Sales-Based Aggregate Rebate Design

Amir Ajorlou    Ali Jadbabaie

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Motivation

Network effect (or positive externality): a consumer’s utility is increasing in the usage level of her peers.

Bob’s WPT*: $1

*WPT: willingness to pay.
Network effect (or positive externality): a consumer’s utility is increasing in the usage level of her peers.

Bob’s WPT*: $1

Bob’s WPT: $2

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Bob’s WPT*: $1

Bob’s WPT: $2

Bob’s WPT: $2.50

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smartphone apps
online games
social networks
online dating
Motivation

Network effect results in a marginal gain in profit.

### No Network effect

**price p**

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$p^* = 1, \Pi^* = 2.$
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$p^* = 1, \Pi^* = 2.$

For a product with no (or weak) network effect

- Can a seller artificially inject externality?
- Is it beneficial to do so?
Group-buying:

- Reduced prices if a minimum number of buyers would make the purchase.
- **Groupon, LivingSocial, Amazon Local, Facebook Marketplace, Eversave, ...**
- **Groupon**: fastest-growing company in the history of web (**Forbes (2010)**).

Benefits of group-buying:

- **economies of networking** (Jing and Xie (2011); Edelman et al. (2016); Zhang et al. (2016), ...).
- **economies of scale** (Monahan (1984); Mourdoukoutas (2012)).
- dealing with **demand uncertainty** (Anand and Aron (2003); Chen and Zhang (2015); Marinesi et al. (2016)).

Mostly in favor of injecting positive externality.

Positive externality via **referral reward programs**: (Aral and Walker (2011); Lobel et al. (2016); Leduc et al. (2017)).
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Positive externality via *referral reward programs*: (Aral and Walker (2011); Lobel et al. (2016); Leduc et al. (2017)).
The other side of the coin:

- *LivingSocial* (Groupon’s main competitor): once valued at $6 billion, recently acquired by Groupon for $0 (Knowledge Wharton (2017)).
- *Amazon Local* (offering similar packages) discontinued in 2015 (Soper (2015)).

Operational value of injecting externality?

- Setting aside the second order beneficial marketing effects.
- Purely resulted from direct effect on the utilities of seller and buyers.
Aggregate Reward Program

How to induce externality?

- Buyers pay price $p$ when making a purchase.
- Get back reward $r(\bar{a})$ at the end of the sales period ($\bar{a}$: sales volume).
- $r(\bar{a})$ increasing (decreasing): positive (negative) externality.

Can such a program be profitable?

A warm-up case:

- Mass 1 of consumers $i \in I = [0, 1]$.
- Valuations $v_i$.
- Average or true value $v$: average valuation of the product in the market.
- $v_i = v + \epsilon_i$, $\epsilon_i \sim N(0, \sigma^2_\epsilon)$: idiosyncratic tastes of consumers.
- Seller offers price $p$ and reward program $r(\cdot)$.
- Payoff of a purchase:

$$u_i = v_i - p + r(\bar{a}).$$
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Can such a program be profitable? (cont’d)

- Purchase decision $a_i \in \{0, 1\}$.
- Profit of seller: $\Pi(p, r(\cdot)) = (p - r(\bar{a}))\bar{a}$.
- **Profit maximizing buyers:** $\bar{a}^*$ sales volume at equilibrium,
  
  $$a_i^* = 1\{u_i > 0\} = 1\{v_i > p - r(\bar{a}^*)\}.$$

- **Effective price** for both is $p - r(\bar{a}^*)$,
  
  $$\Pi(p, r(\cdot)) = \Pi(p - r(\bar{a}^*), 0).$$

Why did it fail?

- Full agreement on foreseen reward $\Rightarrow$ reward $\equiv$ shift in price.

In reality, the average value $v$ is not known:

- Uncertainty in $v$. Differs from variation in tastes ($\epsilon_i$).
- Heterogeneous beliefs on $v$ hence on sales volume.
- Need to revise the model!
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Modeling Uncertainty in Average Value

