8.02 Document **Math**

MATHEMATICS SUPPLEMENT

FOR PHYSICS 8.02

Massachusetts Institute of Technology

(FOR 18.01 GRADUATES)

SERIES APPROXIMATIONS

INTEGRATION

LINE INTEGRALS, DOUBLE AND TRIPLE INTEGRALS

INTEGRATING VECTOR DOT PRODUCTS

CLOSED AND OPEN LINES AND SURFACES

Hale Bradt

FOREWORD

The subject 8.02 makes use of several mathematical tools you may have just learned in 18.01 (*e.g.*, integration in one dimension, linear approximations of functions) and several you may learn shortly in 18.02 (vector dot products, integration along an arbitrary line in space, integration of vector dot products, and double integrals). Unfortunately, all but the first of the latter topics will be covered in 18.02 (or later math courses) *after* they are introduced in 8.02. It is difficult to change the order of presentation in either subject without doing severe damage to the internal logic and consistency of the subject itself. Therefore, the 8.02 lecturer normally presents the necessary mathematical tools as they are needed, and the physics text may also give help in this regard. This has proven to be adequate for past generations of students. (This discussion does not apply, of course, to those of you who have already completed 18.02 or its equivalent.)

This document (prepared for the 8.02 students in previous terms) is an attempt to present the needed concepts and tools in a single and relatively compact unit. The approach is not meant to be rigorous (see 18.02 for this). Instead it appeals to your intuition, for instance, by drawing analogies to 18.01 material. (*E.g.*, double integrals can be viewed as a 2–dimensional analog of the 1–dimensional 'single' integral.). *The examples herein also illustrate the physics encountered in the first few weeks of 8.02.*

In 8.02, it is often the physical meaning of mathematics expressions that the student needs; difficult manipulations (*e.g.,* evaluations of dificult integrals) are rarely encountered. . *So, it is important that you at least lean the meaning of a double integral. In other words, can you describe the meaning of the integral expression in words?*

This presentation should be relatively easy going to 18.01 graduates. However, it is a lot of material in a compact form, so some might find it difficult. If so, do not be alarmed; the ideas will be covered in the physics course as needed. In either case, a single serious reading now and an attempt to do the few problems should help you when you next encounter these ideas. *We suggest strongly that you work through this review during IAP or during the first week of the course*.

A few topics here are a review of 18.01 material. Most of the material is 18.02 material directly useful in the early weeks of 8.02. The discussion on triple integrals (pp. 20–21) is not needed until late in the course. We encourage you to look ahead (or back) in your Math Textbook and Notes and read about (or review) these topics. Usually this would amount only to a page or two for each topic.

This document is a supplement to 8.02. It was prepared by HB. Prof. A. Mattuck of the Mathematics department provided helpful comments.

Hale Bradt
January 2001

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Note: Vectors herein are indicated with bold face characters.

SERIES APPROXIMATIONS

Physicists frequently find it useful to simplify a complicated mathematical expression so that it can be compared directly to another formula or experimental result. Sometimes this serves as a check on the correctness of the complicated expression. Often this simplification amounts to finding the value of a function at a limiting value of the variable, *e.g.,* some function $f(y/b)$ evaluated at $|y/b| << 1$, where b is a constant.

A powerful and useful tool for this is the series expansion of an arbitrary function $f(x)$,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots.$$

where the coefficients are constants. For x sufficiently small, $x \approx 0$ or $|x| << 1$, only the first 1 or 2 terms need be retained; the higher order terms are much smaller and may be neglected. A method of obtaining the coefficients of the first terms is presented in 18.01. The higher order terms may be obtained with the Taylor series, to be presented in 18.02; we do not need them in 8.02.

We list below a few examples that are often useful. We drop the higher order terms in x because they are negligible for sufficiently small values of x; thus these expressions constitute approximations of the true functions. The list includes the only ones you are likely to encounter in 8.02. In fact you will encounter them only once or twice; you will find them quite useful in other courses and labs. To use them, you must cast your mathematical expression into the given form, *e.g.*, for expression (a) below:

$$(b-y)^{-1/3} = b^{-1/3}\left[1+\left(-\frac{y}{b}\right)\right]^{-1/3} \underset{\lim y/b\ <<\ 1}{\longrightarrow} b^{-1/3}\left(1+\frac{y}{3b}\right)$$

(a)	$(1+x)^m$	$\approx$	$1 + mx + \ldots$	for $\lvert x\rvert << 1$
(b)	$\sin x$	$\approx$	$x + \ldots$	for $\lvert x\rvert << 1^*$
(c)	$\cos x$	$\approx$	$1 - x^2/2 + \ldots$	for $\lvert x\rvert << 1^*$
(d)	$\tan x$	$\approx$	$x + \ldots$	for $\lvert x\rvert << 1^*$
(e)	$\ln(1+x)$	$\approx$	$x + \ldots$	for $\lvert x\rvert << 1$
(f)	e^x	$\approx$	$1 + x + \ldots$	for $\lvert x\rvert << 1$
(g)	$f(x)$	$\approx$	$f(0) + f'(0)\, x + [f''(0)/2!]\, x^2 + \ldots$	for $\lvert x\rvert << 1$

$*$ x is angle in radians

In (g), $f'(0)$ and $f''(0)$ are the first and second derivatives of $f(x)$ evaluated at $x = 0$. The terms $f(0) + f'(0)$ express the fact that the graph of $f(x)$ near $x = 0$ is well approximated by its tangent line at $x = 0$. All of these expressions are expansions about $x \approx 0$; refer to 18.01 notes for expansions about other points.

Problem A1: Convince yourself that the above formulae are reasonable in a few cases. For example calculate $(1 + 0.02)^{-2}$ by direct multiplication/division and again with (a). Similarly look up or calculate precise values of sin 3°, cos 3°, and tan 3° and compare to the values obtained from (b), (c), and (d).

Problem A2: Write simplified versions of $(z^2 - c^2)^{-0.2}$ for $z << c$ and for $z >> c$.

Problem A3 (optional): Use (g) to derive three of the equations (a) – (f).

INTEGRATION IN E & M (8.02)

INTRODUCTION

In 8.02, "Line Integrals" and "Double Integrals" are used before they are covered in 18.02. We make use primarily of their *conceptual* meaning and avoid almost entirely the complex computations of the type you probably associate with 'integration'. Here we present the concepts as they are used in 8.02. As stated earlier, this is not intended to be a rigorous presentation. You are encouraged to read your Math text for a more complete explanation.

