A Game Theoretic Approach to Promotion Design in Two-Sided Platforms

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Motivation

Two-sided platforms

- Connects providers to consumers (e.g., auctioneers, credit cards, dating services, ride-sharing platforms)
- Revenue source: commissions, ads, from one side or both.

Ride-sharing platforms

- Two-sided markets that pair customers and drivers.
- Popular platforms: *Uber* and *Lyft*.
- Extensive literature on different aspects of ride-sharing platforms.
- Dynamic pricing (Banerjee et al. 2015), surge pricing (Cachon et al. 2017), matching providers to consumers (Ozkan & Ward 2016), spatial pricing (Bimpikis et al. 2016).
- Bike-sharing systems (Kabra et al. 2015, Henderson et al. 2016).
Motivation

High uncertainty in the number of drivers working at anytime

- Little control of platform over drivers.
- Short-term participation decisions.
- Uncertainty in expected wages:
  - how likely to get called.
  - function of provider’s expectation on demand.
  - total number of active providers.
  - hence their expectation on other providers’ expectations, and so on.
- Can result in too many or too few providers.

Use promotions to influence providers’ availability.
Motivation

A game theoretic approach to promotion design


- Formulate the optimal promotion design as an infinite dimensional convex optimization.

- Characterize the optimal price-dependent promotion.
Model: Full Information Case

- Mass 1 of service providers $i \in I = [0, 1]$.
- Demand of mass $\theta$.
- Price (i.e., hourly average wages) increasing with demand ($\theta$) and decreasing with active providers ($\bar{a}$):
  \[ p = 1 - \bar{a} + \alpha \theta, \]
  WLOG $\alpha = 1$.
- Utility of provider $i$:
  \[ u_i = (1 - \delta)p - v_i, \]
  $\delta$ commission rate (set by platform), $v_i$ provider’s reserved price.
- $v_i = v + \epsilon_i$, $v$ is average reserved price. $\epsilon \sim N(0, \sigma_\epsilon^2)$ the idiosyncratic variation across providers.
- Profit of the platform:
  \[ \Pi(\delta, p, \bar{a}) = \delta p\bar{a}. \]
Model: Full Information Case

Profit maximizing providers:

\[ a_i = 1\{u_i \geq 0\} = 1\{(1 - \delta)p - v_i \geq 0\}. \]

- Equilibrium: a threshold strategy

\[ a_i = 1\{v_i \leq c\}, \]

\( c \) is the reserved value at cutoff.

- Mass of active providers \( \bar{a} = \Phi(\frac{c-v}{\sigma_e}) \).

- Indifference equation \( c = (1 - \delta)p \) for cutoff gives

\[ c = (1 - \delta)(1 - \bar{a} + \theta) = (1 - \delta)(1 - \Phi(\frac{c-v}{\sigma_e}) + \theta). \]

- LHS increasing in \( c \), RHS decreasing in \( c \) \( \Rightarrow \) unique threshold \( c \).

- Can use to analyze the effect of commission rate \( \delta \) on platform’s profit \( \Pi = \delta p\bar{a} \).
Model: Incomplete Information Case

Uncertainty in average of reserved prices ($v$)

- Recall $v_i = v + \epsilon_i$, $v$ is average reserved price.
- $\epsilon \sim N(0, \sigma^2_\epsilon)$: the idiosyncratic variation across providers.
- $v \sim N(\mu, \sigma^2_\mu)$: aggregate variation in reserved prices (common prior).

Provider $i$ reserved price $v_i$: a private observation from $v$.
Posterior belief $v|v_i \sim N(\mu_i, \sigma^2_v)$,

$$
\mu_i = \tau v_i + (1 - \tau)\mu, \quad \tau = \frac{\sigma^2_\mu}{\sigma^2_\epsilon + \sigma^2_\mu}, \quad \sigma^2_v = \frac{\sigma^2_\epsilon \sigma^2_\mu}{\sigma^2_\epsilon + \sigma^2_\mu}.
$$

Solving the updated model:

- Actions maximize expected utilities:
  $$
a_i = 1\{\mathbb{E}_{v|v_i}[u_i] \geq 0\} = 1\{(1 - \delta)\mathbb{E}_{v|v_i}[p] \geq v_i\}.
$$

