Sales-Based Aggregate Rebate Design

Amir Ajorlou Ali Jadbabaie

Laboratory for Information and Decision Systems
Institute for Data, Systems, and Society
Massachusetts Institute of Technology

Motivation

Network effect (or positive externality): a consumer's utility is increasing in the usage level of her peers.

Bob's WPT*: $1

*WPT: willingness to pay.

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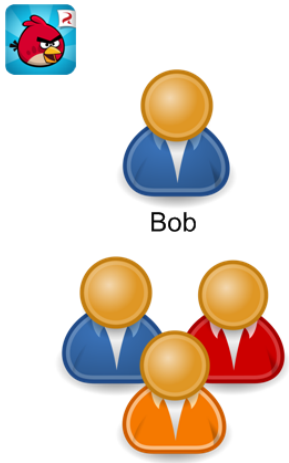
Bob's WPT: $2



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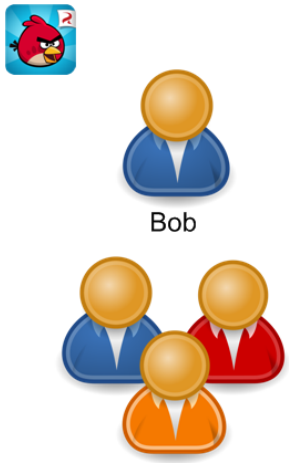

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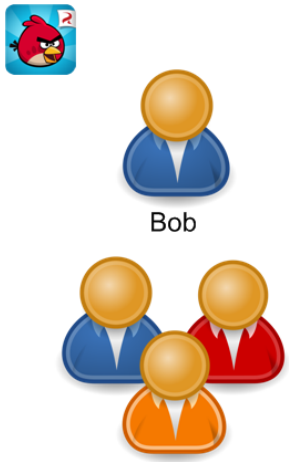

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smartphone apps

WhatsApp

online games

social networks

online dating

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Network effect results in a marginal gain in profit.

No Network effect

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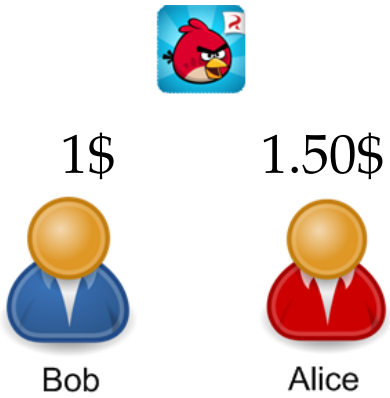

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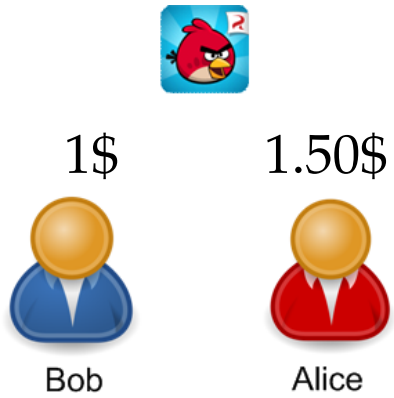

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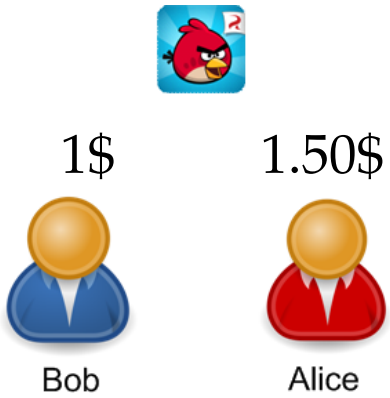

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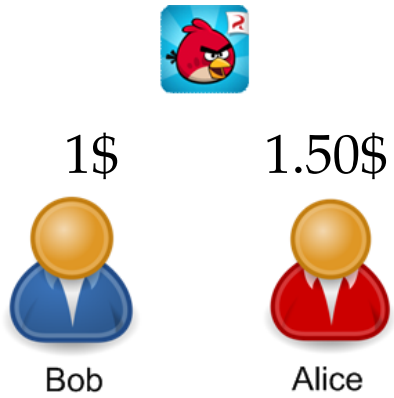

