Self-organized criticality

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We study the concept of the self-organized criticality (SOC) and its application to a wide range of scientific problems with very different backgrounds. In particular, we discuss the Bak-Tang-Wiesenfeld sandpile model which displays SOC behavior and by computing the critical exponent for the two-dimensional model we find the agreement with the known result. Finally, we provide a new example of Zipf's law and discuss the connection of power-laws found in nature to the SOC phenomenon.

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I. INTRODUCTION

The discovery of the self-organized criticality (SOC) is one of ground-breaking achievements of statistical physics in the last couple of decades. Self-organized criticality is a very rich phenomenon as it combines self-organization and criticality to describe complexity. This concept was first introduced by P. Bak and the collaborators in the seminal paper in 1987[1]. SOC is a property of dynamical systems to organize its microscopic behavior to be spatial (and/or temporal) scale independent. That resembles of the critical behavior of the critical point of phase transitions. However, in contrast to the usual phase transitions, the systems displaying SOC do not require external tuning of the control parameters, i.e. *the system evolutions, i.e. organizes itself* into the critical behavior.

Thus, the dynamical system organizes itself into a state with complex, but rather general structure. Complexity arises in the sense that no single characteristic event size exists, i.e. no scale present to guide the system's evolution. Despite the complexity, system exhibits simple statistical properties governed by power laws. For example, the number of events D as a function of its size s (where a big event is less likely to happen compared to a smaller one) is given by,

$$D(s) = As^{-\alpha}, \quad (1)$$

where A is some constant and α is the exponent describing statistical features of a SOC state. Remarkably, some of the exponents α can be same for systems with very different microscopic description.

SOC is typically observed in slowly-driven non-equilibrium systems with extended degrees of freedom and a high level of non-linearity. Many individual examples have been identified since Bak's original paper, but to date there is no known set of general characteristics that guarantee a system to display SOC. Phenomena of strikingly different backgrounds were claimed to exhibit SOC behavior: sandpiles, earthquakes, forest fires, rivers, mountains, cities, literary texts, electric breakdown, motion of magnetic flux lines in superconductors, water droplets on surfaces, dynamics of magnetic domains, growing surfaces, human brains, etc.[2]. The common feature for all mentioned systems is that the temporal and/or spatial power-law correlations extend over several decades where intuitively one may anticipate that the physical laws would vary dramatically.

Empirical observations of power-law distributions of spatially-extended objects have triggered a need for a theoretical explanation. Unfortunately, no unifying mathematical formalism has been elaborated so far and it appears unclear how to identify whether a given system displays SOC behavior or it is something different. Even worse, there exists no generally accepted definition of SOC.

Fortunately, there exist a few mathematical models which seem to display SOC behavior: Bak-Tang-Wiesenfeld sandpile, Olami-Feder-Christensen earthquake model, Lattice Gas model, Critical Forest Fire model, etc. In the present work we describe the original BTW sandpile model which is the first discovered example of a dynamical system displaying SOC[1]. We perform computations for the two-dimensional model using parameters different from the original one. We obtain an exponent which is in a good agreement with the known one, thus again indicating that SOC state is achieved without any need of parameters tuning.

Next, we find another system exhibiting power-law statistical properties. It appears, that distribution of skyscrapers heights obeys power-law. Finally, we discuss the open question which systems exhibiting power-law characteristics can be considered to exhibit SOC behavior.

