Communication with Side Information at the Transmitter

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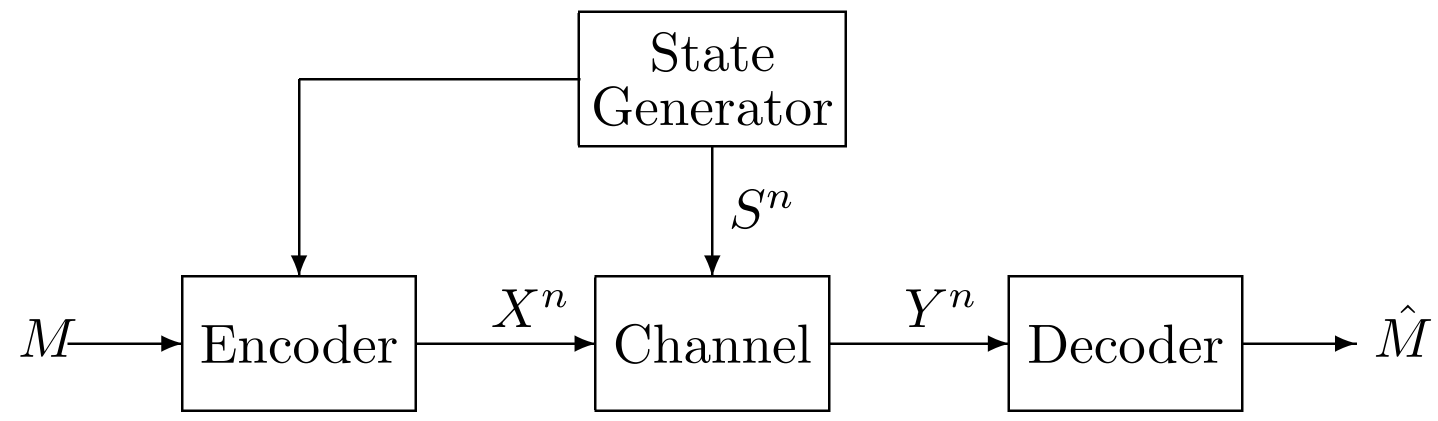
Outline

- Basic model of communication with side info at the transmitter
  - Causal vs. non-causal side information
  - Examples
- Relationship with watermarking and other problems
- Capacity results
- Writing on dirty paper and extensions

Basic Model



- Message M uniformly distributed in $\{1, \ldots, 2^{nR}\}$.
- State vector S^n generated IID according to $p(s)$.
- Channel memoryless according to $p(y|x, s)$.
- Sets $\mathcal{S}$, $\mathcal{X}$, and $\mathcal{Y}$ are finite.