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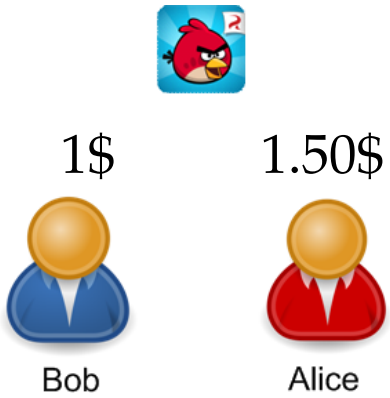

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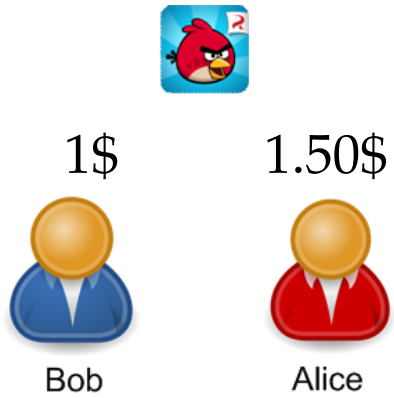

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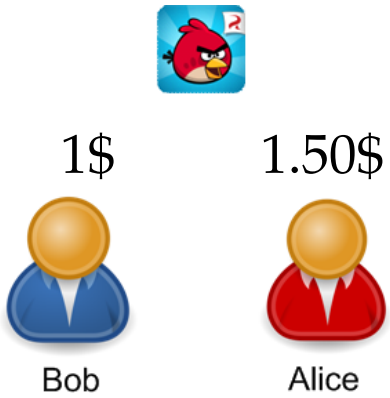

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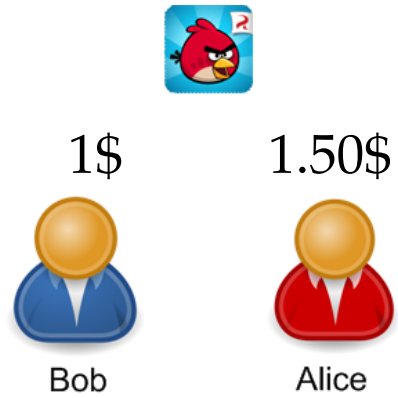

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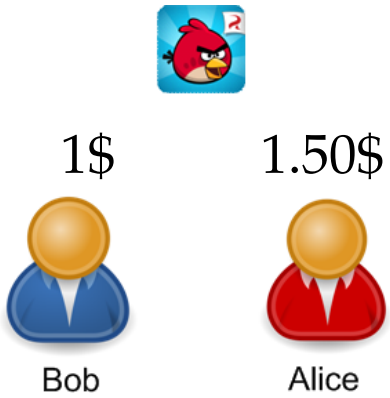

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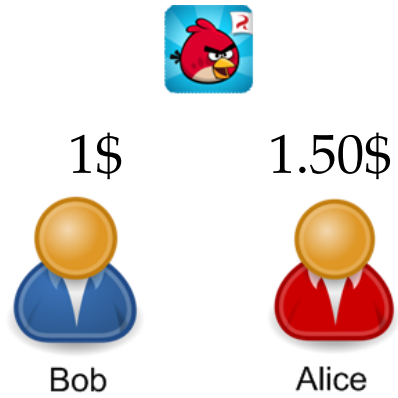

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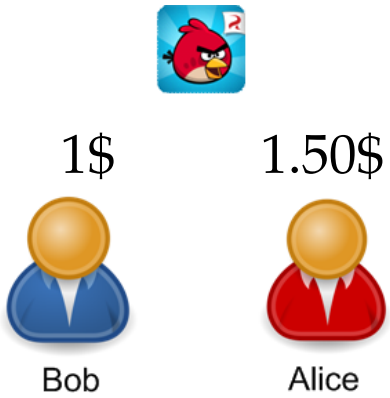

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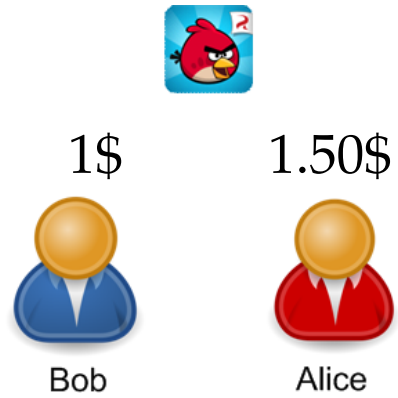

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For a product with no (or weak) network effect

- Can a seller artificially inject externality?
- Is it beneficial to do so?

