Higher-dimensional Jordan-Wigner Transformation

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I first review the standard Jordan-Wigner transformation in 1D, and apply it to the transverse field Ising model as an example. I then discuss how locality is not preserved when one naively generalize the Jordan-Wigner transformation from 1D to higher dimensions, and a way to circumvent this problem by using loop gas states and the parton construction.

I. INTRODUCTION

Duality between different systems oftentimes points to unifying themes that deepen our understanding of statistical models and phases of matter. For example, Kramers-Wannier transformation between the high temperature and low temperature expansion of the 2D Ising model allows one to pinpoint the critical temperature of the Ising transition, and it also points to interesting lattice gauge theory when one apply Kramers-Wannier transformation to the low-temperature expansion of the 3D Ising model [1]. Another example is the Jordan-Wigner (JW) transformation, which is an exact mapping between fermionic and spin (i.e. hard-core boson) systems, which may come as a surprise given the drastically different (anti)commutation relation between fermionic and bosonic operators. Nonetheless, it provides a powerful duality between systems that, at the surface level, might be drastically different; problems that are too difficult in the fermionic description might become more amenable after applying Jordan-Wigner transformation and obtaining the bosonic description of the system, and vice versa. It also provide an ready-made way to generate bosonic/fermionic version of interesting systems.

Originally, it is thought that the existence of such mapping is a peculiarity with 1D systems; however, recently developments in generalizing JW transformation to higher dimension suggest otherwise [2–7]. In this paper, I will go through in detail the formalism as presented in Ref. [6].

II. 1D JW TRANSFORMATION

A. Generalities

The Jordan-Wigner transformation make use of the fact that the dimension of the Hilbert space of a single fermion is 2, which the same as that of a spin-1/2 particle. One can either choose the basis to be in terms of the fermionic occupation basis $|0\rangle, |1\rangle$ or in terms of the spin-1/2 basis: $|\uparrow\rangle, |\downarrow\rangle$. What remains to be done is to establish a correspondence between operators acting on the two Hilbert space.

The most tricky part in establishing the operator correspondence is that bosonic operators at distinct sites commute, whereas fermionic operators at distinct sites anticommute. Therefore, in representing a fermionic operator in terms of bosonic operators, one has to incur the extra cost of making the bosonic operator nonlocal in order to faithfully reproduce the anticommutation relation.

Consider a 1D chain of spinless complex fermion. In the second quantized notation, the occupied state $|1\rangle = f^\dagger|0\rangle$, where $f, f^\dagger$ are fermionic operators that satisfy the following anticommutation relations

$$\left\{f_i, f_j^\dagger\right\} = \delta_{ij}, \qquad \{f_i, f_j\} = 0, \qquad \left\{f_i^\dagger, f_j^\dagger\right\} = 0.$$

One can also express the complex fermion operators in terms of the Majorana operators,

$$f_n^\dagger = \frac{1}{2}(\gamma_n + i\gamma_n'), \qquad f = \frac{1}{2}(\gamma_n - i\gamma_n')$$

which can be viewed as the real and imaginary part of the complex fermionic operator.

In terms of the Majorana operators γ_n, γ_n' for each site n, the 1D JW is given by

$$\gamma_n \leftrightarrow \left(\prod_{i=0}^{n-1} \sigma_i^z\right)\sigma_n^x \tag{1}$$

$$\gamma_n' \leftrightarrow \left(\prod_{i=0}^{n-1} \sigma_i^z\right)\sigma_n^y \tag{2}$$

and in terms of the complex fermion operators,

$$f_n^\dagger = \frac{1}{2}(\gamma_n + i\gamma_n') = \left(\bigotimes_{i=1}^{n-1} \sigma_i^z\right) \otimes \sigma_n^+ \otimes 1 \otimes \cdots \otimes 1 \tag{3}$$

$$f_n = \frac{1}{2}(\gamma_n - i\gamma_n') = \left(\bigotimes_{i=1}^{n-1} \sigma_i^z\right) \otimes \sigma_n^- \otimes 1 \otimes \cdots \otimes 1 \tag{4}$$

where σ^i are the Pauli matrices. Since σ^z anticommute with $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$, it matches up with the anticommutation relation between $f, f^\dagger$ on the fermionic side. On the bosonic side (i.e. right hand side of above equations), the strings of σ^z to the left of the site n is called the JW string. We see that in order to preserve the anticommutation relation between the fermionic operators on the fermionic side of the JW transformation, we need to introduce non-local JW string on the bosonic side. However, since the Hamiltonian is local, the non-local JW string has to cancel out for terms in the Hamiltonian such that locality is still preserved. This is an important property and we would want the higher dimensional generalization of JW transformation to also have this property. We will discuss this further in later sections. Let's first consider an example that showcase the power of Jordan-Wigner transformation.

