

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Physics

Physics 8.01

Fall 2000

Exam 1 Solutions

Friday, September 22, 2000

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FAMILY (Last) NAME

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GIVEN (First) NAME

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Student ID Number

Solutions written by Alan Guth

Your Recitation (check one) →

Instructions:

1. SHOW ALL WORK. All work must be done in this booklet. Extra blank pages are provided.
2. This is a closed book exam.
3. CALCULATORS, BOOKS, and NOTES are NOT ALLOWED.
4. Do all FOUR (4) problems.
5. Exams will be collected 5 minutes before the hour.

NOTE: In all problems, your answer must be expressed **ONLY** in terms of given symbols. Not all the given symbols will necessarily be needed.

Problem	Maximum	Score	Grader
1	25		
2	25		
3	25		
4	25		
TOTAL	100		

R01	MW 1:00	W. Bertozzi	
R02	MW 2:00	W. Bertozzi	
R03	MW 3:00	W. Bertozzi	
R12	TR 1:00	A. Bolton	
R18	TR 9:00	B. Burke	
R19	TR 10:00	B. Burke	
R20	TR 11:00	B. Burke	
R21	TR 2:00	M. Evans	
R22	TR 3:00	M. Evans	
R06	MW 2:00	M. Feld	
R07	MW 3:00	M. Feld	
R08	MW 4:00	M. Feld	
R16	TR 11:00	D. Fernie	
R15	TR 10:00	A. Guth	
R23	TR 11:00	J. Hager	
R24	TR 12:00	J. Hager	
R09	MW 1:00	P. Joss	
R10	MW 2:00	P. Joss	
R11	MW 3:00	P. Joss	
R17	TR 12:00	K. Lee	
R25	TR 1:00	K. Lee	
R26(M)	TR 3:00	McBride/Bove	
R13	TR 2:00	J. Shelton	
R14	TR 3:00	J. Shelton	
R04	MW 1:00	G. Stephans	
R05	MW 2:00	G. Stephans	

EXAM 1
FORMULA SHEET
Friday, September 22, 2000

Equations introduced in Chapter 1:

$$\vec{v} = \frac{d\vec{r}}{dt}; \quad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}; \quad \vec{r}(t_1) = \vec{r}_0 + \int_0^{t_1} \vec{v} dt; \quad \vec{v}(t_1) = \vec{v}_0 + \int_0^{t_1} \vec{a} dt .$$

For *constant* acceleration \vec{a} , if $\vec{r} = \vec{r}_0$ and $\vec{v} = \vec{v}_0$ at time $t = 0$, then

$$\vec{v}(t) = \vec{v}_0 + \vec{a}t$$
$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0t + \frac{1}{2}\vec{a}t^2 .$$

For one-dimensional motion with constant acceleration a :

$$v^2 = v_0^2 + 2a(x - x_0) .$$

For circular motion at constant speed v :

$$a = \frac{v^2}{r} ,$$

where r is the radius of the circle, and the acceleration is directed towards the center of the circle.

If an object has position \vec{r} and velocity \vec{v} , its position and velocity relative to an observer with position \vec{r}_0 and velocity \vec{v}_0 are given respectively by

$$\vec{r}' = \vec{r} - \vec{r}_0 , \quad \vec{v}' = \vec{v} - \vec{v}_0 .$$

Average velocity and acceleration are given by

$$\vec{v}_{\text{average}} = \frac{\Delta\vec{r}}{\Delta t} , \quad \vec{a}_{\text{average}} = \frac{\Delta\vec{v}}{\Delta t} .$$

Equations introduced in Chapter 2:

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} \quad (\text{Newton's second law});$$

$$\vec{\mathbf{F}} = -\frac{GMm}{r^2}\hat{\mathbf{r}} \quad (\text{the gravitational force between two particles});$$

$$\vec{\mathbf{F}} = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}\hat{\mathbf{r}} \quad (\text{the electrostatic force between two particles});$$

$$\vec{\mathbf{F}} = -k\vec{\mathbf{x}} \quad (\text{Hooke's law});$$

$$\frac{d^2x}{dt^2} = -\omega^2x \quad (\text{for a particle near a point of stable equilibrium; equation leads to simple harmonic motion});$$

$$x = A\sin\omega t \quad (\text{a solution to the above equation; any solution can be written this way if we choose } t = 0 \text{ when } x = 0);$$

$$\omega = 2\pi f \quad (\text{relation between angular frequency and frequency});$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \quad (\text{period of an oscillator}).$$