- Recall $v_i = v + \epsilon_i$, $v$ is average or true value.
- $\epsilon_i \sim N(0, \sigma_\epsilon^2)$: the idiosyncratic tastes.
- Consumers beliefs on the true value lean towards their own valuation.
  - common prior $v \sim N(\theta, \sigma_\theta^2)$: variation in average value.
  - consumer $i$ valuation $v_i$: a private observation from $v$.
    Posterior belief $v | v_i \sim N(\mu_i, \sigma_v^2)$,
    
    $$\mu_i = \tau v_i + (1 - \tau)\theta, \quad \tau = \frac{\sigma_\theta^2}{\sigma_\epsilon^2 + \sigma_\theta^2}, \quad \sigma_v^2 = \frac{\sigma_\epsilon^2 \sigma_\theta^2}{\sigma_\epsilon^2 + \sigma_\theta^2}.$$ 

Solving the updated model:

- Actions maximize expected utilities:
  $$a_i = 1\{E_{v | v_i}[u_i] > 0\} = 1\{E_{v | v_i}[v_i + r(\bar{a}) - p] > 0\}.$$ 
- Seller looks at its expected profit over all realizations of $v$:
  $$E_v[\Pi(p, r(\cdot))] = E_v[(p - r(\bar{a}))\bar{a}].$$
Recall \( v_i = v + \epsilon_i \), \( v \) is average or true value.

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Equilibrium Analysis

Equilibria of the induced *global game* (Morris and Shin (1998, 2003); Carlsson and van Damme (1993)) among consumers:

- **Threshold strategy with cutoff** $c$:
  
  $$ a_i = 1 \{ v_i > c \}. $$

- $a_i = 1 \{ v_i + \mathbb{E}_{v|v_i}[r(\bar{a}(v))] > p \} \Rightarrow \text{necessary condition for a threshold strategy to be equilibrium}$
  
  $$ c + \mathbb{E}_{v|v_i=c}[r(\bar{a}(v))] = p, $$

  that is *indifference equation* for cutoff.

- **Standard approach in global games**: ensure *single crossing property* (monotonicity) for expected utilities. Condition becomes sufficient.
Equilibrium Analysis

Lemma

Let $r_{\text{min}}$ and $r_{\text{max}}$ be the minimum and maximum of reward $r(\cdot)$. Then, the expected payoff of adoption is increasing with valuations if

$$r_{\text{max}} - r_{\text{min}} \leq \sqrt{2\pi\sigma_{\epsilon}} \sqrt{1 + \left(\frac{\sigma_{\epsilon}}{\sigma_{\theta}}\right)^2}.$$

Assumption: $r_{\text{max}} - r_{\text{min}} \leq r_{M}$, where $r_{M} = \sqrt{2\pi\sigma_{\epsilon}} \sqrt{1 + \left(\frac{\sigma_{\epsilon}}{\sigma_{\theta}}\right)^2}$.

- Can we relax this assumption? Yes! Converts to an infinite dimensional optimization problem (a cute one).

- $a_i = 1\{v_i > c\}$ is an equilibrium strategy iff:

  $$c + \mathbb{E}_{v|c}[r(\tilde{a}(v))] = p.$$

- Size of buyers for a threshold strategy: $\tilde{a}(v) = \Phi\left(\frac{v-c}{\sigma_{\epsilon}}\right)$.

- There always exists a threshold equilibrium strategy.

Profitable joint price-reward program: yields an expected profit higher than the optimal no reward profit.
Equilibrium Analysis

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Profitable joint price-reward program: yields an expected profit higher than the optimal no reward profit.
Profitable Reward Programs

- The expected utility of the seller:
  \[ E_v[\Pi(p, r(\cdot))] = E_v[(p - r(\bar{a}(v)))\bar{a}(v)]. \]

- A useful alternative:
  \[ E_v[\Pi(p, r(\cdot))] = \int_c^\infty (p - E_v|v_i [r(\bar{a}(v))])\psi(v_i)dv_i, \]

  \[ \psi(v_i) = \frac{\phi(\frac{\theta - v_i}{\sqrt{\sigma^2 + \sigma^2_\theta}})}{\sqrt{\sigma^2 + \sigma^2_\theta}}: \text{ex-ante pdf of valuations.} \]

- A simple partition of expected profit over valuation of buyers.