B. Example: 1D Transverse field Ising model

The Hamiltonian of the 1D Transverse field Ising model (with open boundary condition) is given by

$$H = J\sum_{i=1}^{N-1} X_i X_{i+1} + h\sum_{i=1}^{N} Z_i$$

where X_i, Z_i are Pauli operators at site i. Without the trasverse field term, we can straightforwardly conclude the ground state of the Hamiltonian to be $\otimes_i \left|+\right\rangle_i$, the product state of eigenvector of X_i. However, the presence of the Z transverse field necessitate some quantum fluctuations that render the product state of $\left|+\right\rangle$ no longer the true ground state.

One way to exactly solve the model (and thereby find the true ground state) is to use JW transformation, mapping the bosonic spin operators to fermionic Majorana operators, such that the terms in the Hamiltonian are mapped to fermion bilinears

$$-X_i X_{i+1} \leftrightarrow i\gamma_i' \gamma_{i+1} \tag{5}$$

$$-Z_i \leftrightarrow i\gamma_i \gamma_i' \tag{6}$$

such that the Hamiltonian becomes

$$H = -J\sum_i i\gamma_i' \gamma_{i+1} - h\sum_i i\gamma_i \gamma_i'.$$

One can rewrite this Hamiltonian in terms of complex fermion operators instead of the Majorana operators to obtain

$$H = \sum_i J c_{i+1}^\dagger c_i + h\sum_i c_{i-1}^\dagger c_i^\dagger + h.c.$$

where h can be interpreted as the order parameter of a superconductor at the mean-field level. At this point one realizes this is exactly the Hamiltonian for the Kitaev chain, and it can be exactly diagonalized in the momentum basis, since it is a free fermionic problem. This concludes our discussion on the 1D case, let's proceed to the 2D case.

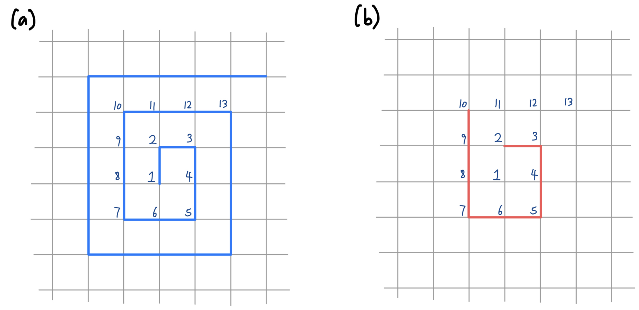

FIG. 1. Naive linearization. (a) Spiral ordering of lattice sites on a 2D square lattice. (b) The hopping term $c_2^\dagger c_{10}$ appears non-local in the bosonic description, as it involves a JW string that need to tranverse all sites between 2 and 10.

III. NAIVE GENERALIZATION: A 2D CASE STUDY

A. Problem

Building on the success of the 1D JW transformation, the minimal effort approach to generalize the JW transformation to higher dimensions is to linearize the higher dimensional system so as to reduce it to an effective one dimensional system.

Let's try this on fermions on the square lattice for concreteness. One can linearize the 2D square lattice by imposing some ordering to the lattice sites in the square lattice. For example, after choosing the origin, we can create a spiral-like ordering of the lattice site as shown in Fig. 1(a).

After linearizing our system into an effective 1D system, we then apply the 1D JW transformation and obtain the bosonic description of the system. However, there is one undesirable properties of this generalization: locality in the 2D lattice is obscured; terms that are local in the 2D square lattice can appear to be highly nonlocal in the 1D description. For example, a nearest neighbor term, such as $c_2^\dagger c_{10}$, in the 2D square lattice across neighboring lines will appear a long string around the square lattice as shown in Fig. 1(b).