1. (Multiple Choice problems; 5 points each part, no partial credit.) If your answer is “none of these”, please state which other answer is the closest to being correct. Also explain why it is not correct.

a) A car of mass m is traveling on a horizontal road with constant velocity. The magnitude of this velocity is a . Assuming the local acceleration of gravity is g , the magnitude of the net force on the car is:

- i) mg ii) ma iii) $mg + ma$ iv) $ma - mg$ v) $mg - ma$
 vi) $\frac{ma + mg}{2}$ vii) 0 viii) none of these

Solution:

(No explanation was necessary for the multiple choice questions, but for pedagogical purposes these solutions will include explanations.) We are told that the car is traveling with a constant velocity. If the velocity is constant there is no acceleration, and hence no net force acting. Therefore (vii) is the correct answer.

b) An astronaut of mass m is a member of the crew of a space shuttle orbiting the earth at an altitude h . If the radius of the earth is R and g is the acceleration due to gravity at the surface of the earth, the magnitude of the gravitational force on the astronaut is:

- i) mg ii) $mg \frac{R}{R+h}$ iii) $mg \frac{(R+h)}{R}$ iv) $mg \frac{R^2}{(R+h)^2}$ v) $mg \frac{(R+h)^2}{R^2}$
 vi) none of these

Solution:

The gravitational force on a point mass m at a radius r from the center of the spherical body of mass M is given by

$$\vec{\mathbf{F}} = -\frac{GMm}{r^2} \hat{\mathbf{r}} ,$$

where $\hat{\mathbf{r}}$ is a unit vector pointing radially outward from the spherical body. Thus, for the astronaut,

$$|\vec{\mathbf{F}}| = \frac{GM_E m}{r_A^2} , \tag{1}$$

where M_E is the mass of the Earth, m is the mass of the astronaut (well-approximated as a point mass), and r_A is the distance of the astronaut from the center of the Earth. If we knew the values of G and M_E we could evaluate Eq. (1) directly, but the problem gives us only choices written in terms of other variables.

When the astronaut is on the ground, at radius R , the gravitational force acting on him has magnitude mg . Thus, Eq. (1) implies

$$\frac{GM_E m}{R^2} = mg . \quad (2)$$

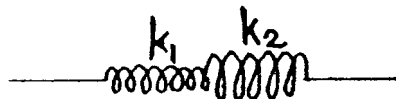
Combining Eqs. (1) and (2), the magnitude of the gravitational force on the astronaut at any radius is given by

$$|\vec{\mathbf{F}}| = \left(\frac{R}{r_A}\right)^2 mg . \quad (3)$$

We are asked for the gravitational force at an altitude h , which means a distance $r_A = R + h$ from the center of the Earth. For this case Eq. (3) is identical to answer (iv), which is the answer.

The answer above is the one that we wanted, but it ignores one subtlety: the surface of the Earth is not exactly an inertial frame. g is defined as the acceleration relative to the surface of the Earth that is experienced by a freely falling body, and mg is then called the weight of the body. In a non-inertial frame mg is not equal to the gravitational force, and therefore Eq. (2) is not exactly true. Thus, a more precise answer would be to check (vi), “none of these,” and to explain this subtlety.

c) Two springs are connected as shown below:



Both obey Hooke's Law. One has spring constant k_1 and the other k_2 . The spring constant of the combined system is:

- i) $k_1 + k_2$ ii) $k_1 k_2$ iii) $\frac{k_1 k_2}{k_1 + k_2}$ iv) $\frac{k_1 + k_2}{k_1 k_2}$ v) $\frac{k_1 + k_2}{2}$
 vi) $2(k_1 + k_2)$ vii) none of these

Solution:

Suppose that the springs are stretched, with spring 1 stretched by a distance $\Delta\ell_1$ beyond its natural length, and spring 2 by a distance $\Delta\ell_2$ beyond its natural length. (The natural length of a spring is the length at which it exerts no force.) If we introduce an x -axis that runs along the springs, with positive x to the right, then

Hooke's law implies that spring 2 exerts a force on the object at its right end given by

$$F_x^{(2,R)} = -k_2 \Delta\ell_2 ,$$

and spring 1 exerts a force on the object at its left end given by

$$F_x^{(1,L)} = k_1 \Delta\ell_1 .$$

Note that although the above formula has no minus sign, it corresponds to the restoring force with which we are familiar—the left end of the stretched spring pulls to the right. It's just a little unfamiliar to write equations to describe the left end of the spring.