Ideas Out There

Group-buying:

- Reduced prices if a minimum number of buyers would make the purchase.
- *Groupon, LivingSocial, Amazon Local, Facebook Marketplace, Eversave, . . .*
- Groupon: fastest-growing company in the history of web (Forbes (2010)).

- Benefits of group-buying:
  - *economies of networking* (Jing and Xie (2011); Edelman et al.(2016); Zhang et al. (2016), . . .).
  - *economies of scale* (Monahan (1984); Mourdoukoutas (2012)).
  - dealing with *demand uncertainty* (Anand and Aron (2003); Chen and Zhang (2015); Marinesi et al. (2016)).
- Mostly in favor of injecting positive externality.

Positive externality via *referral reward programs*: (Aral and Walker (2011); Lobel et al. (2016); Leduc et al. (2017)).

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Ideas Out There

The other side of the coin:

- *LivingSocial* (Groupon's main competitor): once valued at $6 billion, recently acquired by Groupon for $0 (Knowledge Wharton (2017)).
- Groupon's stock dropped from $28 in 2011 to ~ $4 in 2017.
- *Amazon Local* (offering similar packages) discontinued in 2015 (Soper (2015)).

Operational value of injecting externality?

- Setting aside the second order beneficial marketing effects.
- Purely resulted from direct effect on the utilities of seller and buyers.

Aggregate Reward Program

How to induce externality?

- Buyers pay price p when making a purchase.
- Get back reward $r(\bar{a})$ at the end of the sales period ($\bar{a}$: sales volume).
- $r(\bar{a})$ increasing (decreasing): positive (negative) externality.

Can such a program be profitable?

A warm-up case:

- Mass 1 of consumers $i \in I = [0, 1]$.
- Valuations v_i.
- Average or true value v: average valuation of the product in the market.
- $v_i = v + \epsilon_i$, $\epsilon_i \sim N(0, \sigma_\epsilon^2)$: idiosyncratic tastes of consumers.
- Seller offers price p and reward program $r(\cdot)$.
- Payoff of a purchase:

$$u_i = v_i - p + r(\bar{a}).$$

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Aggregate Reward Program: Warm-Up Case

Can such a program be profitable? (cont'd)

- Purchase decision $a_i \in \{0, 1\}$.
- Profit of seller: $\Pi(p, r(\cdot)) = (p - r(\bar{a}))\bar{a}$.
- Profit maximizing buyers: $\bar{a}^*$ sales volume at equilibrium,

$$a_i^* = \mathbf{1}\{u_i > 0\} = \mathbf{1}\{v_i > p - r(\bar{a}^*)\}.$$

- *Effective price* for both is $p - r(\bar{a}^*)$,

$$\Pi(p, r(\cdot)) = \Pi(p - r(a^*), 0).$$

Why did it fail?

- Full agreement on foreseen reward$\Rightarrow$ reward $\equiv$ shift in price.

In reality, the average value v is not known:

- Uncertainty in v. Differs from variation in tastes (ϵ_i).
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Modeling Uncertainty in Average Value

- Recall $v_i = v + \epsilon_i$, v is average or true value.
- $\epsilon_i \sim N(0, \sigma_\epsilon^2)$: the idiosyncratic tastes.
- Consumers beliefs on the true value lean towards their own valuation.
  - common prior $v \sim N(\theta, \sigma_\theta^2)$: variation in average value.
  - consumer i valuation v_i: a private observation from v. Posterior belief $v|v_i \sim N(\mu_i, \sigma_v^2)$,

$$\mu_i = \tau v_i + (1-\tau)\theta, \quad \tau = \frac{\sigma_\theta^2}{\sigma_\epsilon^2 + \sigma_\theta^2}, \quad \sigma_v^2 = \frac{\sigma_\epsilon^2 \sigma_\theta^2}{\sigma_\epsilon^2 + \sigma_\theta^2}.$$

