A Game Theoretic Approach to Promotion Design in Two-Sided Platforms

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Motivation

Two-sided platforms

- Connects providers to consumers (e.g., auctioneers, credit cards, dating services, ride-sharing platforms)
- Revenue source: commissions, ads, from one side or both.

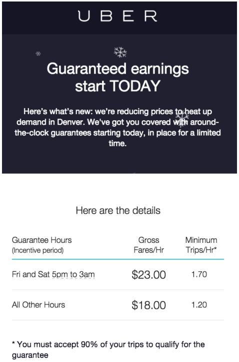
Ride-sharing platforms

- Two-sided markets that pair customers and drivers.
- Popular platforms: *Uber* and *Lyft*.
- Extensive literature on different aspects of ride-sharing platforms.
- dynamic pricing (Banerjee et al. 2015), surge pricing (Cachon et al. 2017), matching providers to consumers (Ozkan & Ward 2016), spatial pricing (Bimpikis et al. 2016).
- Bike-sharing systems (Kabra et al. 2015, Henderson et al. 2016).

Motivation

High uncertainty in the number of drivers working at anytime

- Little control of platform over drivers.
- Short-term participation decisions.
- Uncertainty in expected wages:
  - how likely to get called.
  - function of provider's expectation on demand.
  - total number of active providers.
  - hence their expectation on other providers' expectations, and so on.
- Can result in too many or too few providers.



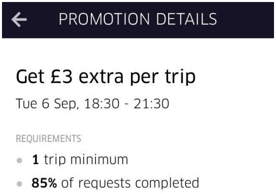
Use promotions to influence providers' availability.

Motivation

A game theoretic approach to promotion design

- *Global games* (Morris & Shin (1998, 2003), Carlsson & van Damme 1993) to model strategic behavior of providers and the heterogeneity of their beliefs on the availability of others.

- Formulate the optimal promotion design as an infinite dimensional convex optimization.
- Characterize the optimal price-dependent promotion.



Model: Full Information Case

- Mass 1 of service providers $i \in I = [0, 1]$.
- Demand of mass θ.
- Price (i.e., hourly average wages) increasing with demand (θ) and decreasing with active providers ($\bar{a}$):

$$p = 1 - \bar{a} + \alpha\theta,$$

  WLOG $\alpha = 1$.
- Utility of provider i:

$$u_i = (1 - \delta)p - v_i,$$

  δ commission rate (set by platform), v_i provider's reserved price.
- $v_i = v + \epsilon_i$, v is average reserved price. $\epsilon \sim N(0, \sigma_\epsilon^2)$ the idiosyncratic variation across providers.
- Profit of the platform:

$$\Pi(\delta, p, \bar{a}) = \delta p \bar{a}.$$

Model: Full Information Case

Profit maximizing providers:

$$a_i = \mathbf{1}\{u_i \geq 0\} = \mathbf{1}\{(1-\delta)p - v_i \geq 0\}.$$

- Equilibrium: a threshold strategy

$$a_i = \mathbf{1}\{v_i \leq c\},$$

  c is the reserved value at cutoff.

- Mass of active providers $\bar{a} = \Phi(\frac{c-v}{\sigma_\epsilon})$.
- Indifference equation $c = (1-\delta)p$ for cutoff gives

$$c = (1-\delta)(1-\bar{a}+\theta) = (1-\delta)(1-\Phi(\frac{c-v}{\sigma_\epsilon})+\theta).$$

- LHS increasing in c, RHS decreasing in $c \Rightarrow$ unique threshold c.
- Can use to analyze the effect of commission rate δ on platform's profit $\Pi = \delta p \bar{a}$.

Model: Incomplete Information Case

Uncertainty in average of reserved prices (v)

- Recall $v_i = v + \epsilon_i$, v is average reserved price.
- $\epsilon \sim N(0, \sigma_\epsilon^2)$: the idiosyncratic variation across providers.
- $v \sim N(\mu, \sigma_\mu^2)$: aggregate variation in reserved prices (common prior).
- Provider i reserved price v_i: a private observation from v. Posterior belief $v|v_i \sim N(\mu_i, \sigma_v^2)$,

$$\mu_i = \tau v_i + (1-\tau)\mu, \quad \tau = \frac{\sigma_\mu^2}{\sigma_\epsilon^2 + \sigma_\mu^2}, \quad \sigma_v^2 = \frac{\sigma_\epsilon^2 \sigma_\mu^2}{\sigma_\epsilon^2 + \sigma_\mu^2}.$$

Solving the updated model:

- Actions maximize expected utilities:

$$a_i = \mathbf{1}\{\mathbb{E}_{v|v_i}[u_i] \geq 0\} = \mathbf{1}\{(1-\delta)\mathbb{E}_{v|v_i}[p] \geq v_i\}.$$

- Platform looks at its expected profit over all realizations of v:

$$\mathbb{E}_v[\Pi(\delta, p(v), \bar{a}(v))] = \mathbb{E}_v[\delta(1 - \bar{a}(v) + \theta)\bar{a}(v)].$$

Equilibrium Analysis

- For a (Bayes Nash) threshold equilibrium strategy $a_i = \mathbf{1}\{v_i \leq c\}$:

$$\mathbb{E}_{v|v_i}[\bar{a}] = \mathbb{E}_{v|v_i}[\Phi(\frac{c-v}{\sigma_\epsilon})] = \Phi(\frac{c-\mu_i}{\sqrt{\sigma_\epsilon^2+\sigma_v^2}}) \Rightarrow$$

$$\mathbb{E}_{v|v_i}[p] = 1 - \Phi(\frac{c-\mu_i}{\sqrt{\sigma_\epsilon^2+\sigma_v^2}}) + \theta.$$

- Indifference equation $c = \mathbb{E}_{v|v_i=c}[(1-\delta)p]$ gives

$$c = (1-\delta)(1-\Phi(\frac{(1-\tau)(c-\mu)}{\sqrt{\sigma_\epsilon^2+\sigma_v^2}}) + \theta).$$

- Unique solution, but not necessarily an equilibrium.
- Standard approach in *global games*: assumptions on signal precisions to ensure single crossing property (monotonicity) for expected utilities.
- May lead to extremely suboptimal design. We avoid it!

Promotion Design

For an optimally designed commission rate, can the platform improve its expected profit further?

- Modeling promotions by a function $r : \mathbb{R}^{+} \cup \{0\} \to \mathbb{R}^{+} \cup \{0\}$.
- Platform compensates active drivers with a (gross value) $r(p)$ if the hourly rate is p.

Updated Utilities:

$$u_i = (1-\delta)(1-\bar{a}(v)+\theta+r(v)) - v_i,$$
$$\Pi = \delta(1-\bar{a}(v)+\theta)\bar{a}(v) - (1-\delta)r(p)\bar{a}(v).$$

- For a threshold strategy $p(v) = 1 - \Phi(\frac{c-v}{\sigma_\epsilon}) + \theta$, so $r(\cdot)$ can be regarded a function of v.
- Motivated by *Uber*'s guaranteed hourly rate promotion, we solve for the optimal promotion decreasing with p (thus with v).

Optimal Promotion Program

We can break the expected profit of the platform into two parts:

$$\mathbb{E}_v[\Pi] = \underbrace{\mathbb{E}_v[\delta(1-\bar{a}(v)+\theta)\bar{a}(v)]}_{\text{commission from target providers}} - \underbrace{\mathbb{E}_v[(1-\delta)r(v)\bar{a}(v)]}_{\text{cost of the promotion program}}$$

Two steps decision making:

- Which group of providers should we target (desired value of c)?
- How to incentivize the target group to become active ($a_i = 1$ for $v_i \leq c$) with minimum spending (choosing $r(\cdot)$)?
- Need to answer the second question first.
- A useful identity:

$$\mathbb{E}_v[r(v)\bar{a}(v)] = \int_{-\infty}^{c} \mathbb{E}_{v|v_i}[r(v)]\psi(v_i)dv_i,$$

$\psi(v_i) = \frac{\phi\left(\frac{\mu - v_i}{\sqrt{\sigma_\epsilon^2+\sigma_\mu^2}}\right)}{\sqrt{\sigma_\epsilon^2+\sigma_\mu^2}}$: ex-ante pdf of reserved price v_i among providers.

- A simple partition of the expected cost as sum of the cost for each active reserved price category.

Optimal Promotion Program

A convex optimization problem:

$$\underset{\substack{r:\mathbb{R}\to\mathbb{R}^+\cup\{0\}\\ r(\cdot)\text{ decreasing}}}{\text{minimize}} \int_{-\infty}^{c} \mathbb{E}_{v|v_i}[r(v)]\psi(v_i)dv_i, \text{ subject to:}$$

$$\mathbb{E}_{v|v_i}[r(v)] + 1 + \theta - \mathbb{E}_{v|v_i}[\bar{a}(v)] - \frac{v_i}{1-\delta} \geq 0 \text{ for } v_i \leq c,$$

$$\mathbb{E}_{v|v_i}[r(v)] + 1 + \theta - \mathbb{E}_{v|v_i}[\bar{a}(v)] - \frac{v_i}{1-\delta} \leq 0 \text{ for } v_i \geq c,$$

where $\mathbb{E}_{v|v_i}[\bar{a}(v)] = \Phi(\frac{c-\mu_i}{\sqrt{\sigma_\epsilon^2+\sigma_v^2}})$.