This is undesirable, as local terms on the fermionic side becomes highly nonlocal on the bosonic side after the JW transformation. We want to find a way to make sure that locality is manifest on both bosonic and fermionic side of the JW transformation.

B. Solution: conceptual level

At a conceptual level, the resolution is to allow for only loop gas state, i.e. state that are invariant under the action of a closed loop (formed by linking the two ends of a JW string); we refer to this as the plaquette constraint. Then, any JW string corresponding to a hopping term

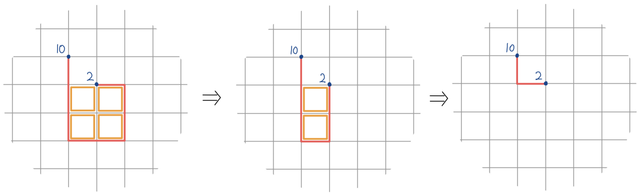

FIG. 2. In a loop gas state, a loose JW string can be deformed and straightened into a shortest-distance string between two sites.

can be straightened to be only within the support of a local term, as shown in Fig. 2.

There is one price we pay for imposing the plaquette constraint: we are restricting the dimension of our Hilbert space to only that spanned by loop gas state. In order to construct an exact duality for the higher-dimensional JW transformation, we expect the Hilbert space dimension to be the same on either of the transformation. Therefore, we need to preemptively enlarge our Hilbert space on the bosonic side, such that after imposing the plaquette constraint, the restricted Hilbert space has the same dimension as that of the fermionic Hilbert space. We will see in later sections that the parton construction enable us to fractionalize particles and enlarge the Hilbert space.

IV. GENERAL THEORY OF 2D JW

As a road map for how higher-dimensional JW transformation is constructed, I will discuss 1) how local fermionic operators are mapped to bosonic operators by using an Lie group exceptional isomorphism, 2) use the parton construction to enlarge the Hilbert space, and 3) impose the plaquette constraints to faithfully represent the fermionic operators while maintaining the locality for fermionic bilinear operators. As a convention, from here onward I will refer to JW transformation in the direction from the fermionic to bosonic system. And for concreteness, we will focus on 2D square lattice with periodic boundary condition.

A. Exceptional isomorphism

Let's first see how to construct the operator correspondence in general. Starting with a single site of spinful electron with 2 complex fermion modes $f^j, f^{\dagger j}$; just as in the 1D case, we can express the complex fermion operators in terms of the Majorana operators:

$$\gamma^{2j-1} = f^j + f^{j\dagger} \qquad \gamma^{2j} = i(f^j - f^{j\dagger})$$

with anticommutation relation $\left\{\gamma^\alpha, \gamma^\beta\right\} = 2\delta^{\alpha\beta}$.

Terms in the Hamiltonian need to be Hermitian and local, so it is generated by fermion bilinear of the form

$$\theta_0^{\alpha\beta} = \frac{i}{2}\gamma^\alpha\gamma^\beta \tag{7}$$

for $\alpha \neq \beta$ (since otherwise by anti-commutation it is trivially 1); the 0 subscript emphasizes that we are in the representation defined using the physical DoF. The fermion bilinears satisfies the commutation relation of the Lie algebra $\mathfrak{so}(4)$:

$$\left[\theta_0^{\alpha\beta}, \theta_0^{\gamma\delta}\right] = i\Big(\delta^{\alpha\delta}\theta_0^{\beta\gamma} - \delta^{\alpha\gamma}\theta_0^{\beta\delta} + \delta^{\beta\gamma}\theta_0^{\alpha\delta} - \delta^{\beta\delta}\theta_0^{\alpha\gamma}\Big).$$

After exponentiation,

$$e^{-i\xi\theta_0^{\alpha\beta}} = \cos\frac{\xi}{2} + \sin\frac{\xi}{2}\gamma^\alpha\gamma^\beta.$$

One may initially expect it to be an element of the Lie group $SO(4)$; however, for $\xi = 2\pi$, the Lie group element is -1 instead of 1, so the Lie group is actually Spin(4) instead of SO(4). Also note that for $\xi = \pi$, the Lie group element is $\gamma^\alpha\gamma^\beta$ itself. So $\gamma^\alpha\gamma^\beta$ can be viewed as either a Lie group or Lie algebra element.