II. BAK-TANG-WIESENFELD SANDPILE MODEL

Consider a flat table of finite size with one grain of sand added per unit time interval so that the system has enough time to equilibrate before the next grain drops down. The grains can be added either randomly or at some fixed position of the table. As a result of friction between the grains the system does not automatically equilibrate to a ground state of flat height profile. Initially, the grains are most likely to stay at the same places where they landed, however as we carry on adding more sand, the height profile becomes steeper and small sand slides or avalanches can occur. If the grain lands on top of other grain it may topple to a lower level overcoming friction due to gravity. This toppling causes local disturbance which does not affect the large-scale picture, i.e. there are no correlations between distant parts of the sandpile. However, as the slope increases, a single grain is more likely to cause other grains to topple and eventually the slope reaches a certain maximal value when the amounts of sand being added and falling off the edges are balanced. Clearly, now the local dynamics no longer governs the process and the avalanches span the entire system leading to complexity. This is the SOC state with its own complex emergent dynamics which cannot be described by local dynamics laws. That is why it is natural to expect that SOC state is robust to the modifications of the systems, which is the crucial requirement for SOC to describe real world. For example, by changing the size of our system (as long as it still stays large), by adding different barriers on the table, by adding some amount of wet sand, the critical state dynamics stays exactly the same. That has been demonstrated on the example of the Bak-Tang-Wiesenfeld (BTW) sandpile model.

To show how the BTW sandpile model works, we consider a 2D flat surface defined by $z(x, y) = 0$ for all x and y (again, this initial condition does not affect final self-organized critical behavior) and start adding a grain of sand at a random position (x, y):

$$z(x, y) \to z(x, y) + 1. \tag{2}$$

If the number of grains $z(x, y)$ on a given site becomes larger than the critical value z_{cr} then there is a redistribution of grains between neighbouring cells, i.e. an avalanche:

$$\begin{aligned} z(x, y) &\to z(x, y) - 4, \\ z(x \pm 1, y) &\to z(x \pm 1, y) + 1, \\ z(x, y \pm 1) &\to z(x, y \pm 1) + 1. \end{aligned} \tag{3}$$

It is straightforward to apply the same logic to d-dimensions where the critical site redistributes $2d$ grains among its neighbours. One can also think of making grains redistribution from the critical cell to be between second-nearest neighbours, be anisotropic, etc. Eventually, all these modifications do not affect avalanche distributions.

The toppling process desribed above for 2D is illustrated in Fig.1. By adding a grain to a central site we cause an avalanche of size $s = 9$. When the edge cell becomes critical some grains fall off the grid and are not involved in the further dynamics. Overall, as a result of the avalanche depicted in Fig.1, the system loses, i.e. dissipates, 5 grains.

Thus, defining simple local dynamics we end up with a slowly-driven non-equilibrium system with extended degrees of freedom (i.e. grains of sand), a high-level of non-linearity and, finally, energy dissipation.

We applied the outlined BTW model to study 2D problem. Figure 2 illustrates the obtained probability distribution $D(s)$ of an avalanche as a function of its size.

1	0	1	0	3	1	0	1	0	3	1	0	1	0	3	1	0	1	0	3
0	1	1	2	2	0	1	1	2	2	0	1	2	2	2	0	1	2	2	2
2	2	3	1	2	2	2	4	1	2	2	3	0	2	2	3	3	1	2	2
1	3	3	3	2	1	3	3	3	2	1	3	4	3	2	1	4	0	4	2
2	0	3	2	3	2	0	3	2	3	2	0	3	2	3	2	0	4	2	3
1	0	1	0	3	1	0	1	0	3	1	0	1	0	3	1	0	1	0	3
0	1	2	2	2	0	2	2	2	2	0	2	2	2	2	0	2	2	2	2
2	4	1	3	2	3	0	2	3	2	3	0	2	3	2	3	0	2	3	3
2	0	3	0	3	2	1	3	1	3	2	1	3	1	4	2	1	3	2	0
2	2	0	4	3	2	2	1	0	4	2	2	1	1	0	2	2	1	1	1

FIG. 1: Ilustration of toppling avalanche in a small 5×5 sandpile. A grain falling at the site with $Z_{cr} = 3$ grains on it at the center of the grid triggers an avalanche with size of nine toppling events, i.e. $s = 9$. As a result of the avalanche, the system loses 5 grains of sand.

We used parameters different from the original paper by having taken 40×40 grid and averaged over 20 samples with total of $N = 5 \times 10^5$ grains added to each sample. By interpolating the statistical data we obtain the power-law distribution Eq.(1) with $A = 0.083$ and $\alpha = 1.098$ which is in a very good agreement with the original results[1]. Thus, parameters do not affect the exponent and there is no scale in the system, which are the key feature of systems displaying SOC behavior. Thus, the BTW model transparently shows how simple local interactions can lead to a complexity with transparent mechanisms of energy dissipation and input.