- Platform looks at its expected profit over all realizations of $v$:
  $$
  \mathbb{E}_v[\Pi(\delta, p(v), \bar{a}(v))] = \mathbb{E}_v[\delta(1 - \bar{a}(v) + \theta)\bar{a}(v)].
  $$
Equilibrium Analysis

- For a (Bayes Nash) threshold equilibrium strategy \( a_i = 1\{v_i \leq c\}: \)

  \[
  \mathbb{E}_{v|v_i}[\bar{a}] = \mathbb{E}_{v|v_i}[\Phi(c - v)] = \Phi\left(\frac{c - \mu_i}{\sigma_v}\right) \Rightarrow \\
  \mathbb{E}_{v|v_i}[p] = 1 - \Phi\left(\frac{c - \mu_i}{\sqrt{\sigma_v^2 + \sigma^2}}\right) + \theta.
  \]

- Indifference equation \( c = \mathbb{E}_{v|v_i=c}[1 - \delta)p \) gives

  \[ c = (1 - \delta)(1 - \Phi\left(\frac{(1 - \tau)(c - \mu)}{\sqrt{\sigma_v^2 + \sigma^2}}\right) + \theta). \]

- Unique solution, but not necessarily an equilibrium.

- Standard approach in global games: assumptions on signal precisions to ensure single crossing property (monotonicity) for expected utilities.

- May lead to extremely suboptimal design. We avoid it!
For an optimally designed commission rate, can the platform improve its expected profit further?

- Modeling promotions by a function $r : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+ \cup \{0\}$.
- Platform compensates active drivers with a (gross value) $r(p)$ if the hourly rate is $p$.

Updated Utilities:

$$u_i = (1 - \delta)(1 - \bar{a}(v) + \theta + r(v)) - v_i,$$
$$\Pi = \delta(1 - \bar{a}(v) + \theta)\bar{a}(v) - (1 - \delta)r(p)\bar{a}(v).$$

- For a threshold strategy $p(v) = 1 - \Phi\left(\frac{c-v}{\sigma_e}\right) + \theta$, so $r(\cdot)$ can be regarded a function of $v$.
- Motivated by Uber’s guaranteed hourly rate promotion, we solve for the optimal promotion decreasing with $p$ (thus with $v$).
Optimal Promotion Program

We can break the expected profit of the platform into two parts:

\[ E \mathbb{E}[\Pi] = \mathbb{E} \left[ \delta(1 - \bar{a}(v) + \theta)\bar{a}(v) \right] - \mathbb{E} \left[ (1 - \delta)\bar{r}(v)\bar{a}(v) \right] \]

- commission from target providers
- cost of the promotion program

Two steps decision making:

- Which group of providers should we target (desired value of \( c \))?  
- How to incentivize the target group to become active \( (a_i = 1 \text{ for } v_i \leq c) \) with minimum spending (choosing \( r(\cdot) \))?  
- Need to answer the second question first.
- A useful identity:

\[ E \mathbb{E}[r(v)\bar{a}(v)] = \int_{-\infty}^{c} E \mathbb{E}_{v_i}[r(v)] \psi(v_i) dv_i, \]

\[ \phi \left( \frac{\mu - v_i}{\sqrt{\sigma^2 + \sigma^2 \mu}} \right) = \int_{-\infty}^{c} E \mathbb{E}_{v_i}[r(v)] \psi(v_i) dv_i, \]

\[ \psi(v_i) = \frac{\phi \left( \frac{\mu - v_i}{\sqrt{\sigma^2 + \sigma^2 \mu}} \right)}{\sqrt{\sigma^2 + \sigma^2 \mu}} : \text{ex-ante pdf of reserved price } v_i \text{ among providers.} \]

- A simple partition of the expected cost as sum of the cost for each active reserved price category.
Optimal Promotion Program

A convex optimization problem:

$$\text{minimize } \int_{-\infty}^{c} \mathbb{E}_{v_i}[r(v)]\psi(v_i)dv_i, \text{ subject to:}$$

$$\mathbb{E}_{v_i}[r(v)] + 1 + \theta - \mathbb{E}_{v_i}[\bar{a}(v)] - \frac{v_i}{1 - \delta} \geq 0 \text{ for } v_i \leq c,$$

$$\mathbb{E}_{v_i}[r(v)] + 1 + \theta - \mathbb{E}_{v_i}[\bar{a}(v)] - \frac{v_i}{1 - \delta} \leq 0 \text{ for } v_i \geq c,$$

where $$\mathbb{E}_{v_i}[\bar{a}(v)] = \Phi\left(\frac{c - \mu_i}{\sqrt{\sigma_e^2 + \sigma_v^2}}\right).$$