Now consider the right end of spring 1 and the left end of spring 2, both of which are attached to the same small piece of metal that joins the two springs. According to Hooke's law, the forces that each spring exerts on this small piece of metal are:

$$F_x^{(1,R)} = -k_1 \Delta\ell_1 ,$$

$$F_x^{(2,L)} = k_2 \Delta\ell_2 .$$

But in equilibrium the net force on this piece of metal must be zero, so

$$k_1 \Delta\ell_1 = k_2 \Delta\ell_2 .$$

Since $k_1 \Delta\ell_1$ and $k_2 \Delta\ell_2$ are equal, we can use the symbol F to represent either one of them.

Now we want to think of the two springs as one combined system of length

$$\ell = \ell_1 + \ell_2 ,$$

which exerts a force

$$F_x^R = -F$$

at its right end and

$$F_x^L = F$$

at its left end. Note that the change in length of the combined system from its natural length is

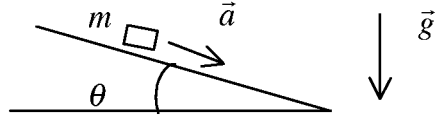
$$\Delta\ell = \Delta\ell_1 + \Delta\ell_2 .$$

The effective spring constant for the combined system is then

$$k = \frac{F}{\Delta\ell} = \frac{F}{\Delta\ell_1 + \Delta\ell_2} = \frac{F}{\frac{F}{k_1} + \frac{F}{k_2}} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} = \frac{k_1 k_2}{k_1 + k_2} .$$

So the correct answer is (iii).

- d) A block of mass m slides down a frictionless slope inclined at an angle θ to the horizontal.

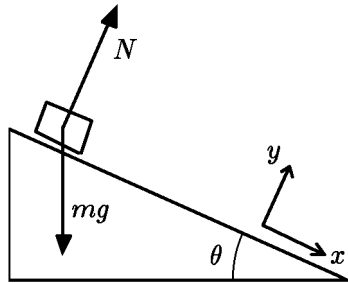


The magnitude of the acceleration of the block is observed to be a . From this observation you can conclude that the magnitude of the local acceleration due to gravity is:

- i) $ma \sin \theta$ ii) $ma \cos \theta$ iii) $\frac{ma}{\sin \theta}$ iv) $\frac{ma}{\cos \theta}$ v) $a \sin \theta$
 vi) $a \cos \theta$ vii) $\frac{a}{\sin \theta}$ viii) $\frac{a}{\cos \theta}$ ix) none of these

Solution:

Inclined planes are treated most easily with a coordinate system which is inclined at the same angle. The forces acting on the block are gravity, of magnitude mg and downward, and also the normal force of the slope on the block, which is perpendicular to the plane and has whatever magnitude is needed to prevent the block from penetrating the surface. We call its magnitude N . Drawing a force diagram,



one can resolve the forces acting to find

$$F_x = mg \sin \theta$$

$$F_y = N - mg \cos \theta .$$

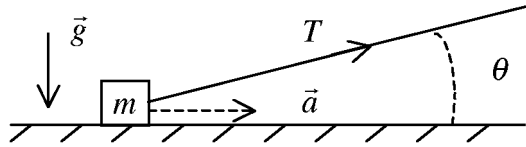
The normal force N adjusts itself so that there is no acceleration in the y -direction, so $N = mg \cos \theta$. The acceleration down the slope is in the x -direction, with

$$a_x = \frac{F_x}{m} = g \sin \theta .$$

We are given a_x and asked to find g , so

$$g = \frac{a_x}{\sin \theta} \implies \text{The correct answer is } \boxed{\text{(vii)}} .$$