The exponent we have obtained by counting s after each grain added does not take into account other two important properties of an avalanche: the lifetime and the linear size. Generally, one can use hyperscaling hypothesis to obtain other exponents for the probability density of linear sizes, $D(l)$, as well as for the probability density of lifetimes, $D(t)$. Importantly, the set of the exponents will in general depend on the dimension d. However, as it turns out, using numerical computation it has been showed that for $d > d_c = 4$ the exponents become d-independent[2]. Thus, it indicates that the critical dimension is 4, above which the mean field value for the exponents is valid. For example, the mean-field value for α is 1.5 compared to 1.1 and 1.3 for two- and three-dimensional BTW models respectively.

III. SKYSCRAPERS HEIGHT DISTRIBUTION: ZIPF'S LAW AND SELF-ORGANIZED CRITICALITY CONNECTION

About 60 years ago G. K. Zipf showed that many human-related systems display power-law distributions[4]. For example, the distribution of cities by their size can be well fitted by the power-law. Many other geographical systems obey power-law distribution, such as mountains, rivers (Horton-Strahler law), etc. Furthermore, Zipf also analysed many literary texts (no matter either it was a solid book or a stack of newspapers) by counting the most frequent words

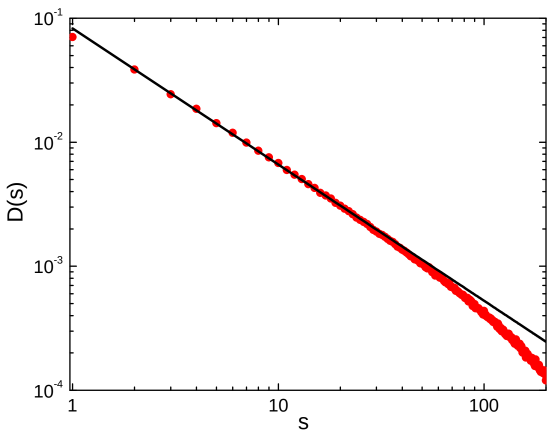

FIG. 2: Distribution of avalanche sizes $D(s)$ as a function of the size s. The data points correspond to an average over 20 samples of 40×40 grids with $N = 5 \times 10^5$ grains added to each. The power-law distribution has the exponent $\alpha = 1.098$. By taking a larger N we can increase the range where the fitting function works well, what was checked for one sample with $N = 2 \times 10^6$.

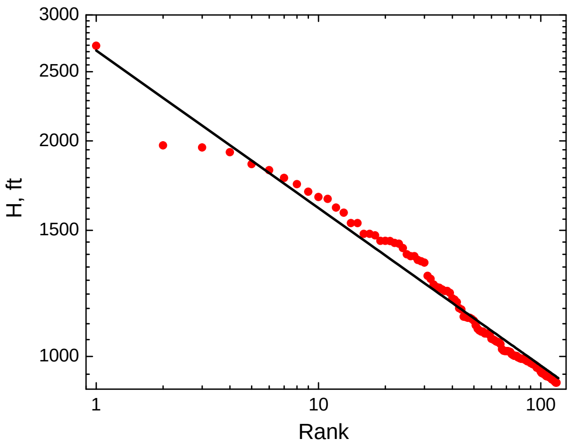

FIG. 3: Ranking of skyscrapers by height. The curve shows the number of skyscrapers which exceed a given height, or equivalently, the relative ranking of skyscrapers versus their height. The fitted power-law distribution has exponent $\alpha = 0.22$.

used in it and plotted the frequency given word was used as a function of the word's rank. A power-law regularity was observed for many literary texts with diverse backgrounds. This is similar to the distribution of earthquakes and the empirical Gutenberg-Richter law, although, clearly the phenomena described are just absolutely different.