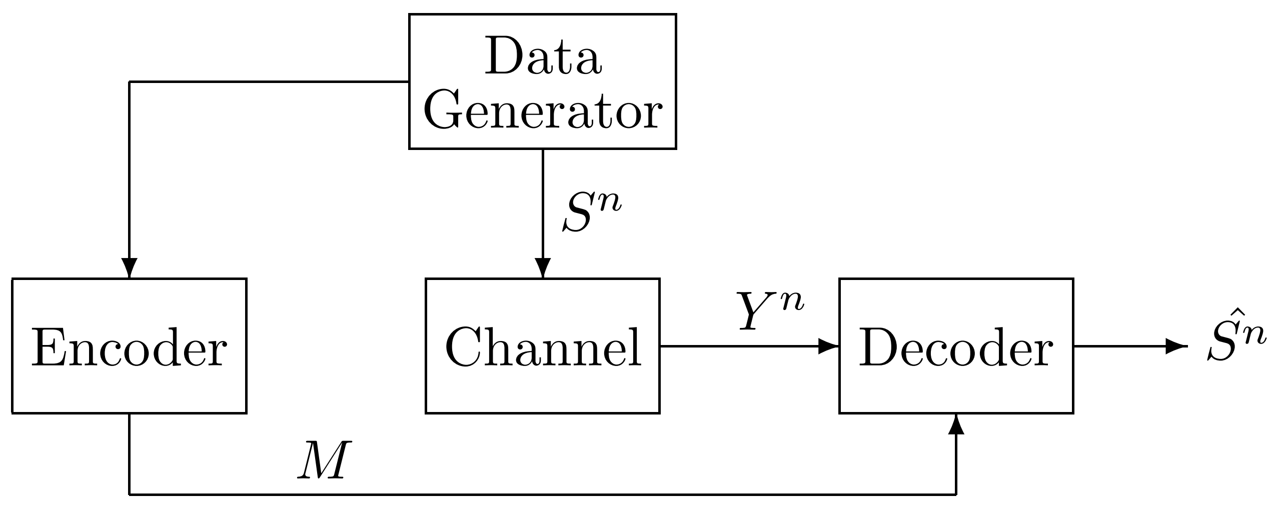
Types of side information

1. Causal side information: x_i depends only on m and s^i.
   - Denote capacity with C_c.
2. Non-causal side information: x^n depends on m and s^n.
   - In particular, x_i depends on m and s^n (the entire state sequence) for all i.
   - Denote capacity with C_{nc}.

Comments:

- $C_{nc} \geq C_c$.
- Non-causal assumption relevant for watermarking.

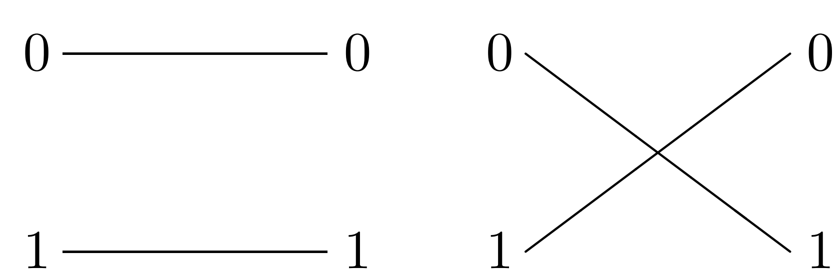
Comparison with last week



- Diagram of "lossy" source coding with side information.
- "Lossless" would require another encoder for Y^n.
- Encoder has non-causal side information.

Example 1

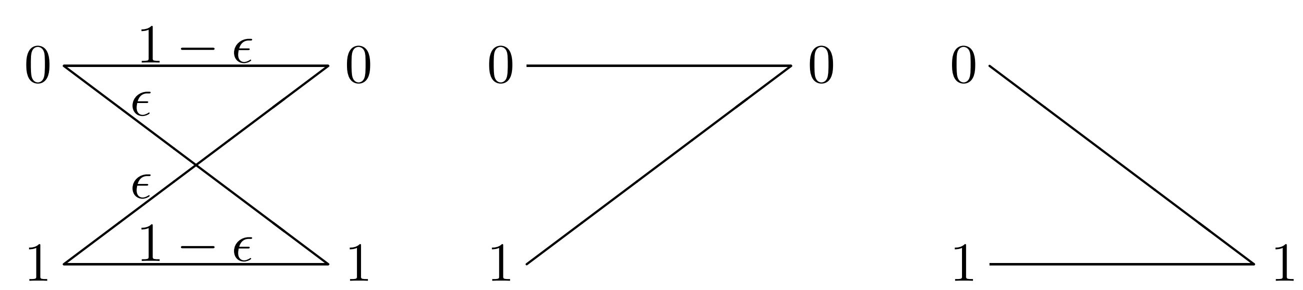
State:	0	1
Prob:	$1-p$	p



- $\mathcal{S} = \mathcal{X} = \mathcal{Y} = \{0, 1\}$.
- $Y_i = X_i + S_i \mod 2$.
- $C_c = C_{nc} = 1$.
- With no side information, capacity is $1 - h(p)$.