- e) A large mass m , at the end of a rope, is pulled on level ice. The mass of the rope and the friction between the mass and the ice are negligible. The rope makes an angle θ to the horizontal. If the mass has an acceleration \vec{a} and the acceleration due to gravity is \vec{g} , the tension T in the rope is:



- i) $\frac{ma}{\cos \theta}$ ii) $\frac{ma}{\sin \theta}$ iii) $ma \cos \theta$ iv) $ma \sin \theta$ v) $mg \sin \theta$
vi) $mg \cos \theta$ vii) $\frac{ma}{\cos \theta} + mg \sin \theta$ viii) none of these

Solution:

Defining a coordinate system in which the x -axis is horizontal, the acceleration is entirely in the x -direction: $\vec{a} = [a, 0, 0]$. Both gravity and the normal force are strictly in the y (vertical) direction, so they do not contribute. The tension force must be resolved, so

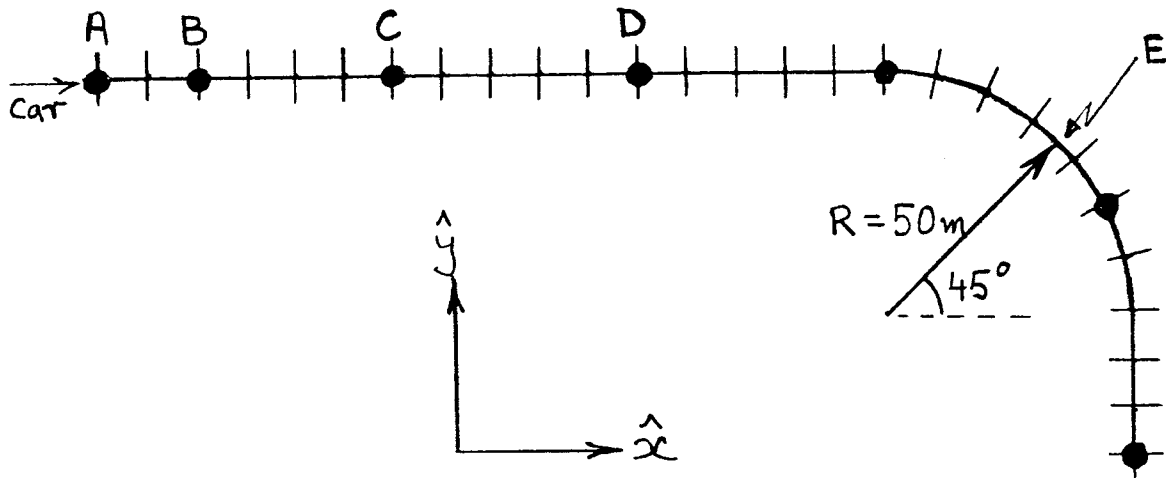
$$F_x = T \cos \theta = ma .$$

We are asked for T , so

$$T = \frac{ma}{\cos \theta} \implies \text{The correct answer is } \boxed{(i)} .$$

2. (25 points)

The figure below illustrates a stretch of a horizontal road. The markers (\perp) indicate distance along the road separated by 10 m. At A a car enters the stretch of road. It accelerates uniformly up to point C, and then continues moving at constant speed. The dots show the position of the car at one second intervals. Using the coordinate system shown, write down expressions for the velocity and acceleration of the car at points B, D, and E.

**Solution:**

In uniform acceleration the velocity changes linearly with time, so the average velocity over any time interval is precisely equal to the instantaneous velocity at the midpoint of the time interval. So the velocity at B is same as the average velocity between A and C, which are one second earlier and later than B, respectively. The distance between A and C is 6 markers, or 60 m, so the average speed during this interval is

$$\frac{60 \text{ m}}{2 \text{ s}} = 30 \frac{\text{m}}{\text{s}} .$$

The direction is along the x-axis, so

$$\vec{v}_B = [30, 0, 0] \frac{\text{m}}{\text{s}} .$$

To find the acceleration, we can determine the velocity at 0.5 s before and after B. The velocity at 0.5 s before B is equal to the average velocity between A and B, so

$$\vec{v}(t = t_B - 0.5 \text{ s}) = [20, 0, 0] \frac{\text{m}}{\text{s}} ,$$

and

$$\vec{v}(t = t_B + 0.5 \text{ s}) = [40, 0, 0] \frac{m}{s} .$$

The acceleration, which is uniform from A to C, is then

$$\vec{a}_B = \frac{\Delta \vec{v}}{\Delta t} = \frac{([40, 0, 0] - [20, 0, 0]) \frac{m}{s}}{1 \text{ s}} = \boxed{[20, 0, 0] \frac{m}{s^2} .}$$