You should already be familiar with 1–dimensional "single" integrals. The Line Integral is a natural generalization of this concept. The Double Integral is another generalization of the Single Integral. We choose to begin with the Single Integral as a review. In 8.02, the vector dot product is used as the integrand of line and double integrals. We therefore include some discussion of this. (You learn about dot products in the 1st week of 18.02.) Although not required until very late in 8.02, we conclude with a bit about "Triple Integrals" because it is a natural extension of the Double Integral.

You will find that our approach to integration is to view it almost exclusively as a *summation*. Seeing an integral sign in 8.02 should cause you to think "Oh, here is a summation; I know what it means even if I can't solve it". (The idea of integration as 'inverse differentiation' is a useful concept, of course, for actually carrying out the integration operation; however we do not stress it here.)

ONE–DIMENSIONAL ('LINE') INTEGRALS

Integral as summation along a straight line

The concept of integration as a summation of quantities is now illustrated. The length L of a straight line running in the x direction from $x = a$ to $x = b$ (Fig. 1) can be calculated from a sum of the length elements dx,

(1) $$L = \sum_i \Delta x_i$$

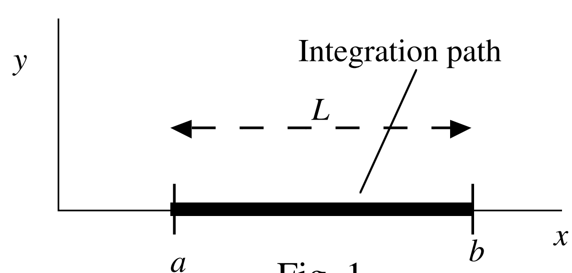

Fig. 1

In the *limit* ($\Delta x \to dx$), the summation sign becomes the integral sign,

$$(2) \qquad L = \int_a^b dx = b - a \qquad \text{(meters)*}$$

One wag has been known to say that if you write (1) fast enough, the Σ will begin to look like the integral sign in (2). When you see the integral sign, it is perfectly correct to think 'summation' (*i.e.,* Eq. 2) where, of course, the summed intervals are infinitessimally small.

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*We occasionally give appropriate SI units in this document as a guide; most or all of the equations herein are independent of the choice of units.

The Single Integral can also be used to sum a given quantity along the line. In E&M, the concept of electric charge Q (units 'Coulomb' or 'C') is introduced. For example, individual electrons carry a fixed negative charge of 1.6×10^{-19} C. If lots of charges (*e.g.*, electrons) are distributed along a line, one can define a linear charge density $\lambda(x)$ (C/m). If the charges are uniformly distributed, λ is a constant, and the total charge Q along the line would then be

$$(3) \qquad Q = \int_a^b \lambda \; dx = \lambda \int_a^b dx = \lambda \; (b - a) \qquad \text{(Coulombs)}$$

Note that the integration process indeed does yield the obviously correct answer you would obtain by direct multiplication.

Summation of a function f(x) along a straight line

Integration becomes necessary if one wishes to sum a given physical quantity $f(x)$ that is a function of position x. For example, let the charge density $\lambda(x)$ be the following function of x,

$$(4) \qquad \lambda(x) = k\,x^2 \qquad \text{(Coulomb/meter)}$$

where k is a constant. The total charge Q is then

$$(5) \qquad Q = \int_a^b \lambda(x)\,dx = \int_a^b k\,x^2\,dx = k\,\frac{x^3}{3}\,\Bigg|_a^b = k\,\frac{\left(b^3 - a^3\right)}{3} \qquad \text{(Coulombs)}$$

Here it is not obvious that the answer is correct. You could adopt specific values of a, b, and k and do a stepwise summation to convince yourself the answer in (5) is approximately correct. Or rather, you could review the proof of the calculus theorem that the summation in the limit over the interval $a < x < b$ is indeed the equivalent of the integration,

$$(6) \qquad \int_a^b \lambda(x)\,dx \equiv \lim_{\substack{\Delta x \to 0 \\ m \to \infty}} \sum_{i=1}^{m} \lambda(x)\,\Delta x_i \qquad (\text{for } a \le x \le b)$$

Line integral: a single integration along an arbitrary path

An integral can equally well be taken along a line that is not straight. The path can lie in a plane as shown in Fig. 2 or it can be a curve in 3–dimensional space. In Fig. 2, **ds** is a *displacement* vector with magnitude that is the differential *length* element (analogous to dx) and with the direction of the displacement at the given position.

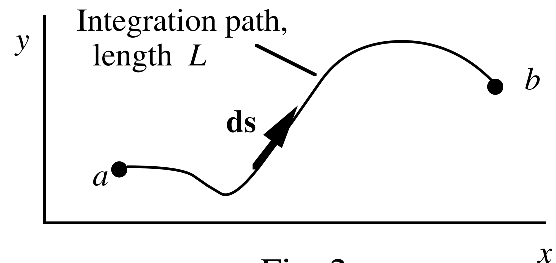

Fig. 2

We could sum by integration the magnitudes of differential lengths |**ds**| to get the total length L of the line much as we did in (2). Similarly, we could calculate the total charge Q much as we did in (3) and (5).

The formalism would be somewhat different than in (2), (3) and (5) because we would have to properly describe the magnitude of **ds** in terms of its components dx and dy. You are not likely to encounter this particular problem in 8.02, so we move right on to the case where the integrand of the line integral is a vector dot product.

Vector dot product

The quantity $dW = \mathbf{F} \cdot \mathbf{ds}$ is a vector dot product; it represents the increment of work dW done by the force **F** that is exerted on an object when the object is displaced the distance and direction indicated by the vector **ds**. The definition of the dot product is (See Fig. 3)

(7) $\quad \mathbf{F} \cdot \mathbf{ds} \equiv |\mathbf{F}|\,|\mathbf{ds}| \cos\theta$

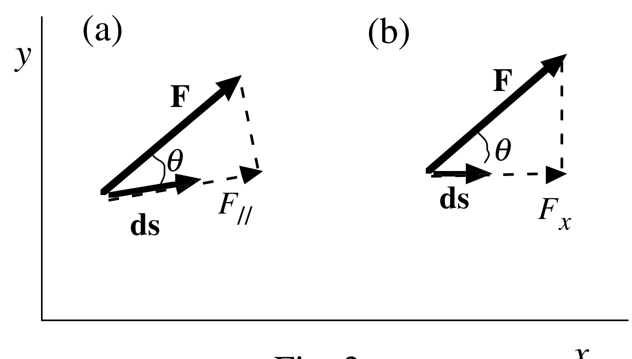

Fig. 3

where θ is the angle between the vectors. Thus the dot product is the projection of the force along the path times the path element, *e.g.*, the product $F_{||}$ |**ds**| in Fig. 3a, or if the displacement is in the x direction, $F_x dx$ (Fig. 3b).