Solving the updated model:

- Actions maximize expected utilities:

$$a_i = \mathbf{1}\{\mathbb{E}_{v|v_i}[u_i] > 0\} = \mathbf{1}\{\mathbb{E}_{v|v_i}[v_i + r(\bar{a}) - p] > 0\}.$$

- Seller looks at its expected profit over all realizations of v:

$$\mathbb{E}_v[\Pi(p, r(\cdot))] = \mathbb{E}_v[(p - r(\bar{a}))\bar{a}].$$

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Equilibrium Analysis

Equilibria of the induced *global game* (Morris and Shin (1998, 2003); Carlsson and van Damme (1993)) among consumers:

- Threshold strategy with cutoff c:

$$a_i = \mathbf{1}\{v_i > c\}.$$

- $a_i = \mathbf{1}\{v_i + \mathbb{E}_{v|v_i}[r(\bar{a}(v))] > p\} \Rightarrow$ necessary condition for a threshold strategy to be equilibrium

$$c + \mathbb{E}_{v|v_i=c}[r(\bar{a}(v))] = p,$$

  that is indifference equation for cutoff.

- Standard approach in *global games*: ensure single crossing property (monotonicity) for expected utilities. Condition becomes sufficient.

Equilibrium Analysis

Lemma

Let $r_{\min}$ and $r_{\max}$ be the minimum and maximum of reward $r(\cdot)$. Then, the expected payoff of adoption is increasing with valuations if

$$r_{\max} - r_{\min} \leq \sqrt{2\pi}\sigma_\epsilon \sqrt{1 + (\frac{\sigma_\epsilon}{\sigma_\theta})^2}.$$

Assumption: $r_{\max} - r_{\min} \leq r_M$, where $r_M = \sqrt{2\pi}\sigma_\epsilon \sqrt{1 + (\frac{\sigma_\epsilon}{\sigma_\theta})^2}$.

- Can we relax this assumption? Yes! Converts to an infinite dimensional optimization problem (a cute one).
- $a_i = \mathbf{1}\{v_i > c\}$ is an equilibrium strategy iff:

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- Size of buyers for a threshold strategy: $\bar{a}(v) = \Phi(\frac{v-c}{\sigma_\epsilon})$.
- There always exists a threshold equilibrium strategy.

Profitable joint price-reward program: yields an expected profit higher than the optimal no reward profit.

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Profitable Reward Programs

- The expected utility of the seller:

$$\mathbb{E}_v[\Pi(p, r(\cdot))] = \mathbb{E}_v[(p - r(\bar{a}(v)))\bar{a}(v)].$$

- A useful alternative:

$$\mathbb{E}_v[\Pi(p, r(\cdot))] = \int_c^{\infty} (p - \mathbb{E}_{v|v_i}[r(\bar{a}(v))])\psi(v_i)dv_i,$$

$\psi(v_i) = \frac{\phi(\frac{\theta - v_i}{\sqrt{\sigma_\epsilon^2 + \sigma_\theta^2}})}{\sqrt{\sigma_\epsilon^2 + \sigma_\theta^2}}$: ex-ante pdf of valuations.

- A simple partition of expected profit over valuation of buyers.
- Plug in $p = c + r_c$, $r_c = \mathbb{E}_{v|c}[r(\bar{a}(v))]$: reward expected at cutoff,

$$\mathbb{E}_v[\Pi(p, r(\cdot))] = \mathbb{E}_v[\Pi(c, 0)]$$

$$+ \int_c^{\infty} (\mathbb{E}_{v|c}[r(\bar{a}(v))] - \mathbb{E}_{v|v_i}[r(\bar{a}(v))])\psi(v_i)dv_i,$$

with $\mathbb{E}_v[\Pi(c, 0)]$: expected utility of price c with no reward.

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Profitable Reward Programs

Lemma

For an equilibrium strategy $a_i = \mathbf{1}\{v_i > c\}$, $\mathbb{E}_v[\Pi(p, r(\cdot))] > \mathbb{E}_v[\Pi(c, 0)]$ if and only if the ex-ante expected reward paid per purchase is less than the reward expected at the cutoff.