The Lie group Spin(4) has an exceptional isomorphism

$$\mathrm{Spin}(4) = \mathrm{SU}_s(2) \times \mathrm{SU}_c(2) \tag{8}$$

such that given the spinful electron, we can decompose it into two decoupled sectors: s stands for the spin sector, and c stands for the charge sector. $\mathrm{SU}_c(2)$ is the even-parity sector spanned by the states $|0\rangle$ and $f^{\uparrow\dagger}f^{\downarrow\dagger}|0\rangle$ (both are spin singlet); whereas $\mathrm{SU}_s(2)$ is the odd-parity sector spanned by the states $f^{\uparrow\dagger}|0\rangle$ and $f^{\downarrow\dagger}|0\rangle$ (both transforms as spin-1/2). The generators of the two sectors are

$$\tau_0^i = \frac{1}{2}\begin{pmatrix} f^{\uparrow\dagger} & f^{\downarrow\dagger}\end{pmatrix}\sigma^i\begin{pmatrix} f^\uparrow \\ f^\downarrow\end{pmatrix} \tag{9}$$

$$\chi_0^i = \frac{1}{2}\begin{pmatrix} f^{\uparrow\dagger} & f^{\downarrow}\end{pmatrix}\sigma^i\begin{pmatrix} f^\uparrow \\ f^{\downarrow\dagger}\end{pmatrix} \tag{10}$$

where σ^i are again the Pauli matrices, τ_0^i are the generators for the $\mathrm{SU}_s(2)$ sector, and χ_0^i are the generators for the $\mathrm{SU}_c(2)$ sector. The generators are chosen to satisfy $[\sigma^i, \chi^j] = 0$, i.e. the two sectors are decoupled. This is reasonable since unitary transformations generated by fermion bilinears doesn't change fermion parity, so the two sectors do not mix. The motivation for the naming of the two sectors is as follows: the odd-parity sector transforms as usual spin-1/2, whereas for the even-parity, since rotation changes the particle number, it is a "charged" rotation.

The exceptional isomorphism is the key bridge that gives rise to the Jordan-Wigner transformation: we can map between fermionic operator that generates Spin(4) to bosonic operators that generates SU$_s(2)$ and SU$_c(2)$.

B. Parton construction

Next, we use parton construction to enlarge our Hilbert space in anticipation of the plaquette constraints we will impose later. Parton construction is a way of fractionalizing particles into constituents; In the current case, we will fractionalize a spinful electron into a spinon and a chargeon/holon. So we express the generators of the two SU(2) sectors as

$$\tau^i = \frac{1}{2}\begin{pmatrix} u^\dagger & d^\dagger \end{pmatrix} \sigma^i \begin{pmatrix} u \\ d \end{pmatrix} \tag{11}$$

$$\chi^i = \frac{1}{2}\begin{pmatrix} c^\dagger & h^\dagger \end{pmatrix} \sigma^i \begin{pmatrix} c \\ h \end{pmatrix} \tag{12}$$

where u, d are the spinons that carry the original spin-1/2 of the electron but not its charge; c, h are also fermionic, but they are spinless and carry electric charge ± 1. Since we have introduced 4 fermionic modes u, d, c, h, so the starting Hilbert space is $2^4 = 16$. Effectively, we have embedded our original 4-dimensional Hilbert space (two from spin and two from fermion occupancy) into the 16-dimensional Hilbert space of these partons.

The parton fermion parity at a site $\mathbf{r}$ is defined as

$$\Gamma_{\mathbf{r}} \equiv e^{i\pi(u^\dagger u + d^\dagger d + c^\dagger c + h^\dagger h)} = e^{-i2\pi\theta^{\alpha\beta}} \tag{13}$$

for any $\alpha \neq \beta$, and Γ satisfies

$$\Gamma^2 = 1,$$

so it eigenvalues are ± 1. We choose to restrict our Hilbert space to just the eigensubspace of $\Gamma = -1$, in order to maintain the fermionic statistics (which follows from $e^{-i2\pi\theta^{\alpha\beta}} = -1$ as that defines the Spin(4) instead of SO(4) group). This reduces the Hilbert space dimension from 16 to 8. The Hilbert space dimension will be further halved when nontrivial constraints from the lattice after imposing the plaquette constraints with multiple sites.