From C onward the speed is constant, so we can find it by using the 1 s interval between C and D, which is 5 markers or 50 m long. So the speed is 50 m/s. At D the velocity is to the right, and unchanging. Since the speed and direction are unchanging at D, the acceleration is zero:

$$\boxed{\vec{v}_D = [50, 0, 0] \frac{m}{s} , \quad \vec{a}_D = 0 .}$$

At E we have uniform speed with circular motion. The acceleration has magnitude

$$a = \frac{v^2}{R} = \frac{(50 \frac{m}{s})^2}{50 \text{ m}} = 50 \frac{m}{s^2} .$$

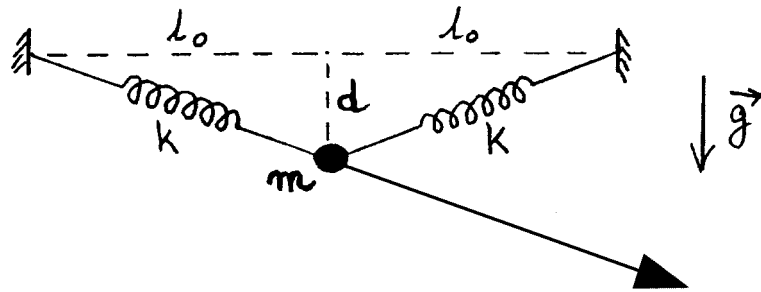
The velocity is at 45° downward and to the right. The acceleration is toward the center of the circle, which is 45° downward and to the left. Using $\sin 45^\circ = \cos 45^\circ = 1/\sqrt{2}$, one has

$$\boxed{\vec{v}_E = \frac{1}{\sqrt{2}} [50, -50, 0] \frac{m}{s} , \quad \vec{a}_E = \frac{1}{\sqrt{2}} [-50, -50, 0] \frac{m}{s^2} .}$$

3. (25 points)

A mass m is stationary, supported from two identical massless springs as shown. Each spring obeys Hooke's Law and has unstretched length ℓ_0 and constant k . Suddenly the left spring breaks. Immediately after the spring breaks, the mass accelerates with acceleration \vec{a} .

a) On the diagram show the direction of \vec{a} . Explain in words, or by calculations, what is the direction of \vec{a} and how you established it.



Solution:

In equilibrium before the spring breaks, the force that the left spring exerts on the mass is exactly canceled by the combined force of the other spring and gravity. Once the spring breaks the mass feels only the combined force of gravity and the right spring, but this force must still be equal in magnitude but opposite in direction to the force that the left spring exerted before it broke. The left spring was stretched and therefore exerted a force along its length upward and to the left, so the force of gravity and the right spring must sum to a force that points along the left spring, but in the direction downward and to the right.

b) Write an expression for the magnitude of \vec{a} .

Solution:

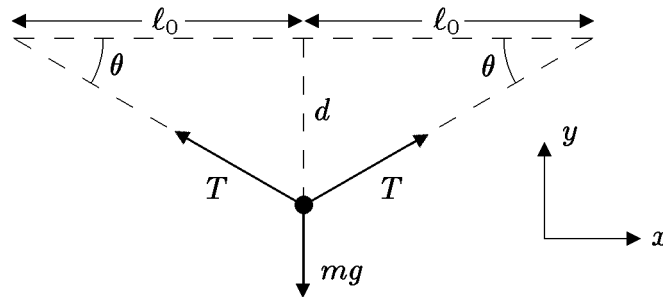
By the previous argument, the magnitude of the net force acting after the spring breaks is the same as the magnitude of the force that the left spring exerted before it broke. By Hooke's law, this magnitude is just

$$|\vec{F}| = k \Delta \ell = k \left(\sqrt{\ell_0^2 + d^2} - \ell_0 \right) .$$

The acceleration is then

$$|\vec{a}| = \frac{|\vec{F}|}{m} = \frac{k}{m} \left(\sqrt{\ell_0^2 + d^2} - \ell_0 \right) .$$