For a threshold strategy $a_i = \mathbf{1}\{v_i > c\}$ and non-constant decreasing $r(\cdot)$ the expected rewards are strictly decreasing with valuations:

$$v_1 > v_2 \Rightarrow \mathbb{E}_{v|v_1}[r(\bar{a}(v))] < \mathbb{E}_{v|v_2}[r(\bar{a}(v))],$$

and are increasing with valuations for increasing reward.

Theorem

Every decreasing non-constant reward function $r(\cdot)$ can be joined by an appropriate price p such that the joint price-reward program is profitable. On the other hand, there exists no profitable increasing reward program.

Optimal Aggregate Reward Program

Optimal reward program:

$$\mathbb{E}_v[\Pi(p, r(\cdot))] = \mathbb{E}_v[\Pi(c, 0)] + \Delta\Pi(c, r(\cdot))$$

Two steps decision making:

- Choosing the target consumers (controlled by threshold c).
- Reward program extracting the maximum profit from them.

Optimal reward program for equilibrium strategy $a_i = \mathbf{1}\{v_i > c\}$:

$$\Delta\Pi(c, r(\cdot)) = \int_c^\infty (\mathbb{E}_{v|c}[r(\bar{a}(v))] - \mathbb{E}_{v|v_i}[r(\bar{a}(v))])\psi(v_i)dv_i,$$

- Regroup the terms as

$$\Delta\Pi(c, r(\cdot)) = \Phi(\frac{\theta - c}{\sqrt{\sigma_\epsilon^2 + \sigma_\theta^2}}) \int_{-\infty}^\infty r(\bar{a}(v))(q(v|c) - q(v|v_i \geq c))dv,$$

where $q(v|v_i)$ is the likelihood of the true value v observing a valuation v_i.

Popularity-Guaranteed Reward Program

- Values for which $q(v|c) > q(v|v_i \geq c)$ contribute to the profit while those less likely at cutoff are harmful.
- A unique value v_c at which

$$q(v_c|c) = q(v_c|v_i \geq c).$$

- True values $v > v_c$ are less likely at cutoff while values $v < v_c$ are more likely.
- Optimal reward program:

$$r_c^*(\bar{a}) = r_M \mathbf{1}\{\bar{a} \leq a_c\},$$

where $a_c = \Phi(\frac{v_c - c}{\sigma_\epsilon})$.

- Pays a fixed rebate if the product fails to reach certain level of popularity.
- We can then solve for the optimal target sector.
- The optimal reward program improves both the welfare and seller's profit.