  - Plug in \( p = c + r_c, \) \( r_c = E_v|v [r(\bar{a}(v))]: \text{reward expected at cutoff,} \)
    \[ E_v[\Pi(p, r(\cdot))] = E_v[\Pi(c, 0)] + \int_c^\infty (E_v|v [r(\bar{a}(v))] - E_v|v_i [r(\bar{a}(v))])\psi(v_i)dv_i, \]

  with \( E_v[\Pi(c, 0)]: \text{expected utility of price } c \text{ with no reward.} \)
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    \[ + \int_c^\infty (E_v|c[r(\bar{a}(v))] - E_v|v_i[r(\bar{a}(v))])\psi(v_i)dv_i, \]
  with \( E_v[\Pi(c, 0)] \): expected utility of price \( c \) with no reward.
Lemma

For an equilibrium strategy $a_i = 1\{v_i > c\}$, $\mathbb{E}_v[\Pi(p, r(\cdot))] > \mathbb{E}_v[\Pi(c, 0)]$ if and only if the ex-ante expected reward paid per purchase is less than the reward expected at the cutoff.

For a threshold strategy $a_i = 1\{v_i > c\}$ and non-constant decreasing $r(\cdot)$ the expected rewards are strictly decreasing with valuations:

$$v_1 > v_2 \Rightarrow \mathbb{E}_{v|v_1}[r(\bar{a}(v))] < \mathbb{E}_{v|v_2}[r(\bar{a}(v))],$$

and are increasing with valuations for increasing reward.

Theorem

Every decreasing non-constant reward function $r(\cdot)$ can be joined by an appropriate price $p$ such that the joint price-reward program is profitable. On the other hand, there exists no profitable increasing reward program.
Optimal Aggregate Reward Program

Optimal reward program:

$$
E_v[\Pi(p, r(\cdot))] = E_v[\Pi(c, 0)] + \Delta \Pi(c, r(\cdot))
$$

Two steps decision making:

- Choosing the target consumers (controlled by threshold $c$).
- Reward program extracting the maximum profit from them.

Optimal reward program for equilibrium strategy $a_i = 1\{v_i > c\}$:

$$
\Delta \Pi(c, r(\cdot)) = \int_c^\infty (E_{v|c}[r(\bar{a}(v))] - E_{v|v_i}[r(\bar{a}(v))])\psi(v_i)dv_i,
$$

- Regroup the terms as

$$
\Delta \Pi(c, r(\cdot)) = \Phi\left(\frac{\theta - c}{\sqrt{\sigma^2 + \sigma^2}}\right) \int_{-\infty}^\infty r(\bar{a}(v))(q(v|c) - q(v|v_i \geq c))dv,
$$

where $q(v|v_i)$ is the likelihood of the true value $v$ observing a valuation $v_i$. 
Popularity-Guaranteed Reward Program

- Values for which \( q(v|c) > q(v|i \geq c) \) contribute to the profit while those less likely at cutoff are harmful.

- A unique value \( v_c \) at which
  \[
  q(v_c|c) = q(v_c|i \geq c).
  \]

- True values \( v > v_c \) are less likely at cutoff while values \( v < v_c \) are more likely.

- Optimal reward program:
  \[
  r^*_c(\bar{a}) = r_M 1\{\bar{a} \leq a_c\},
  \]
  where \( a_c = \Phi(\frac{v_c-c}{\sigma_c}) \).

- Pays a fixed rebate if the product fails to reach certain level of popularity.

- We can then solve for the optimal target sector.

- The optimal reward program improves both the welfare and seller’s profit.
Figure: Ex-post profit and net prices for different realizations of the true value $v$, for a sample case with $\theta = 5$, $\sigma_\epsilon = \sigma_\theta = 1$.

- For the optimal joint program, net price is $p^* = 5.85$ for $v > v_c = 4.61$, and $p^* - r_M = 2.31$ for $v < 4.61$. Average profit: 4.41.
- For the optimal no reward case, $p_0^* = 3.86$ with an average profit of 3.05.
Figure: Net price expected at valuation $v_i$ for the optimal joint price-reward program for $\theta = 5, \sigma_\epsilon = \sigma_\theta = 1$. Threshold is $c = 2.97$. 

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Concluding Remarks and Future Directions

- Positive Network effect results in a marginal gain in profit.
- Artificially inducing such an effect using an aggregate reward program may be harmful.
- Sellers with a established customer base (e.g., Amazon, Facebook), and a fixed marginal cost (many digital goods and services).
- A new generation of rebate programs.
- Pays back a fixed rebate if the product fails to reach certain level of popularity.
- Possible to keep track of the users’ statistics specially for digital goods and services.
- Induces price discrimination over valuations.
- Next: relax assumption on precisions by formulating as an infinite dimensional optimization problem.
- Dual program for ride-sharing platforms to cope with high uncertainty in the availability of service providers.
Thank You!