In general, the dot product may be written in cartesian coordinates as (see your text):

(8) $\quad \mathbf{F} \cdot \mathbf{ds} \equiv F_x \, dx + F_y \, dy + F_z \, dz$

The cartesian components of the displacement vector **ds** are dx, dy, and dz.

Dot product in polar coodinates

In polar coordinates in the (r,θ) plane, the dot product of $\mathbf{F} \cdot \mathbf{ds}$ can be shown to be (see your text):

(9) $\quad \mathbf{F} \cdot \mathbf{ds} = F_r \, dr + F_\theta \, r \, d\theta$

This form of the dot product is particularly useful when we deal with 'central' (radial) forces, like Newton's law of gravitation, or electric fields. Central forces by definition have only radial components, *i.e.*, $F_\theta = 0$.

Problem B1: Show that Eq. 9 is correct, from an examination of Figs. 4a,b. The vector **ds** represents the displacement; its (scalar) components are shown. You may base your argument upon a simple analogy to Eq. 8.

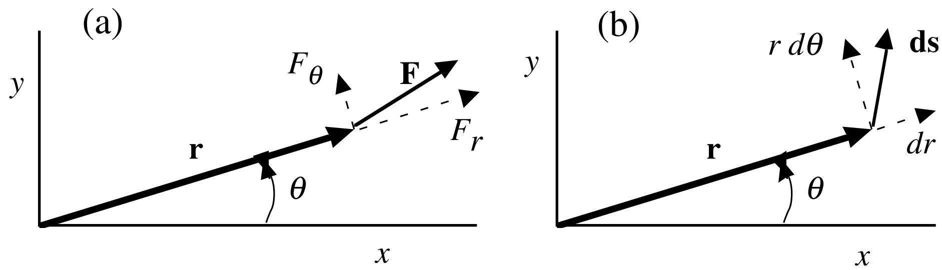

Fig. 4

Line integral of a dot product

In 8.01, you learned of force fields wherein a force vector $\mathbf{F}(x,y,z)$ could be assigned to every point in space. [In Electromagnetism, we also have vector fields, such as the electric field vector $\mathbf{E}(x,y,z)$] You will learn about it early in 8.02. A typical integration with which you may be familiar is the calculation of the potential energy at point b relative to point a, $U(b) - U(a)$. By definition, this is the negative of the work done by a conservative force on a body as the body is moved from point a to point b (Fig. 5a). In its most general form, the definition makes use of the dot product **F·ds**,

$$U(b) - U(a) \equiv -\int_a^b \mathbf{F}\cdot\mathbf{ds} \qquad \text{(Joules)} \tag{10}$$

This formula can be applied to any arbitrary path in 3–dimensional space. You will note that the integral amounts to a summation of the elements of work dW along the path traversed.

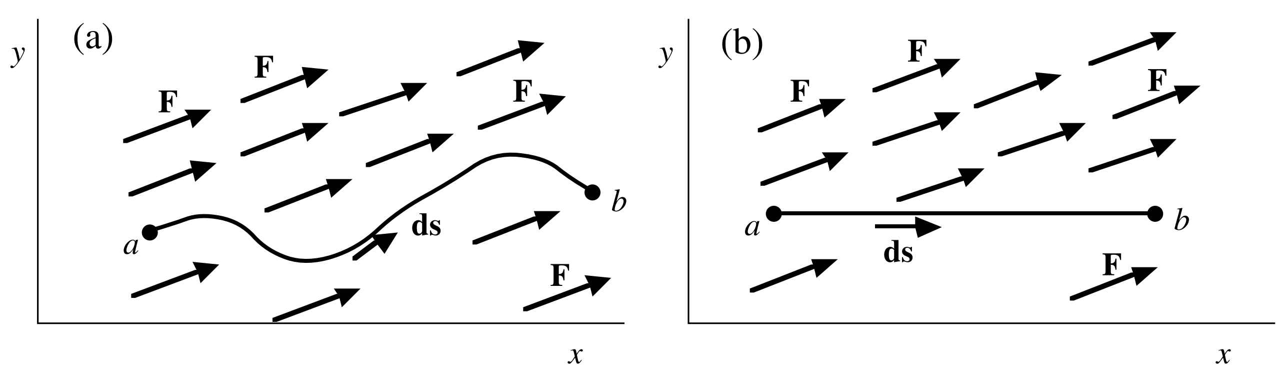

Fig. 5

Displacements in the x direction

If the displacement of the body is in the x direction, as shown in Fig. 5b (so $dy = dz = 0$), *or* if the force is in the x direction (so $F_y = F_z = 0$), Eq. 10 becomes, using (8):

(11) $$U(b) - U(a) = -\int_a^b F_x\,dx \qquad (e.g., \text{ for } dy = dz = 0)$$

Now the integration in (11) is simply the sum of the products $F_x\,dx$, *i.e.,* the sum of all the elements of work, which in turn is the total work done during the displacement from a to b. This is what we intended when we wrote (10). We have thus demonstrated that (10) has the intended meaning and that it can easily reduce to a simple 'normal' integral with a scalar argument. The moral is to not let a dot product inside an integral cause panic; it can be immediately converted to scalar notation and solved a la 18.01.

Displacements in the radial direction

If the motion is only along the radial direction (so $d\theta = 0$) *or* if the force is solely in the radial direction (so $F_\theta = 0$), the integral (10) may be written, with the help of (9):

(12)

$$U(b) - U(a) = -\int_a^b F_r\,dr \qquad \text{(Joules)}$$

($e.g.$, for $F_\theta = 0$ or $d\theta = 0$)

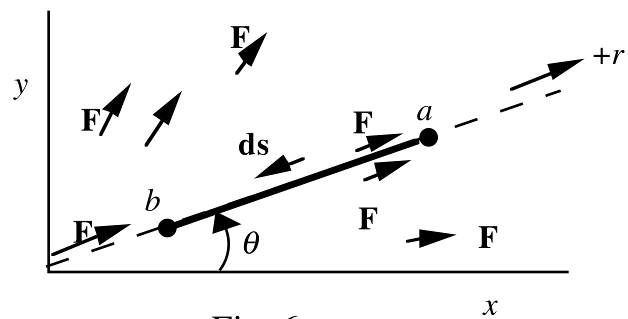

Fig. 6

In Fig. 6, we illustrate a displacement in the *negative* radial direction (from a to b) in the presence of a radial force field. Equation 12 applies to this situation.