There is another possible answer to this problem, since the information given was really more than is necessary to solve the problem. Even if the spring constant k were not given, one could find the magnitude $T \equiv |\vec{\mathbf{F}}|$ of the force exerted by each spring from the following force diagram:



Although θ is not given, one can see from the diagram (using the Pythagorean theorem) that

$$\sin \theta = \frac{d}{\sqrt{\ell_0^2 + d^2}} \quad \cos \theta = \frac{\ell_0}{\sqrt{\ell_0^2 + d^2}} .$$

Before the spring breaks, in equilibrium, there is no force in the vertical direction, so

$$F_y = 2T \sin \theta - mg = 0 ,$$

so

$$T = \frac{mg}{2 \sin \theta} = \frac{mg}{2} \frac{\sqrt{\ell_0^2 + d^2}}{d} .$$

Thus, the magnitude of the acceleration can be written

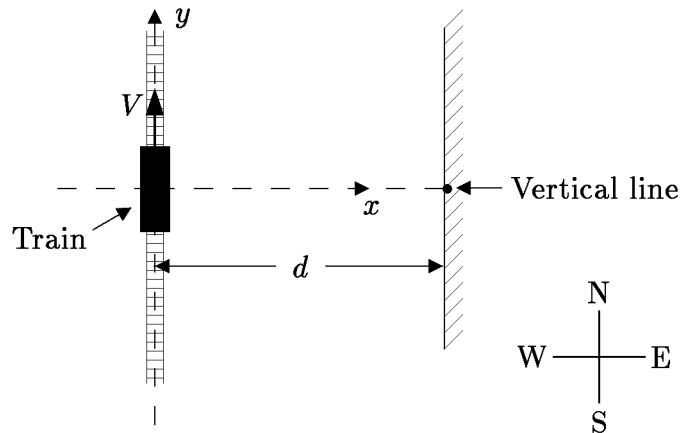
$$|\vec{\mathbf{a}}| = \frac{T}{m} = g \frac{\sqrt{\ell_0^2 + d^2}}{2d} .$$

4. (25 points)

A toy train moves due north at a constant speed V along a straight track which is parallel to the wall of a room. The wall is to the east of the track at a distance d . There is a toy dart gun on the train with its barrel fixed in a plane perpendicular to the motion of the train. The gun points at an angle θ to the horizontal. There is a vertical line drawn on the wall, stretching from floor to ceiling, and the dart gun is fired at the instant when the line is due east of the gun. If the dart leaves the gun at speed W relative to the gun, derive an expression for the distance by which the dart misses the vertical line. That is, find how far north or south of the vertical line is the point at which the dart hits the wall. Assume that the dart hits the wall without first bouncing off the floor or ceiling.

Solution:

To describe the trajectory, it is useful to adopt a coordinate system. As shown in the diagram, we let the train tracks run along the y axis with the positive y -direction as north, with the x -axis pointing to the east. The wall lies at $x = d$, and the vertical line lies at $(x, y) = (d, 0)$. The z -axis points up.



The dart is then fired from the origin, with initial velocities in the x - and z -directions given by resolving the dart gun velocity \vec{W} :

$$v_{0,x} = W \cos \theta, \quad v_{0,z} = W \sin \theta.$$

But the dart gun is moving with the train, so the dart has an initial velocity in the y -direction

$$v_{0,y} = V.$$

There is no acceleration in either the x - or y -directions, so the trajectory of the dart obeys

$$x = v_{0,x}t = Wt \cos \theta$$

$$y = v_{0,y}t = Vt.$$

The motion in the z -direction is irrelevant to this problem, as long as we are told that the dart does not hit the ceiling or floor. The dart hits the wall when $x = d$, which occurs at time

$$Wt \cos \theta = d \implies t = \frac{d}{W \cos \theta} .$$

Thus, the y -coordinate when the dart hits the wall is

$$y = Vt = d \frac{V}{W \cos \theta} .$$

The dart therefore hits the wall to the north of the vertical line by this amount.