Here we provide a new example of such a power-law distribution found in the real world: Fig.3 displays the height H distribution of the skyscrapers as a function of skyscraper's rank (data used from Ref.[5]). By fitting the data points with Eq.(1) we obtain $A = 2676$ft, and $\alpha = 0.22$. Clearly, this power-law distribution will have a lower boundary cut-off.

Of course, the skyscrapers construction is an extremely complex system with a dramatically large degrees of freedom. Moreover, a very large number of external technological and economic factors influence the probability of constructing a skyscraper of a given height. Thus, one can hardly think of a mathematical formalism which could describe such a complex problem. Moreover, it appears to be impossible to change the parameters of the problem and see if the power-law distribution exhibits the same exponent. This problem is general for many other systems that obey Zipf's law and the border between the SOC behavior and other explanations is very blurry. Per Bak tends to link any power law that occurs in a potentially self-organizing system with the principle of SOC[3]. Of course, one can think of strong local dynamics interactions between people which results in such global avalanche effect as large city creation (in analogy to BTW sandpile model), however as long as no persuasive models of self-organized criticality for Zipf's laws are constructed, the question remains open.

A relatively recent study specifically discusses the relation of scaling laws and indication of SOC in urban systems [6]. The conclusion the authors reach is that the urban systems display three spatial and temporal SOC signs. For example, for cities these three SOC indications are the number law, the population size law, and the area law. However, it is difficult to judge either the system of interest exhibits SOC or not, but, perhaps, more importantly is to reveal that *both physical and human systems conform to the same mathematical rules.*

Here we would like to mention that in contrast to the BTW model with simple local interactions, literary texts (which obey Zipf's law) have complicated non-local correlations. In our previous work we showed that the theory of additive Markov chains with long-range memory can be used to describe the correlation properties of literary texts[7]. By studying the coarse grained literary texts we concluded that their organization is complex due to long-range memory, in contrast to the BTW model where local correlations lead to complexity. That can be explained by the existence of both short- and long-range correlations, where the former one can be referred to as the grammatical one, whereas the latter kind of correlations may be semantic, i.e. due to a general idea spanning the whole text. Thus, non-local correlations can also play an important role in systems which obey power-law statistics and claimed to exhibit SOC behavior.

Fortunately, despite no agreement exists today about the way to define either the system is in SOC state or not, the renormalization group tools seem to be a fair judge for mathematical models exhibiting SOC. The RG analysis of sandpile-like model found an attractive fixed point, thus fulfilling the SOC demand for scale-invariance and irrelevance of tuning parameters. In contrast to that, the RG analysis of the Critical Forest Fire model has found a repulsive fixed point, what means that there is a specific scale in the system. That has been explained by another demand for *threshold* for SOC state to occur: there must be a large number of static metastable configurations, what is not the case for the Critical Forest Fire model. It has been concluded, that the existence of local thresholds is a necessary, but not a sufficient, condition for SOC.

IV. CONCLUSIONS

In the present work we have reviewed the concept of self-organized criticality which describes a complex self-organized state of a system with general, spatially- and/or temporally-independent, i.e. scale-free, characteristic behavior. We have also described the Bak-Tank-Weisenfel sandpile model which was the first dynamical system to exhibit SOC. We have carried out the computations for BTW model with parameters different from those used in the original paper and obtained the same power-law distributions for 2D case, what is in agreement with the idea of scale-invariance of SOC state. Finally, we review Zipf's law and provide a new example of skyscraper's height distribution which obeys the power-law statistics. Finally, we discuss the connection between Zipf's law and SOC and find that there is no agreement reached so far whether power law distributions found in nature are related to SOC or not. That is a direct consequence of SOC to be rather phenomenological, than constructive definition. We mention, that models with local interactions (such as BTW) as well as models with non-local (literary texts) correlations may lead to power-law distributions. Eventually, we briefly discuss the results of RG analysis applied to some models: the mathematical model (or real system) must allow for a large number of metastable states to exist to be a candidate for SOC behavior.

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