A common force (in mechanics *and* in electromagnetism) is an inverse squared *radial* force, *i.e.,* $F_r \sim 1/r^2$ and $F_\theta = 0$. If this force is repulsive, unlike gravity but like the electric force between 2 like charges, it may be written:

(13) $$\mathbf{F} = F_r\,\hat{\mathbf{u}}_\mathrm{r} = +\frac{k}{r^2}\,\hat{\mathbf{u}}_\mathrm{r}$$

where the + sign indicates that the force is in the positive radial direction (Fig. 6), k is a constant and $\hat{\mathbf{u}}_\mathrm{r}$ is a unit radial vector. The potential energy is usually referenced to the potential at infinity so the potential energy at radius r becomes, from (12) and (13):

(14) $$U(r) - U(\infty) = -\int_\infty^r +k\ r^{-2}\,dr \qquad (\text{since } F_\theta = 0)$$

It is important for problem solving to note the relative positions of the symbols a and b on the left and right sides of (12). The a appears as the lower limit and in the second or reference value of U, and the b as the upper limit and the first or desired value of U. The same relative positions of r and ∞ are adhered to in (14). Also note that the correct sign was used for the force component F_r in the integral of (14). The integration limits automatically take into account the direction of displacement. The integration operation then gives the correct sign for the potential energy.

Solving (14) by the usual rules of integration, and adopting the convention that $U(\infty) = 0$, we obtain

(15) $$U(r) = +k/r$$

This result tells us that the potential energy is positive for all radii less than infinity. This is in accord with our ideas of potential energy. When a test body is carried from infinity to a finite radius against a repulsive force, the system ends up with stored (*i.e.,* positive) potential energy. We have demonstrated that our procedures regarding signs, etc. lead us to the correct result, at least in this case.

In electromagnetism, we will encounter the above formalism but will define somewhat different terms, *e.g.* the electric force per unit charge, called the electric field **E**, and the potential energy *per unit charge*, called the potential V. These are related with an equation identical in form to (12):

(16) $$V(b) - V(a) = -\int_a^b E_r\, dr \qquad \text{(Joules)} \qquad (e.g.,\text{ for } E_\theta = 0)$$

Integral around a closed path

A line integral may be taken around a complete *closed* path, C_1, a path that ends where it begins (*e.g.* at A). Such a path is the boundary of a surface (Fig. 7). An integral around a closed path is indicated by a little circle on the integral sign. For example, the work W by a force on a body that is carried around the arbitrary closed path C_1 shown in Fig. 7 is, symbolically,

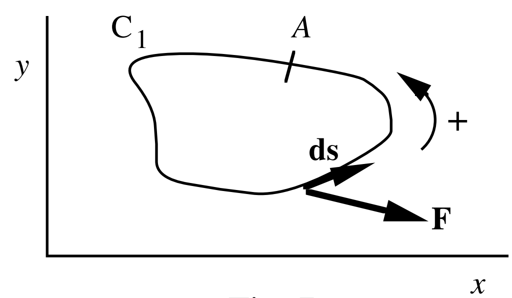

Fig. 7

(17) $$W = \oint_{C_1} \mathbf{F}\cdot\mathbf{ds} \qquad \text{(Joules)}$$

Often in 8.02, you are asked to understand the physical meaning of an integral; it is a logical statement and need not be a difficult calculation. The meaning of (17) is: "the quantity W is the sum of the elements of work done by force **F** on a body that is displaced along the individual elements **ds** that constitute the complete closed path". As you know, if a force is conservative (like gravity or an electrostatic force), the work done around a closed path is zero; hence $W = 0$. Other forces or quantities (see below) can give non–zero values.

Example: Line integral around a circular path

If in fact, you are asked to actually carry out an integration, the vector field will probably be ridiculously simple, *e.g.* a force with only an azimuthal component, F_θ (see Fig. 8; compare to Fig. 4a). An example might be the (non–conservative) force exerted by a whirlpool on a boat (if the radial component is negligible). If the azimuthal force is a function of radius, but not of azimuth, it could have the form:

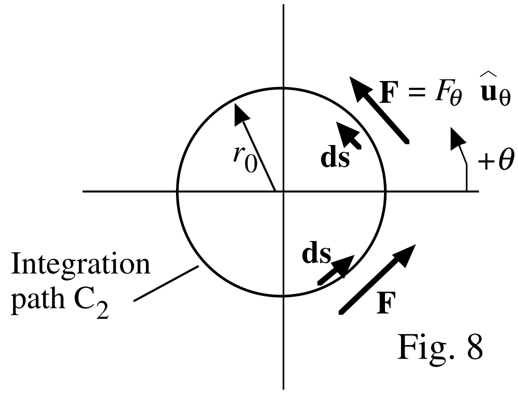

Fig. 8

(18) $$\mathbf{F} = F_\theta\ \hat{\mathbf{u}}_\theta = +k\,r^{-1}\,\hat{\mathbf{u}}_\theta$$

In (18), $\hat{\mathbf{u}}_\theta$ is a unit vector in the azimuthal direction, k is a constant, and the '+' indicates that the force is in the positive azimuthal direction. If we are asked to integrate the quantity **F·ds** around a circular path C_2 at constant radius r_0 (Fig. 8), we have from (9) and (18):

(19) $$W = \oint_{C_2} \mathbf{F}\cdot\mathbf{ds} = \int_0^{2\pi} F_\theta\ r_0\,d\theta = \int_0^{2\pi} +k\,r_0^{-1}\,r_0\,d\theta = k\,r_0^{-1}\,r_0\int_0^{2\pi} d\theta = +2\pi\,k$$

We do not obtain zero because net work is done by the *non*–conservative force as the body is carried around the path. Since the body was carried around the circle in the direction of the force vector, the work W (by the force) is expected to be positive; this agrees with the result we obtained.

As in our earlier calculation (Eq. 14), we obtained the correct sign by using the correct sign for the force in (19). A positive direction was adopted (counterclockwise 'ccw' in Fig. 8), and that convention was used for the force component F_θ (Eq. 18) and for the direction of increasing angle θ. Thus the ccw direction of angular displacement in our problem (Fig. 8) is positive, and the limits on the integral reflect this. They indicate that the body moves from $\theta = 0$ to $\theta = 2\pi$ (If the positive direction had been defined as clockwise, the integration limits for ccw displacements would be, *e.g.*, 0 to -2π or 2π to 0.)

For any azimuthal force F_θ, independent of θ, the integral (19) can be written,

(20) $$W = \oint_{C_2} \mathbf{F}\cdot\mathbf{ds} = F_\theta\ r_0\int_0^{2\pi} d\theta = F_\theta\ 2\pi\,r_0 \qquad (F_r = 0)$$

This expression (20) is transparently the work done by the force as the body moves around the circle; it is simply the constant azimuthal force multiplied by the distance around the circular path. You see therefore that the integral (19) takes on a very simple form (20) if the force **F** is azimuthal and axially symmetric.