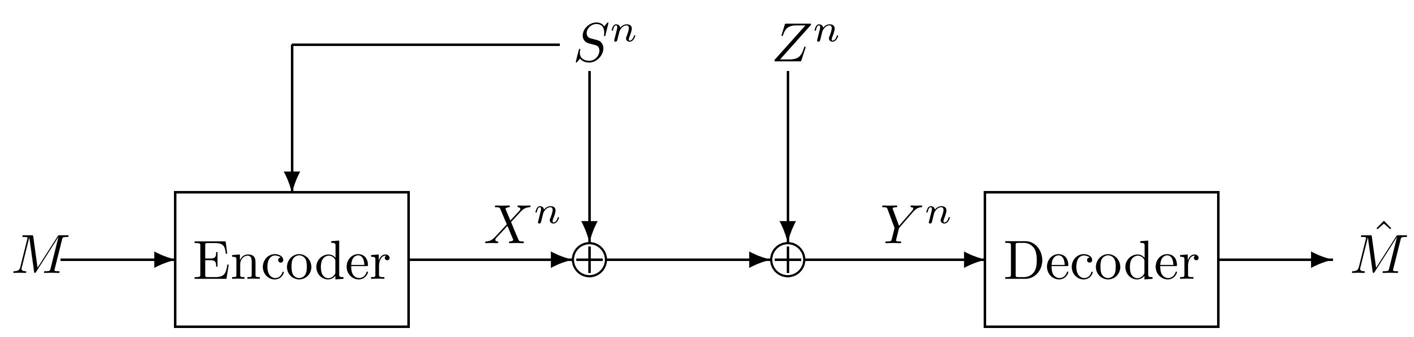
Example 2 : Memory with defects

State:	a	b	c
Prob:	$1-p$	$p/2$	$p/2$



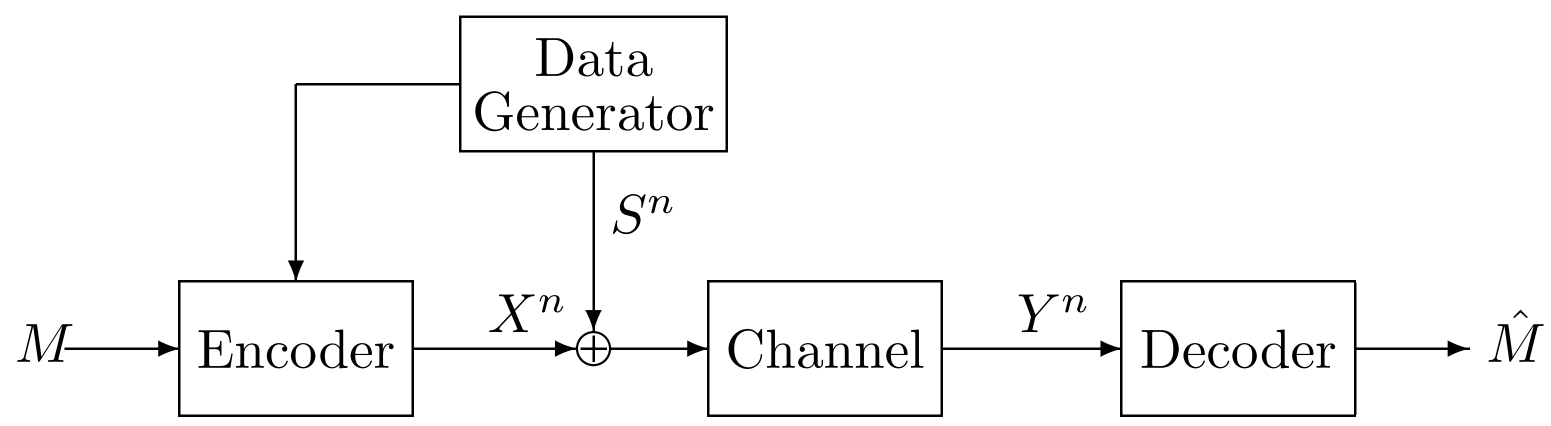
- $\mathcal{S} = \{a, b, c\}$, $\mathcal{X} = \mathcal{Y} = \{0, 1\}$.
- We will see that $C_{nc} > C_c$.