- $\lambda^* : \mathbb{R} \to \mathbb{R}$ lagrange multipliers for the optimal promotion $r^*(\cdot)$,

$$\int_{-\infty}^{+\infty} \lambda^*(v_i)(\mathbb{E}_{v|v_i}[r^*(v)] + 1 + \theta - \mathbb{E}_{v|v_i}[\bar{a}(v)] - \frac{v_i}{1-\delta})dv_i = 0, \quad \text{(CS)}$$

  where $\lambda^*(v_i)(v_i - c) \geq 0$.
- Also, there are finite number of active constraints.
- Only need to look at the dual function $q(\lambda)$ at λ with finite nonzero coordinates.

Optimal Promotion Program

- For any finite set of valuations $\mathcal{V} = \{v_1 \leq v_2 \leq \ldots \leq v_n\}$, let

$$q(\lambda, \mathcal{V}, r(\cdot)) = \int_{-\infty}^{c} \mathbb{E}_{v|v_i}[r(v)]\psi(v_i)dv_i + \sum_{v_k \in \mathcal{V}} \lambda(v_k)(\mathbb{E}_{v|v_k}[r(v)] + 1 + \theta - \mathbb{E}_{v|v_k}[\bar{a}(v)] - \frac{v_k}{1-\delta})$$

- $\lambda(v_k)(v_k - c) \geq 0.$
- Dual function: $q(\lambda, \mathcal{V}) = \min_{\substack{r:\mathbb{R}\to\mathbb{R}^+\cup\{0\} \\ r(\cdot) \text{ decreasing}}} q(\lambda, \mathcal{V}, r(\cdot)).$
- We can write $q(\lambda, \mathcal{V}, r(\cdot))$ as

$$q(\lambda, \mathcal{V}, r(\cdot)) = \int_{-\infty}^{+\infty} r(v) g(v, \lambda(\mathcal{V}))dv + \sum_{v_k \in \mathcal{V}} \lambda(v_k)(1 + \theta - \mathbb{E}_{v|v_k}[\bar{a}(v)] - \frac{v_k}{1-\delta}),$$

$$g(v, \lambda, \mathcal{V}) = \int_{-\infty}^{c} \psi(v_i)\eta(v|v_i)dv_i + \sum_{v_k \in \mathcal{V}} \lambda(v_k)\eta(v|v_k).$$

- $\eta(v|v_i) = \frac{\phi(\frac{v-\mu_i}{\sigma_v})}{\sigma_v}$ is pdf of $v|v_i$.

Optimal Promotion Program

Characterizing the dual function:

Lemma

Define,

$$\begin{aligned} G(w,\lambda,\mathcal{V}) &= \int_{-\infty}^{w} g(v,\lambda,\mathcal{V})dv \\ &= \int_{-\infty}^{c} \Phi(\frac{w-\mu_i}{\sigma_v})\psi(v_i)dv_i + \sum_{v_k\in\mathcal{V}} \lambda(v_k)\Phi(\frac{w-\mu_k}{\sigma_v}). \end{aligned}$$

Then,

$$q(\lambda,\mathcal{V}) = \begin{cases} \sum_{v_k\in\mathcal{V}} \lambda(v_k)(1+\theta-\Phi(\frac{c-\mu_k}{\sqrt{\sigma_\epsilon^2+\sigma_v^2}}) - \frac{v_k}{1-\delta}), & \textit{if } \min_{w\in\mathbb{R}\cup\{\pm\infty\}} G(w,\lambda,\mathcal{V}) = 0, \\ -\infty, & \textit{otherwise}. \end{cases}$$

- Note that $G(-\infty,\lambda,\mathcal{V}) = 0$ for all λ and $\mathcal{V}$. Therefore, $\min_{w\in\mathbb{R}\cup\{\pm\infty\}} G(w,\lambda,\mathcal{V}) \leq 0$.