Line integral for magnetic field (anticipated)

In E&M, the magnetic field vector **B** due to a current in a long straight wire is azimuthal and decreases as $1/r$ where r is the distance from the wire (Fig. 9). The field lines are circular about the wire, and the integral $\oint \mathbf{B}\cdot d\mathbf{s} = 2\pi\, r\, B_\theta$ is directly related to the current in the wire (Ampere's Law). We will therefore have occasion to evaluate integrals such as (20). However, the result will not be 'work' because the magnetic field **B** is *not* a force; it is a magnetic field.

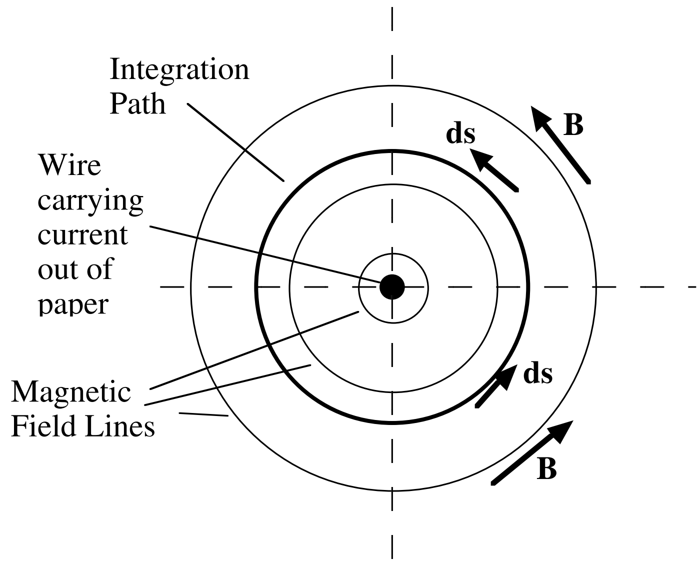

Fig. 9

DOUBLE ('SURFACE') INTEGRALS

Double integral as a summation

It is possible to calculate the area A of a surface with 'double' integrals just as we were able to calculate the length of a line with a 'single' integral. Thus, an area A in the x–y plane would have the value:

$$A = \iint_{\text{surface}} dx\ dy \equiv \lim_{\substack{\Delta x_i \to 0 \\ \Delta y_j \to 0}} \sum_i \sum_j \Delta x_i\, \Delta y_j \quad (\text{m}^2) \tag{21}$$

where the summation is over all the little area elements indicated by the products $\Delta x_i\ \Delta y_j$ that make up the surface of interest (Fig. 10) and where the integration limits (not given) represent the boundary of the surface. The summation (right–hand term) is obviously a correct description of the total area, although it is incomplete in that the summation limits that define the boundaries are not indicated. Equation (21) defines the meaning of 'double integral'. We show next that operationally it simply amounts to two one-dimensional integrations.

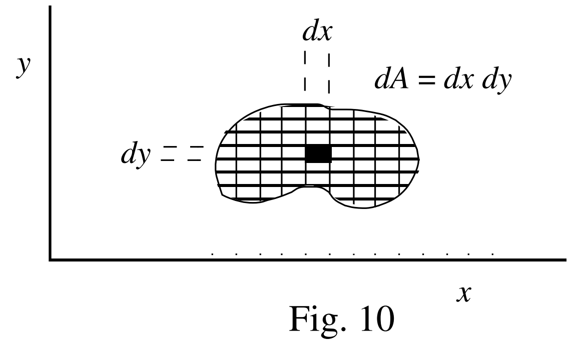

Fig. 10

Area of a rectangle

Mathematically, the double integral simply represents two sequential integrations, as follows:

(22) $$A = \iint dx\ dy \equiv \int \left[\int dx \right] dy$$

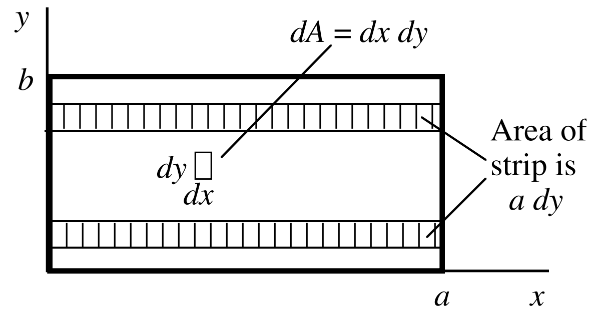

Fig. 11

where the integral inside the square brackets is carried out first. As a simple example, let us obtain the area A of the rectangle (dark boundary) shown in Fig. 11. We first install limits on the integration to define the boundaries of the surface of interest and then proceed to integrate.

(23) $$A = \int_0^b \int_0^a dx\, dy \equiv \int_0^b \left[\int_0^a dx \right] dy = \int_0^b a\, dy = a \int_0^b dy = a\, b$$

The result is patently the correct answer, the width times the height (Fig. 11). We have therefore demonstrated that this operational definition of the double integral gives the correct answer, thus validating, for this case, the identity of (21) and the operational definition of (22).

It is thus appropriate to think of a double integral as the *sum of the integrands* (*e.g.*, the product *'dx dy'* in Eq. 21) evaluated at each incremental location on a 2–dimensional *surface*. (For the single integral, the integrand was evaluated at each location along a *line*.)

The process of carrying out the 2 integrals in (23) can be visualized as two summations. The integration over x is the addition of all the elements $dx\, dy$ in a row at a fixed y (Fig. 11). The area of a row is '$a\, dy$' in our example. The integration over y is then the sum of the rows.

In the most general case, the boundaries of the surface are not constant values, *e.g.* the upper limit a on the inner integral of (23) could be a function of y, viz: $a(y)$. Operationally, one would proceed exactly as above. The function $a(y)$ would end up as part of the integrand of the outer 'y' integral. [In the more general case (*e.g.,* Fig. 10), the lower limit of the inner integral would also be a function of y. The limits on the outer integral would be the extreme y values of the surface.]

Summation of charge on a surface

The integrand of a double integral can contain more information than just the area of each element. For instance, it can contain the charge density $\sigma(x,y)$ (Coulomb/m^2). If $\sigma(x,y)$ is multiplied by the area element $dx\, dy$ at each (x,y) coordinate, we have the element of charge, $dQ = \sigma(x,y)\, dx\, dy$. The total charge Q on a surface area of interest can be summed with an integration. For our rectangular area (Fig. 11), we have

(24) $$Q = \int_0^b \int_0^a \sigma(x,y)\ dx\ dy \qquad \text{(Coulombs)}$$

How does one argue that (24) is the correct formulation to give us the total charge? Again, we fall back on the summation equivalence of (21). The integrand of (24) is the product of the charge density, σ, times the area, $dx\ dy$, of the element. Clearly the summation of these integrands is the desired total charge.