1. *Local terms*

So far, we have restricted our attention to just a single site. Let's summarise the JW transformation for such on-site terms. There are 6 possible fermion bilinear $i\gamma^\alpha\gamma^\beta$ for $1 \leq \alpha, \beta \leq 4$ (only nonzero for $\alpha \neq \beta$), and one quartic term $\gamma^1\gamma^2\gamma^3\gamma^4$ that could appear in the Hamiltonian; we can obtain them as elements in the Spin(4) Lie group by doing a π rotation:

$$i\gamma^\alpha\gamma^\beta \mapsto \Theta^{\alpha\beta} \equiv \sqrt{\Gamma} e^{-i\pi\theta^{\alpha\beta}} \tag{14}$$

this is the bosonized operator of the fermionic bilinear. As for the quartic term, the identification is $\gamma^1\gamma^2\gamma^3\gamma^4 = e^{-i\pi\theta^{12}} e^{-i\pi\theta^{34}}$.

After above bosonization, we can check that the algebraic relations between $\Theta^{\alpha\beta}$ is the same as that of fermionic bilinear $i\gamma^\alpha\gamma^\beta$, provided that we are in the $\Gamma = -1$ subspace:

$$\Theta^{\alpha\beta}\Theta^{\beta\gamma} = \Gamma\Theta^{\beta\gamma}\Theta^{\alpha\beta} \qquad \text{for distinct } \alpha, \beta, \gamma, \delta$$
$$[\Theta^{\alpha\beta}, \Theta^{\gamma\delta}] = 0 \qquad \text{for distinct } \alpha, \beta, \gamma, \delta$$

Let's illustrate with a few common on-site terms. The electron number operator $n = \sum_\sigma f^{\sigma\dagger} f^\sigma$ is mapped to

$$n = 1 - \frac{1}{2}(i\gamma^1\gamma^2 + i\gamma^3\gamma^4) \mapsto 1 - \frac{1}{2}(\Theta^{12} + \Theta^{34}).$$

The on-site interaction term is mapped to

$$(n-1)^2 = \frac{1}{2}\left[(i\gamma^1\gamma^2)(i\gamma^3\gamma^4) + 1\right] \mapsto \frac{1}{2}(\Theta^{12}\Theta^{34} + 1)$$

2. *Hopping terms*

We first assign an orientation to each edge of the square lattice, such that it is now a directed graph. For fermionic bilinear that are across two nearest neighbor sites $\mathbf{r}$ and $\mathbf{r}'$, we map it to a bilocal operator

$$i\hat{\gamma}^\alpha_{\mathbf{r}}\hat{\gamma}^\beta_{\mathbf{r}'} \leftrightarrow \pm\hat{\Lambda}^{\alpha\alpha}_{\mathbf{r}}\hat{\Lambda}^{\beta\beta}_{\mathbf{r}'} \tag{15}$$

The sign of the bilocal operator is determined as follows: if the direction of the arrow from $\mathbf{r}$ to $\mathbf{r}'$ matches with the orientation of the edge on the directed graph, then we assign the $+$ sign; we assign the $-$ sign if the direction is opposite.

There are a few properties we require $\Lambda^{\alpha\alpha}$ to satisfy, as detailed in Ref. [6]. One property worth highlighting is that we require $\Lambda^{\alpha\alpha}_r$ to commute with the parton parity Γ_r, since we need to be able to restrict to the Γ_- eigensubspace.

Next, we consider how hopping terms beyond nearest neighbors are mapped under JW transformation. For example, for a fermionic hopping term $i\gamma^3_r\gamma^4_{r-\hat{x}+\hat{y}}$, we can insert identities $\gamma^2_{\mathbf{r}} = 1$ to obtain

$$i\gamma^3_{\mathbf{r}}\gamma^4_{\mathbf{r}-\hat{\mathbf{x}}+\hat{\mathbf{y}}} = (-i)^3\left(i\hat{\gamma}^3_{\mathbf{r}}\hat{\gamma}^2_{\mathbf{r}}\right)\left(i\hat{\gamma}^2_{\mathbf{r}}\hat{\gamma}^1_{\mathbf{r}+\hat{\mathbf{y}}}\right)\left(i\hat{\gamma}^1_{\mathbf{r}+\hat{\mathbf{y}}}\hat{\gamma}^3_{\mathbf{r}+\hat{\mathbf{y}}}\right)\left(i\hat{\gamma}^3_{\mathbf{r}+\hat{\mathbf{y}}}\hat{\gamma}^4_{\mathbf{r}-\hat{\mathbf{x}}+\hat{\mathbf{y}}}\right)$$