- $$\lambda^* : \mathbb{R} \rightarrow \mathbb{R}$$ lagrange multipliers for the optimal promotion $$r^*(\cdot),$$

$$\int_{-\infty}^{+\infty} \lambda^*(v_i)(\mathbb{E}_{v_i}[r^*(v)] + 1 + \theta - \mathbb{E}_{v_i}[\bar{a}(v)] - \frac{v_i}{1 - \delta})dv_i = 0, \quad (CS)$$

where $$\lambda^*(v_i)(v_i - c) \geq 0.$$

- Also, there are finite number of active constraints.

- Only need to look at the dual function $$q(\lambda)$$ at $$\lambda$$ with finite nonzero coordinates.
For any finite set of valuations \( \mathcal{V} = \{v_1 \leq v_2 \leq \ldots \leq v_n\} \), let

\[
q(\lambda, \mathcal{V}, r(\cdot)) = \int_{-\infty}^{c} \mathbb{E}_{v|v_i}[r(v)]\psi(v_i)dv_i + \sum_{v_k \in \mathcal{V}} \lambda(v_k)(\mathbb{E}_{v|v_k}[r(v)] + 1 + \theta - \mathbb{E}_{v|v_k}[\bar{a}(v)] - \frac{v_k}{1 - \delta})
\]

\[
\lambda(v_k)(v_k - c) \geq 0.
\]

**Dual function:** \( q(\lambda, \mathcal{V}) = \min_{r: \mathbb{R} \rightarrow \mathbb{R}^+ \cup \{0\}} q(\lambda, \mathcal{V}, r(\cdot)). \)

We can write \( q(\lambda, \mathcal{V}, r(\cdot)) \) as

\[
q(\lambda, \mathcal{V}, r(\cdot)) = \int_{-\infty}^{+\infty} r(v)g(v, \lambda(\mathcal{V}))dv + \sum_{v_k \in \mathcal{V}} \lambda(v_k)(1 + \theta - \mathbb{E}_{v|v_k}[\bar{a}(v)] - \frac{v_k}{1 - \delta})
\]

\[
g(v, \lambda, \mathcal{V}) = \int_{-\infty}^{c} \psi(v_i)\eta(v|v_i)dv_i + \sum_{v_k \in \mathcal{V}} \lambda(v_k)\eta(v|v_k).
\]

\[
\eta(v|v_i) = \frac{\phi\left(\frac{v - \mu_i}{\sigma_v}\right)}{\sigma_v} \text{ is pdf of } v|v_i.
\]
Lemma

Define,

\[
G(w, \lambda, V) = \int_{-\infty}^{w} g(v, \lambda, V) dv \\
= \int_{-\infty}^{c} \Phi\left(\frac{w - \mu_i}{\sigma_v}\right) \psi(v_i) dv_i + \sum_{v_k \in V} \lambda(v_k) \Phi\left(\frac{w - \mu_k}{\sigma_v}\right).
\]

Then,

\[
q(\lambda, V) = \begin{cases} 
\sum_{v_k \in V} \lambda(v_k)(1 + \theta - \Phi\left(\frac{c - \mu_k}{\sqrt{\sigma^2 + \sigma_0^2}}\right) - \frac{v_k}{1-\delta}), & \text{if } \min_{w \in \mathbb{R} \cup \{\pm \infty\}} G(w, \lambda, V) = 0, \\
-\infty, & \text{otherwise.}
\end{cases}
\]

Note that \(G(-\infty, \lambda, V) = 0\) for all \(\lambda\) and \(V\). Therefore,

\[
\min_{w \in \mathbb{R} \cup \{\pm \infty\}} G(w, \lambda, V) \leq 0.
\]
Optimal Promotion Program

Characterizing the dual optimal solution:

- For a maximizer \((\lambda^*, V^*)\), \(\min_{w \in \mathbb{R} \cup \{\pm \infty\}} G(w, \lambda^*, V^*) = 0\) (lemma).
- \(W^*\): minimizers of \(G\).
- \(\delta: \mathbb{R} \rightarrow \mathbb{R}\), a feasible variation \((\delta(v_i)(v_i - c) \geq 0 \text{ for } v_i \notin V^*)\).
- If \(<\delta(v_i), \Phi(w - \mu_i)\rangle \geq 0\) for all \(w \in W^*\) (dual cone), then

\[
\Delta q = \langle \delta(v_i), 1 + \theta - \frac{c - \mu_i}{\sqrt{\sigma^2 + \sigma^2_v}} - \frac{v_i}{1 - \delta} \rangle \leq 0.
\]

Applying (generalized) Farkas’ lemma, we get the following result.