One can proceed with the integration if the function $\sigma(x,y)$ is known. Let us assume a simple function, *e.g.*, $\sigma(x,y) = k\,xy$, where k is a constant. Then,

(25) $$Q = \int_0^b \left[\int_0^a k\ xy\ dx\right] dy = \int_0^b \left[k\ y \int_0^a x\ dx\right] dy = \int_0^b \left[k\ y\,\frac{a^2}{2}\right] dy$$

Note that the y is treated as a constant during the x integration; as discussed above, this integration is a summation of elements along a row at fixed y (Fig. 11). Continuing with the y integration (the summation of rows):

(26) $$Q = k\,\frac{a^2}{2}\int_0^b y\ dy = \frac{k\,a^2\,b^2}{4} \qquad \text{(Coulombs)}$$

This is the desired charge on the surface.

Problem B2: What is the charge Q on the rectangular surface of Fig. 11 if the charge density is $\sigma(x,y) = k\ e^{-x/(2a)}\ e^{-y/(2b)}$ where k is a constant?

Surface integral in polar coordinates

Sometimes one encounters a situation where the geometry cries out for polar coordinates. Suppose for instance we wish to calculate the total charge Q on a circular plate of radius R (Fig. 12), where the charge density is a function of radius r but is independent of azimuth θ, *e.g.* $\sigma(r,\theta) = +\,k\,r^2$ (Coulomb/m^2) where again k is a constant. The problem will be greatly simplified if the area element is written in terms of r and θ, namely $dA_1 = r\,d\theta\ dr$ (Fig. 12a).

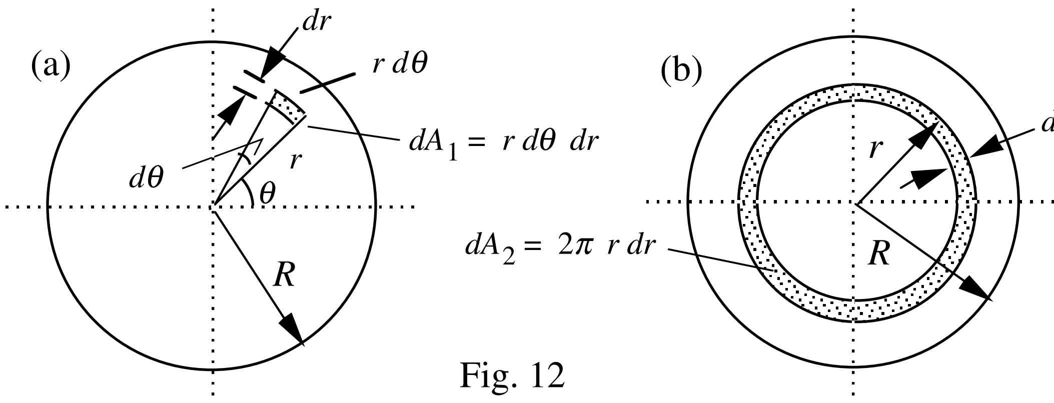

Fig. 12

The summation of the charge elements, $\sigma(r,\theta)\, dA_1$, then takes the form:

$$(27) \qquad Q = \iint_{\substack{\text{Circular}\\ \text{surface}}} \sigma(r,\theta)\, dA_1 = \int_0^R \left[\int_0^{2\pi} k\, r^2\, r\, d\theta\right] dr = \int_0^R k\, r^3 \left[\int_0^{2\pi} d\theta\right] dr$$

where the limits on the integrals define the circular surface over which we are summing the charge. Continuing,

$$(28) \qquad Q = \int_0^R k\, r^3\, 2\pi\; dr = \frac{2\pi\, k\, R^4}{4} \qquad \text{(Coulombs)}$$

Note that the variable r is held constant during the θ integration. For most practical problems, the order of integration is immaterial; the integration over radius r could have been the 'inner' integral which is performed first.

Surface charge obtained with a single integral

The example given in (27) and (28) would be disguised as a 'Single' integral in most freshman physics texts, even though it really is a 2–dimensional surface integral! This is possible because of the circular symmetry of both the geometry and the function being integrated, $\sigma(r,\theta) \rightarrow \sigma(r) = kr^2$. One can then write

$$(29) \qquad Q = \int_0^R \sigma(r)\; dA_2 = \int_0^R \sigma(r)\, 2\pi\, r\, dr$$

where the area element, $dA_2 = 2\pi\, r\, dr$, is simply the area of an annular ring of width dr at radius r (Fig. 12b). The integrand, $\sigma(r)\; 2\pi\, r$ dr, is thus the charge on the ring, and the integration over r is the summation over all the rings. Substitution of $\sigma(r) = kr^2$ into (29) immediately yields the result given in (28). The following problem does *not* have this azimuthal symmetry so a 2–dmensional surface integral *must* be used.

Problem B3: The charge distribution on a circular plate of radius R is a function of both azimuth and radius, $\sigma(r,\theta) = k\, r\ (1 - \sin\theta)$ (Coulomb/m^2). What is the total charge Q (Coulombs) on the plate?

Volume of a solid with a double integral.

Another example of the double integral is the calculation of volume. The function multiplying the area element $dx\,dy$ in the integrand is the height $z = f(x,y)$ of a solid. The sum (integral) of the integrands then gives the volume of the solid. We will see below that a 3–dimensional integral is another (equivalent) way to obtain volumes.

Again, we remind the reader that an integral is often written simply as a logical statement of summation; it doesn't necessarily have to be integrated. If one must solve an intractable integral, a computer can be programmed to tediously (but rapidly) do all the required sums, one–by–one. You will notice that the examples used in this document and in 8.02 are carefully set up to have easy–to–integrate functions or symmetries that simplify the integration.

Closed surfaces

Closed surfaces can be defined, analogously to closed paths. These are surfaces that completely enclose a volume, like a basketball or a cylinder closed at both ends. (Water can't pour out, and if the material is opaque you can't peek in.) There is nothing that precludes us from treating a closed surface in the same manner as the 'open' surfaces described above. One can integrate over it to find its area, the total charge on it, or the electric field due to charge distributed over it.

A basic closed surface is a sphere of radius R. We can write the integral for the calculation of its total area, and of course we already know the answer. Thus:

(30)
$$A = \oiint_{\substack{\text{sphere of}\\ \text{radius R}}} dA = 4\,\pi\,R^2$$

Note that we again use the loop to indicate closure. The loop on the double integral indicates a closed s*urface.*

*Area of a sphere (*This subsection may be skipped.)

The result in (30) can be obtained by writing out the area element on a sphere in spherical coordinates (see your text), in polar coordinates where θ is the *polar angle* and ϕ the *azimuthal angle*,

(31)
$$dA = R^2 \sin\theta\ d\theta\ d\phi$$

and by integrating over the entire sphere. We do not work this out here. [Your math text may reverse the roles of θ and ϕ. Physicists usually use the definitions given just above.]