Popularity-Guaranteed Reward Program

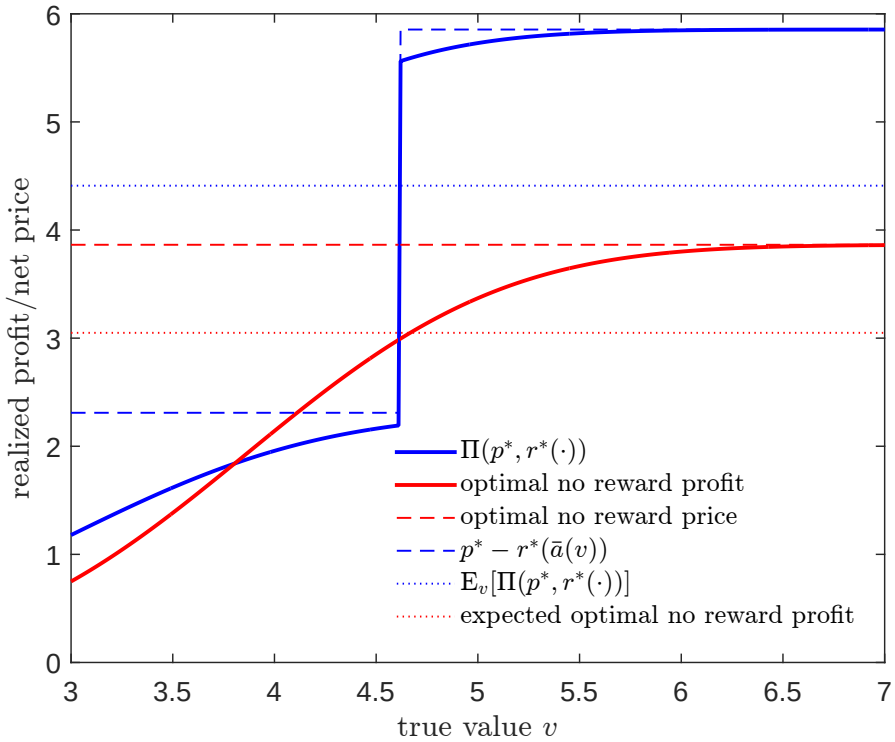

Figure: Ex-post profit and net prices for different realizations of the true value v, for a sample case with $\theta = 5$, $\sigma_\epsilon = \sigma_\theta = 1$.

- For the optimal joint program, net price is $p^* = 5.85$ for $v > v_c = 4.61$, and $p^* - r_M = 2.31$ for $v < 4.61$. Average profit: 4.41.
- For the optimal no reward case, $p_0^* = 3.86$ with an average profit of 3.05.

Popularity-Guaranteed Reward Program

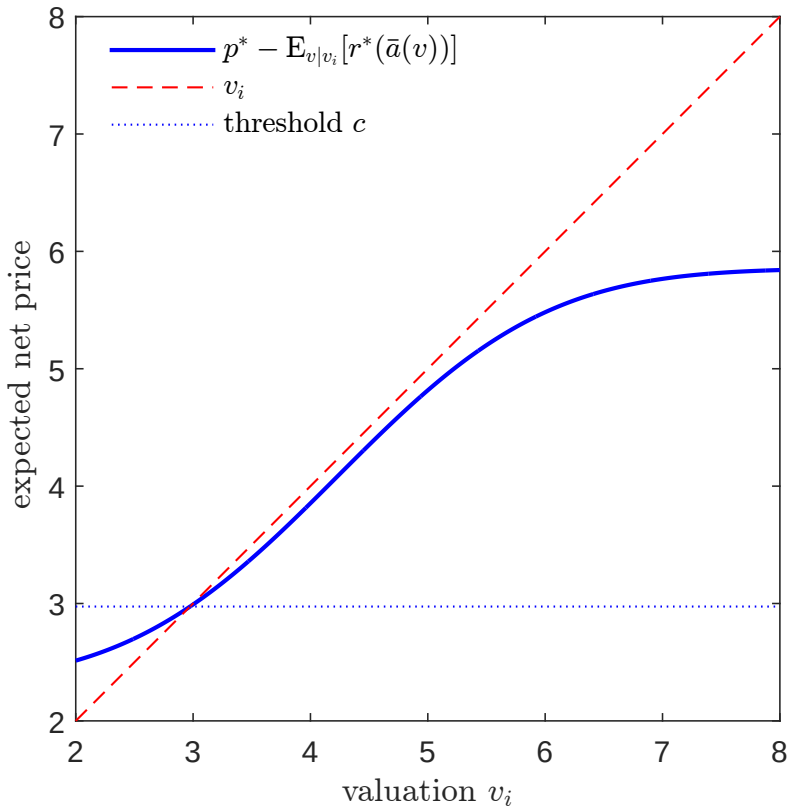

Figure: Net price expected at valuation v_i for the optimal joint price-reward program for $\theta = 5$, $\sigma_\epsilon = \sigma_\theta = 1$. Threshold is $c = 2.97$.

Concluding Remarks and Future Directions

- Positive Network effect results in a marginal gain in profit.
- Artificially inducing such an effect using an aggregate reward program may be harmful.
- Sellers with a established customer base (e.g., *Amazon, Facebook*), and a fixed marginal cost (many digital goods and services).
- A new generation of rebate programs.
- Pays back a fixed rebate if the product fails to reach certain level of popularity.
- Possible to keep track of the users' statistics specially for digital goods and services.
- Induces price discrimination over valuations.
- Next: relax assumption on precisions by formulating as an infinite dimensional optimization problem.
- Dual program for ride-sharing platforms to cope with high uncertainty in the availability of service providers.

Thank You!