where we note that each term within a pair of parantheses is entierh on-site or nearest-neighbor, which has a bosonic analog, so we can map to the corresponding bosonic operators:

$$i\gamma^3_{\mathbf{r}}\gamma^4_{\mathbf{r}-\hat{\mathbf{x}}+\hat{\mathbf{y}}} \mapsto \sqrt{\hat{\Gamma}_{\mathbf{r}}}^{\dagger}\left(\sqrt{\hat{\Gamma}_{\mathbf{r}+\hat{\mathbf{y}}}}^{\dagger}\right)^2 \hat{\Theta}^{32}_{\mathbf{r}}\left(\hat{\Lambda}^{22}_{\mathbf{r}}\hat{\Lambda}^{11}_{\mathbf{r}+\hat{\mathbf{y}}}\right)\hat{\Theta}^{13}_{\mathbf{r}+\hat{\mathbf{y}}}\left(-\hat{\Lambda}^{33}_{\mathbf{r}+\hat{\mathbf{y}}}\hat{\Lambda}^{44}_{\mathbf{r}-\hat{\mathbf{x}}+\hat{\mathbf{y}}}\right) = -\left(\sqrt{\hat{\Gamma}_{\mathbf{r}}}\hat{\Theta}^{32}_{\mathbf{r}}\hat{\Lambda}^{22}_{\mathbf{r}}\right)\left(\hat{\Gamma}_{\mathbf{r}+\hat{\mathbf{y}}}\hat{\Lambda}^{11}_{\mathbf{r}+\hat{\mathbf{y}}}\hat{\Theta}^{13}_{\mathbf{r}+\hat{\mathbf{y}}}\hat{\Lambda}^{33}_{\mathbf{r}+\hat{\mathbf{y}}}\right)\hat{\Lambda}^{44}_{\mathbf{r}-\hat{\mathbf{x}}+\hat{\mathbf{y}}}$$

The mapping of the phase factor i is a bit more subtle: as $\hat{\Gamma}_{\mathbf{r}}$ is spatially dependent, so the position dependence is determined by where the identity $\gamma_{\mathbf{r}}^2 = 1$ is inserted.

The 2nd rewriting (organized by the site labels) motivate us to define

$$\hat{\Lambda}^{\alpha\beta}_{\mathbf{r}} \equiv \sqrt{\hat{\Gamma}_{\mathbf{r}}}^{\dagger}\hat{\Theta}^{\alpha\beta}_{\mathbf{r}}\hat{\Lambda}^{\beta\beta}_{\mathbf{r}}; \tag{16}$$

$$\hat{\Phi}^{\alpha\beta}_{\mathbf{r}} \equiv \sqrt{\hat{\Gamma}_{\mathbf{r}}}\hat{\Lambda}^{\alpha\alpha}_{\mathbf{r}}\hat{\Lambda}^{\alpha\beta}_{\mathbf{r}} = \hat{\Gamma}_{\mathbf{r}}\hat{\Lambda}^{\alpha\alpha}_{\mathbf{r}}\hat{\Theta}^{\alpha\beta}_{\mathbf{r}}\hat{\Lambda}^{\beta\beta}_{\mathbf{r}} \tag{17}$$

where $\hat{\Phi}^{\alpha\beta}_{\mathbf{r}} = \hat{\Gamma}_{\mathbf{r}}\hat{\Phi}^{\alpha\beta}_{\mathbf{r}}$, such that the JW transformation on the fermionic hopping term can be expressed simply as

$$i\gamma^3_{\mathbf{r}}\gamma^4_{\mathbf{r}-\hat{\mathbf{x}}+\hat{\mathbf{y}}} \mapsto -\hat{\Lambda}^{32}_{\mathbf{r}}\hat{\Phi}^{13}_{\mathbf{r}+\hat{\mathbf{y}}}\hat{\Lambda}^{44}_{\mathbf{r}-\hat{\mathbf{x}}+\hat{\mathbf{y}}}$$

We can view $\hat{\Phi}^{13}_{\mathbf{r}+\hat{\mathbf{y}}}$ as the JW string that connects between the two sites at $\mathbf{r}$ and $\mathbf{r}-\hat{\mathbf{x}}+\hat{\mathbf{y}}$. In general, for longer-ranged hopping term, it would consist of more $\hat{\Phi}^{\alpha\beta}$ terms, corresponding to an enlongated JW string that connects two farther-off sites.