Problem B4 (Optional, advanced problem): Carry out the integration in Eq. (30) to obtain the area of the sphere. Use the area element given in (31), after satisfying yourself that it is correct. Note that the azimuthal angle ϕ must be integrated from 0 to 2π radians and that the polar angle θ must be integrated from 0 to π radians in order to properly sum over the entire surface. Also note that the radius R is the same for each surface element; thus it should be treated as a constant in the integration.

Example: Charge on a sphere

A straightforward problem is the determination of the total charge Q on a sphere if the charge is distributed uniformly over the entire sphere, *i.e.,* the charge density σ is independent of θ and ϕ. The integral and its solution are, from (30):

(32) $$Q = \oint\!\!\oint_{\substack{\text{sphere of} \\ \text{radius R}}} \sigma\, dA \;=\; \sigma \oint\!\!\oint_{\substack{\text{sphere of} \\ \text{radius R}}} dA \;=\; \sigma\, 4\pi\, R^2$$

The constancy of σ allowed us to remove it from the integral. The value of the double integral is known from high school geometry (*i.e.,* from Eq. 30). Thus, it is not necessary to grind through the integral of the above problem (B4). Note that here we used the integral signs solely as a logical symbol for summation. This example is typical of 8.02 problems.

Closed surface consisting of several parts

Some slightly more complicated problems may be solved by breaking the surface integral (summation) into 2 or more integrations (summations). For instance, suppose a closed surface consists of 2 hemispheres of different radii but common centers and a flat annular surface connecting the two (Fig. 13). Let the radii for the two hemispheres be R_1 and R_2, their surface charge densities (C/m^2) to be σ_1 and σ_2, respectively, and the surface charge density on the flat surface be σ_3. The total charge, generally speaking, is

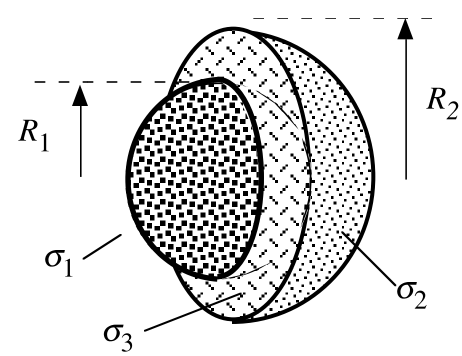

Fig. 13

(33) $$Q = \oint\!\!\oint_{\substack{\text{3-part} \\ \text{closed} \\ \text{surface}}} \sigma(\theta,\phi)\;\; dA$$

This integral over the 3–part closed surface may be rewritten as the sum of 3 separate integrals, one for each portion of the surface. This is clearly valid because (33) itself is a sum. The integrand for each will include one of the constants: σ_1, σ_2, or σ_3:

$$Q = \iint_{\substack{\text{hemisphere}\\ \#1}} \sigma_1 \, dA_1 + \iint_{\substack{\text{hemisphere}\\ \#2}} \sigma_2 \, dA_2 + \iint_{\text{annulus}} \sigma_3 \, dA_3 \qquad (34)$$

Each of these terms is the sum of the elemental charges on one of the 3 surfaces. Note the absence of loops on the integral signs; each of the 3 integrations is over an *open* surface.

The constancy of the charge densities permits us to proceed.

$$Q = \sigma_1 \iint_{\substack{\text{hemisphere}\\ \#1}} dA_1 + \sigma_2 \iint_{\substack{\text{hemisphere}\\ \#2}} dA_2 + \sigma_3 \iint_{\text{annulus}} dA_3 \qquad (35)$$

The integrals are now trivial; each represents an area you learned in high school. Thus there is no need to carry out explicit integrations.

$$Q = \sigma_1 \, 2\pi R_1^2 + \sigma_2 \, 2\pi R_2^2 + \sigma_3 \, \pi (R_2^2 - R_1^2) \qquad (36)$$

Problem B5: Carry out a similar calculation for a closed surface with cylindrical sides of radius R, height h, and charge density σ_s, and with flat closed ends of charge density σ_e. What is the total charge Q on the closed surface?

Integral of dot product over a surface

Another important use of the double integral is to sum a vector dot product over a surface. For example, in 8.02, the electric 'flux' Φ_E emerging from an arbitrary surface (open or closed) will be defined as follows:

$$\Phi_E \equiv \iint_{\substack{\text{arbitrary}\\ \text{surface}}} \mathbf{E \cdot dA} \qquad (37)$$

The electric field vector **E** has a direction and magnitude at each point in space. A vector **dA** is defined for each surface element; its magnitude is the value of the area (in m^2) of the element and its direction is *normal* to the element (Fig. 14a). (The magnitude is the quantity we heretofore called dA.) The vector **dA** for a closed surface is always chosen to be directed outward (Figs. 14a,b) rather than inward. For an open surface, it is chosen to be in the direction arbitrarily chosen as 'positive'. The integral (37), as written, is simply the sum of the values of **E·dA** evaluated at each element of the surface. You will learn early in 8.02 the reason this summation is called a flux.

How does one evaluate such an integral as (37)? A dot product, as usual, may be written as the product of the magnitude of one of the vectors (*i.e.,* $dA \equiv |\mathbf{dA}|$) times the value of the projection (component) of the other vector **E** along the direction of the first vector.

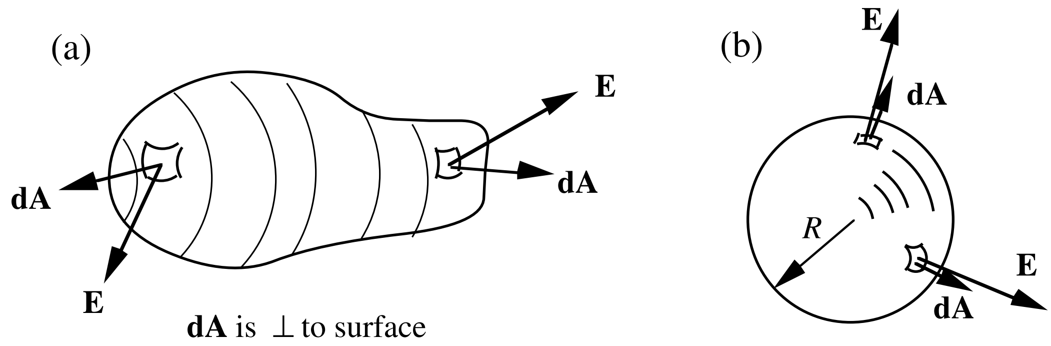

Fig. 14

In our case, $\mathbf{E}\cdot\mathbf{dA} = E_n dA$, where E_n is the component of **E** parallel to **dA**, *i.e.,* normal to the surface element in question, and where the value of **E** is taken at the location of that surface element. In general, the orientation of each surface element is likely to be different as is the direction of **E**. Nevertheless, at each surface element, $E_n dA$ has the same value as the vector dot product (which itself is a scalar) at that position. The integral (37) thus can be written without vector notation:

$$\Phi_E = \iint\limits_{\substack{\text{arbitrary}\\ \text{surface}}} E_n \, dA \tag{38}$$