**Lemma**

\((\lambda^*, V^*)\) is an optimal solution to the dual problem if and only if

i) \(\min_{w \in \mathbb{R} \cup \{\pm \infty\}} G(w, \lambda^*, V^*) = 0\).

ii) There exists \(r_j \geq 0, j = 1, \ldots, |W^*|\) such that for the function

\[
g(v_i) = 1 + \theta - \Phi\left(\frac{c - \mu_i}{\sqrt{\sigma^2 + \sigma^2_v}}\right) - \frac{v_i}{1 - \delta} + \sum_{j=1}^{|W^*|} r_j \Phi\left(\frac{w_j - \mu_i}{\sigma_v}\right),
\]

\(g(v_i) \geq 0\) for \(v_i \leq c\), \(g(v_i) \leq 0\) for \(v_i \geq c\), and \(g(v_i) = 0\) for \(v_i \in V^*\).
We can use the previous lemma to find an optimality condition for primal solutions.

**Lemma**

A decreasing promotion program is optimal if and only if

i) it is feasible, piece-wise constant, i.e., \( r(v) = \sum_{j=1}^{l} 1\{v \leq w_j\} \), and

ii) if \( V \) is the set of active constraints, then there exist \( \{\lambda(v_k)\}_{v_k \in V} \) such that

\[
\min_{w \in \mathbb{R} \cup \{-\infty, +\infty\}} G(w, \lambda, V) = 0 \quad \text{with} \quad W = \{w_1, \ldots, w_l\} \quad \text{as its minimizers.}
\]

**Which threshold values are feasible?**

- \( c \) is feasible if and only if there is a promotion \( r(\cdot) \) such that

\[
\mathbb{E}_{v|v_i}[r(v)] + 1 + \theta - \Phi \left( \frac{c - \mu_i}{\sqrt{\sigma^2_v + \sigma^2_\varepsilon}} \right) - \frac{v_i}{1 - \delta} > 0 \quad \text{for} \quad v_i \leq c,
\]

\[
\mathbb{E}_{v|v_i}[r(v)] + 1 + \theta - \Phi \left( \frac{c - \mu_i}{\sqrt{\sigma^2_v + \sigma^2_\varepsilon}} \right) - \frac{v_i}{1 - \delta} \leq 0 \quad \text{for} \quad v_i \geq c.
\]

- \( \mathbb{E}_{v|v_i}[r(v)] \geq 0 \), so a necessary condition is

\[
1 + \theta - \Phi \left( \frac{c - \mu_i}{\sqrt{\sigma^2_v + \sigma^2_\varepsilon}} \right) - \frac{v_i}{1 - \delta} \leq 0 \quad \text{for} \quad v_i \geq c.
\]
Which threshold values are feasible? (cont’d)

- For a feasible $c$, the best response of an agent with reserved price $v_i \geq c$ to a threshold strategy with cutoff $c$ used by everyone else is to not participate.

- Also sufficient: can be realized using a promotion of the form 
  \[ r(v) = r_0 + r_1 \mathbb{1}_{v \leq w_0} \]
  with $r_0, r_1 \geq 0$ (proof by construction).

Applying the optimality condition derived for the primal solution, we can show the following main result:

**Theorem**

The optimal decreasing promotion program maximizing the expected profit of the platform is a combination of (i) a bonus at all prices, and (ii) a bonus only at low prices, i.e., 
\[ r(p) = r_0 + r_1 \mathbb{1}_{p \leq p_0} \]
for some $r_0, r_1, p_0 \geq 0$. 
Conclusions

- High uncertainty in the number of active providers in ride-sharing platforms.
- Promotions as a means to influence providers’ availability.
- Global games to model the strategic interaction of providers and heterogeneity of their beliefs on active drivers.
- Formulate the optimal price-dependent promotion in the infinite dimensional convex optimization framework.
- Optimal decreasing promotion: combination of a bonus at all prices and a low-price-only bonus.
Thank You!