Optimal Promotion Program

Characterizing the dual optimal solution:

- For a maximizer $(\lambda^*, \mathcal{V}^*)$, $\min_{w \in \mathbb{R} \cup \{\pm\infty\}} G(w, \lambda^*, \mathcal{V}^*) = 0$ (lemma).
- W^*: minimizers of G.
- $\delta : \mathbb{R} \to \mathbb{R}$, a feasible variation ($\delta(v_i)(v_i - c) \geq 0$ for $v_i \notin \mathcal{V}^*$).
- If $< \delta(v_i), \Phi(\frac{w - \mu_i}{\sigma_v}) > \geq 0$ for all $w \in W^*$ (dual cone), then

$$\Delta q = < \delta(v_i), 1 + \theta - \Phi(\frac{c - \mu_i}{\sqrt{\sigma_\epsilon^2 + \sigma_v^2}}) - \frac{v_i}{1 - \delta} > \leq 0.$$

Applying *(generalized) Farkas' lemma*, we get the following result.

Lemma

$(\lambda^*, \mathcal{V}^*)$ is an optimal solution to the dual problem if and only if

i) $\min_{w \in \mathbb{R} \cup \{\pm\infty\}} G(w, \lambda^*, \mathcal{V}^*) = 0$.

ii) There exists $r_j \geq 0$, $j = 1, \ldots, |W^*|$ such that for the function

$$g(v_i) = 1 + \theta - \Phi(\frac{c - \mu_i}{\sqrt{\sigma_\epsilon^2 + \sigma_v^2}}) - \frac{v_i}{1 - \delta} + \sum_{j=1}^{|W^*|} r_j \Phi(\frac{w_j - \mu_i}{\sigma_v}),$$

$g(v_i) \geq 0$ for $v_i \leq c$, $g(v_i) \leq 0$ for $v_i \geq c$, and $g(v_i) = 0$ for $v_i \in V^*$.

Optimal Promotion Program

We can use the previous lemma to find an optimality condition for primal solutions.

Lemma

A decreasing promotion program is optimal if and only if
i) it is feasible, piece-wise constant, i.e., $r(v) = \sum_{j=1}^{l} \mathbf{1}\{v \leq w_j\}$*, and*
ii) if $\mathcal{V}$ *is the set of active constraints, then there exist* $\{\lambda(v_k)\}_{v_k \in \mathcal{V}}$ *such that* $\min_{w \in \mathbb{R} \cup \{\pm\infty\}} G(w, \lambda, \mathcal{V}) = 0$ *with* $W = \{w_1, \ldots, w_l\}$ *as its minimizers.*

Which threshold values are feasible?

- c is feasible if and only if there is a promotion $r(\cdot)$ such that

$$\mathbb{E}_{v|v_i}[r(v)] + 1 + \theta - \Phi\left(\frac{c - \mu_i}{\sqrt{\sigma_\epsilon^2 + \sigma_\nu^2}}\right) - \frac{v_i}{1-\delta} \geq 0 \text{ for } v_i \leq c,$$

$$\mathbb{E}_{v|v_i}[r(v)] + 1 + \theta - \Phi\left(\frac{c - \mu_i}{\sqrt{\sigma_\epsilon^2 + \sigma_\nu^2}}\right) - \frac{v_i}{1-\delta} \leq 0 \text{ for } v_i \geq c.$$

- $\mathbb{E}_{v|v_i}[r(v)] \geq 0$, so a necessary condition is

$$1 + \theta - \Phi\left(\frac{c - \mu_i}{\sqrt{\sigma_\epsilon^2 + \sigma_\nu^2}}\right) - \frac{v_i}{1-\delta} \leq 0 \text{ for } v_i \geq c.$$

Optimal Promotion Program

Which threshold values are feasible?(cont'd)

- For a feasible c, the best response of an agent with reserved price $v_i \geq c$ to a threshold strategy with cutoff c used by everyone else is to not participate.
- Also sufficient: can be realized using a promotion of the form $r(v) = r_0 + r_1 \mathbf{1}\{v \leq w_0\}$ with $r_0, r_1 \geq 0$ (proof by construction).

Applying the optimality condition derived for the primal solution, we can show the following main result:

Theorem

The optimal decreasing promotion program maximizing the expected profit of the platform is a combination of (i) a bonus at all prices, and (ii) a bonus only at low prices, i.e., $r(p) = r_0 + r_1 \mathbf{1}\{p \leq p_0\}$ *for some* $r_0, r_1, p_0 \geq 0$.

Conclusions

- High uncertainty in the number of active providers in ride-sharing platforms.
- Promotions as a means to influence providers' availability.
- Global games to model the strategic interaction of providers and heterogeneity of their beliefs on active drivers.
- Formulate the optimal price-dependent promotion in the infinite dimensional convex optimization framework.
- Optimal decreasing promotion: combination of a bonus at all prices and a low-price-only bonus.

Thank You!