Dot product evaluated over a sphere

A trick for the evaluation of the integral (38) is to choose (if the problem permits) electric fields and surfaces such that E_n, the normal component of the electric field **E**, has a constant value at every position on the surface. Then E_n may be removed from the integral. A simple example is a spherical closed surface of radius R with a point-like charge at the center. Since **E** is radial, **E** is normal to the surface with a constant magnitude everywhere on the surface (Fig. 14b).

$$\Phi_E = \oiint\limits_{\substack{\text{sphere}\\ \text{radius } R}} \mathbf{E} \bullet \mathbf{dA} = \oiint\limits_{\substack{\text{sphere}\\ \text{radius } R}} E_n \, dA = E_n \oiint\limits_{\substack{\text{sphere}\\ \text{radius } R}} dA = E_n \, 4\pi R^2 \tag{39}$$

One could now substitute the radial dependence of E_n , *i.e.*, $E_n = k\, r^{-2}$, into this result.

Sometimes the closed surface is arranged so part of it is everywhere parallel to the **E** field; hence $E_n = 0$. The closed surface would thus consist of two or more open surfaces, some with $E_n = 0$ and one (typically) with E_n constant over the entire (open) surface. The integral (38) is broken into two or more pieces, one for each of the open surfaces, and those with $E_n = 0$ give zero. The other yields a value for E_n, as you will learn.

Remember that flux may be defined for both open and closed surfaces. We present a closed–surface example because closed surfaces are a fundamental part of Gauss's Law which you will encounter in the 2nd week of the course.

END of MATERIAL NEEDED IN EARLY WEEKS OF 8.02

TRIPLE INTEGRALS
(Useful later in course.)

Volume (triple) integrals are used very little in 8.02. If you do encounter them, it will probably be quite late in the course, after you have covered the material in 18.02. Nevertheless, for completeness, we describe them here briefly.

Volume calculated with a triple integral

A triple (3–dimensional) integral is a direct extension of the ideas presented above. For instance, the volume V of an arbitrary solid (Fig. 15a) may be written

(40) $$V = \iiint_{\substack{\text{arbitrary}\\ \text{volume}}} dV = \iiint_{\substack{\text{arbitrary}\\ \text{volume}}} dx\, dy\, dz$$

The limits to the integrals would consist of functions that define the boundaries. This formulation can be shown to be equivalent to the way volumes are calculated with double integrals.

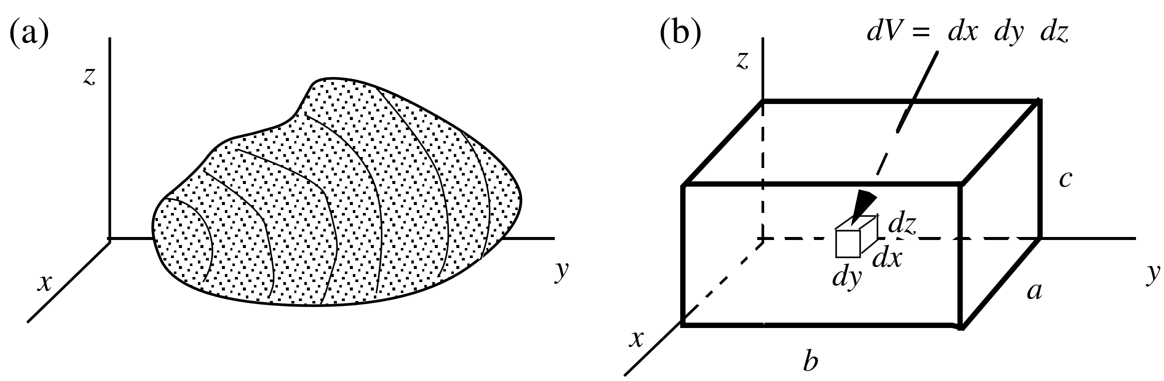

Fig. 15

A trivial example of a volume calculation is a box with sides *a, b*, and *c* (Fig. 15b). In this case the volume calculation is:

(41) $$V = \iiint_{\substack{\text{volume}\\ \text{of box}}} dV = \int_0^c \int_0^b \int_0^a dx\, dy\, dz = \int_0^c \left\{ \int_0^b \left[\int_0^a dx \right] dy \right\} dz$$

Carrying out the integration, inner integral in square brackets first, easily yields the obviously correct answer.

(42) $$V = a\ b\ c$$

Charge in a volume

The total charge in a solid may be calculated analogously to our 2–dimensional calculations. If $\rho(x,y,z)$ is the charge volume density (C/m^3), for the volume shown in Fig. 15b,

(43) $$Q = \int_0^c \int_0^b \int_0^a \rho\,(x,y,z)\; dV = \int_0^c \int_0^b \int_0^a \rho\,(x,y,z)\; dx\,dy\,dz \qquad \text{(Coulombs)}$$

This integration is simply a summation of the charge elements, evaluated at each volume element. If $\rho(x,y,z)$ is a well behaved function that can be integrated, one can proceed to determine the total charge. If, for instance, the density is uniform (constant) throughout the volume, $\rho(x,y,z) = \rho_0$, then, from (43):

(44) $$Q = \int_0^c \int_0^b \int_0^a \rho_0\,(x,y,z)\; dV = \rho_0 \int_0^c \int_0^b \int_0^a dV = \rho_0\, abc$$

The result, of course, is as expected.

The following problem applies to the case when $\rho(x,y,z)$ is not constant. The integration however is straightforward.

Problem B6 (Optional): The volume density of charge (C/m^3) in the rectangular volume of Fig. 15b is $\rho(x,y,z) = k\, x^2 y^3 z^{1/2}$ where k is a constant. What is the total charge Q contained in the volume?

To close out this discussion, we present here the volume element dV in *spherical* coordinates; it should be derived and illustrated in your math text.

(45) $$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

(Again, we adopt the physics convention that θ is the polar angle and ϕ the azimuthal angle.)

The final problem makes use of (45) and provides exercise in the evaluation of a 3–dimensional integral.

Problem B7 (Optional, advanced problem): Find the volume of a sphere of radius R. Use the volume element (45) and evaluate the integral: $V = \int\int\int dV$ over the entire volume of the sphere.

END