.

From (13), we get

\[
\int_{\mathbb{R}} \left( \frac{\phi\left(\frac{v_c - \mu_c}{\sigma_v}\right)}{\sigma_v} - \frac{\phi\left(\frac{v_c - \mu_i}{\sigma_v}\right)}{\sigma_v} \right) \frac{\phi\left(\frac{\theta - v_i}{\sqrt{\sigma^2 + \sigma^2}}\right)}{\sqrt{\sigma^2 + \sigma^2}} dv_i = 0. \tag{15}
\]

A useful identity here is

\[
\frac{\phi\left(\frac{v_c - \mu_i}{\sigma_v}\right)}{\sigma_v} = \frac{\phi\left(\frac{v_c - \mu_i}{\sigma_v}\right)}{\sigma_v} \frac{\phi\left(\frac{\theta - v_i}{\sigma_v}\right)}{\sqrt{\sigma^2 + \sigma^2}}, \tag{16}
\]

which simply follows from the Bayes rule. Incorporating this into (15), we can obtain

\[
\frac{\phi\left(\frac{v_c - \mu_c}{\sigma_v}\right)}{\sigma_v} \Phi\left(\frac{\theta - c}{\sqrt{\sigma^2 + \sigma^2}}\right) = \frac{\phi\left(\frac{v_c - \theta}{\sigma_v}\right)}{\sigma_v} \Phi\left(\frac{v_c - c}{\sigma_v}\right).
\]
Using (16) with \((c, \mu_c)\) in place of \((v_i, \mu_i)\) then yields
\[
\frac{\phi(v_c-c)}{\sigma_c} = \frac{\phi(\frac{\theta-v_i}{\sqrt{\sigma^2+\sigma^2_\theta}})}{\sqrt{\sigma^2+\sigma^2_\theta}}.
\]
(17)

This characterizes the value \(v_c\) associated with the optimal reward program for a given threshold value \(c\). The optimal joint price-reward program can then be obtained by maximizing the expected utility of the firm under the optimal reward program, as given in (14) over \(c\). A helpful observation here is to note that the optimality of the reward program \(r^*_c(\bar{a}) = r_M 1\{\bar{a} \leq a_c\}\) given \(c\), already ensures that \(\frac{\partial}{\partial v_c} \mathbb{E}[\Pi(p^*_c, r^*_c(\cdot))] = 0\). This simplifies deriving the first order condition for the optimality of the threshold \(c\) as it only requires calculating the partial derivative of \(\mathbb{E}[\Pi(p^*_c, r^*_c(\cdot))]\) with respect to \(c\). It is easy to see that
\[
\frac{\partial}{\partial c} \mathbb{E}[\Pi(p^*_c, r^*_c(\cdot))] = \frac{\partial}{\partial c} \mathbb{E}[\Pi(c, 0)] - r_M \frac{\partial}{\partial c} \mu_c \int_{c}^{\infty} \phi(u_c - \mu_c) \frac{\phi(\frac{\theta-v_i}{\sqrt{\sigma^2+\sigma^2_\theta}})}{\sqrt{\sigma^2+\sigma^2_\theta}} dv_i
\]
\[
= \frac{\partial}{\partial c} \mathbb{E}[\Pi(c, 0)] - \sqrt{2\pi} \phi(\frac{u_c - \mu_c}{\sigma_v}) \Phi(\frac{\theta - c}{\sqrt{\sigma^2+\sigma^2_\theta}}),
\]
where we have used the identity \(r_M = \sqrt{2\pi} \sigma_c \sqrt{1 + (\frac{\sigma_v}{\sigma_\theta})^2} = \sqrt{2\pi} \frac{\sigma_v}{\tau}\). Therefore, the optimality condition is
\[
\frac{\partial}{\partial c} \mathbb{E}[\Pi(c, 0)] = \sqrt{2\pi} \phi(\frac{u_c - \mu_c}{\sigma_v}) \Phi(\frac{\theta - c}{\sqrt{\sigma^2+\sigma^2_\theta}}).
\]
(18)

On the other hand, \(\mathbb{E}[\Pi(c, 0)] = c \Phi(\frac{\theta-c}{\sqrt{\sigma^2+\sigma^2_\theta}})\) and hence
\[
\frac{\partial}{\partial c} \mathbb{E}[\Pi(c, 0)] = \Phi(\frac{\theta - c}{\sqrt{\sigma^2+\sigma^2_\theta}}) - c \frac{\phi(\frac{\theta-c}{\sqrt{\sigma^2+\sigma^2_\theta}})}{\sqrt{\sigma^2+\sigma^2_\theta}}.
\]

Incorporating this into (18), we can reach at
\[
1 - \sqrt{2\pi} \phi\left(\frac{v_c - \mu_c}{\sigma_c}\right) \frac{\phi(\frac{\theta-c}{\sqrt{\sigma^2+\sigma^2_\theta}})}{c} = \Phi\left(\frac{\theta-c}{\sqrt{\sigma^2+\sigma^2_\theta}}\right),
\]
which together with (17) proves (9). Finally, we use the indifference equation (2) to find the
associated optimal price as

\[ p^*_c = c + r_M \Phi\left(\frac{v_c - \mu_c}{\sigma_v}\right). \]

**Proof of Proposition 1.** As shown in the text above the proposition, for the optimal no reward price \( p^*_0 \) and the cutoff value \( c^* \) for the optimal price-reward program characterized in Theorem 2, we have \( c^* < p^*_0 \). Therefore, the joint program clearly increases the sales volume. The payoff of \( a_i = 0 \) is zero, so the aggregate welfare resulted from a joint price-reward program \((p, r(\cdot))\) with an associated cutoff \( c \) on the buyers valuations will be

\[
W(p, r(\cdot)) = \Pi(p, r(\cdot)) + \int_{c}^{\infty} u_i \phi\left(\frac{v_i - v}{\sigma_{\epsilon}}\right) dv_i
\]

\[
= \int_{c}^{\infty} v_i \phi\left(\frac{v_i - c}{\sigma_{\epsilon}}\right) dv_i,
\]

where we have used the fact that \( p - r(\overline{a}) \) transfers between the seller and buyers. As a result,

\[
\frac{\partial}{\partial c} W(p, r(\cdot)) = -c \frac{\phi\left(\frac{v - c}{\sigma_{\epsilon}}\right)}{\sigma_{\epsilon}} < 0,
\]

implying a higher welfare under the proposed reward program noting that \( c^* < p^*_0 \).