C. Plaquette constraints

There are multiple ways of connecting two sites on a square lattice. At the moment, they would correspond to different JW strings that act on different sites, so the bosonized state we obtain might not be the same. To put all JW strings connecting two sites on the same footing, we impose the plaquette constraint on the bosonic side

$$\hat{\Phi}^{24}_{\mathbf{r}}\hat{\Phi}^{32}_{\mathbf{r}+\hat{\mathbf{x}}}\hat{\Phi}^{13}_{\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}}}\hat{\Phi}^{41}_{\mathbf{r}+\hat{\mathbf{y}}} = -1$$

which corresponds to the identity $\gamma^2 = 1$ on the 4 sites of interests on the fermionic side.

As mentioned earlier, there is a nice physical interpretation of the plaquette constraint: the only states that satisfy it are loop gas states. The key to representing the JW string in higher-dimension is that we restrict our attention to loop gas states. That's because given a loop gas state, any open string (that corresponds to a JW string) linking between two sites can be straightened to the shortest-distance (i.e. taunt) string that link between two sites. So we can always make a slack, non-local string into a taunt string, such that the extent of the JW string (on the bosonic side) can always be made to be within the support of the hopping term (on the fermionic side). Locality is thus preserved.

D. Sanity-check: Hilbert space dimension

Let's do a careful checking of the dimension of the Hilbert space on both sides of the higher-dimensional JW transformation. Let there be L^2 sites in the square lattice (thus there are L^2 plaquettes).

The origin Hilbert space is a spinful electron, so the dimension of the singlet-site Hilbert space is 4: two from the occupancy number of the site ($n = 0, 1$), and two from the spin degree of freedom ($s = \uparrow, \downarrow$). So the dimension of the physical fermionic Hilbert space is $4L^2$.

On the bosonic side, we started out with a $2^4 = 16$ dimenisonal parton Hilbert space for each site, since we have 4 different types of partons: u, d, c, h, and each is a fermion that contributes 2 dimensions to the Hilbert space. So the initial parton Hilbert space dimension is $16L^2$. We restrict ourselves to the Γ^- eigensubspace (at each site) in order to preserve the fermion anticommutation relation. This reduce the dimension of the Hilbert space to $8L^2$.

Then, when linking different lattice site, we further impose the plaquette constraint, requiring that all plaquettes (and henceforth all closed loops) operators evaluate to identity. One need to be careful in counting the number of independent plaquette constraints. Notice that the product of all plaquette gives the identity, so there are in fact only $L^2 - 1$ independent plaquette constraints! This halves the Hilbert space of each site except for one, giving the dimension to be $4(L^2 - 1) \times 8$. Then we note that since we impose periodic boundary condition on the square lattice, the system is actually on a torus. There are two non-contractible loops along the torus that give two additional constraints, reducing the dimension of Hilbert space to $4L^2/2$! This seem to restrict the Hilbert space too much, but we recognize that fixing both of the non-contractible loops actually also fixes the physical fermion parity, so it makes sense we only get half of the physical fermion Hilbert space. We get the other half by another choice of the non-contractible loop constraint, which correspond to the other physical fermion parity sector.

V. DISCUSSIONS

In conclusion, I have reviewed the general procedure of constructing the Jordan-Wigner transformation for any fermionic Hamiltonian on a 2D square lattice. The advantage of the above JW procedure is that symmetries are manifest on the bosonic side after the mapping. An important application of higher dimensional JW transformation is for simulating generic fermionic problems which can have sign problems. Sign problem renders fermionic systems impossible to be simulated using Monte Carlo algorithms; however, by applying the Jordan Wigner transformation, one can map the fermionic problem to the corresponding bosonic spin problem, which can be simulated using the wide variety of algorithms available for bosonic systems.

For more details, Ref. [6] discuss concrete examples of JW transformation on interacting fermionic Hamiltonian to their bosonic analog. Ref. [7] discuss a slight generalization where the procedure can be applied to any lattice with an even coordination number.

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