Example 3 : Writing on Dirty Paper



- S^n is IID $\mathcal{N}(0, Q)$.
- Z^n is IID $\mathcal{N}(0, P)$.
- X^n subject to power constraint of P.
- Will show that $C_{nc} = \frac{1}{2} \log \left(1 + \frac{P}{N}\right)$.

Relationship with watermarking



- S^n is original data (e.g. Led Zeppelin song)
- M is information to embed (e.g owner ID number)
- Encoder restricted in choice of X^n.
- Non-causal side information reasonable assumption.
- Might want more general model for "Channel".

Other related problems

Different types of side information:

- At any combination of encoder and decoder.
- Noisy or compressed versions of state sequence.

Different state generators:

- Non-memoryless.
- Non-probabilistic – the arbitrarily varying channel.
- One probabilistic choice then fixed – the compound channel.
- Current state depending on past inputs.

Applications:

- Wireless – fading channels.
- Computer memories.

Capacity results

1. Causal case:

$$C_c = \max_{p(u),\ f:\mathcal{U}\times\mathcal{S}\mapsto\mathcal{X}} I(U;Y),$$

   where U is an auxiliary random variable with $|\mathcal{U}| \leq |\mathcal{Y}|$ and

$$p(s,u,x,y) = \begin{cases} p(s)p(u)p(y|x,s) & \text{if } x = f(u,s) \\ 0 & \text{otherwise} \end{cases}.$$

2. Non-causal case:

$$C_{nc} = \max_{p(u|s),\ f:\mathcal{U}\times\mathcal{S}\mapsto\mathcal{X}} I(U;Y) - I(U;S),$$

   where $|\mathcal{U}| \leq |\mathcal{X}| + |\mathcal{S}|$ and

$$p(s,u,x,y) = \begin{cases} p(s)p(u|s)p(y|x,s) & \text{if } x = f(u,s) \\ 0 & \text{otherwise} \end{cases}.$$

Comments on Capacity Results

- $C_c \leq C_{nc}$.
  - If not, then we are in trouble.
  - Same objective function, but different feasible regions.
- Compare C_{nc} with rate distortion region for "lossy" source coding with side information. Given $p(s, y)$,

$$R(D) = \min_{\substack{p(u|s),\ f:\mathcal{U}\times\mathcal{Y}\mapsto\mathcal{S},\\ E[d(S,f(U,Y))]\leq D}} I(U;Y) - I(U;S),$$

  where $p(u, s, y) = p(s, y)p(u|s)$, which gives the Markov condition $(Y \multimap S \multimap U)$.

Achievability : Causal Side Information

- Larger DMC – Input $\mathcal{X}^{\mathcal{S}}$ and output $\mathcal{Y}$.
- Each input letter is a function from $\mathcal{S}$ to $\mathcal{X}$.
- Only need to use $|\mathcal{Y}|$ of the $|\mathcal{X}|^{|\mathcal{S}|}$ input letters.
- Auxiliary RV U indexes the input letters.
- Example: Memory with defects
  - $t_0(s) = 0$ for all s, $\Pr(Y = 0) = (1-\epsilon)(1-p) + p/2$.
  - $t_1(s) = 1$ for all s, $\Pr(Y = 1) = (1-\epsilon)(1-p) + p/2$.
  - Any other function from $\mathcal{S}$ to $\mathcal{X}$ gives one of these distributions on $\mathcal{Y}$.
  - $C_c = 1 - h(p/2 + \epsilon(1-p))$.

Converse : Causal Side Information

Let $U(i) = (M, S^{i-1})$.

- $(M, Y^{i-1}) -\!\!\circ\!\!- U(i) -\!\!\circ\!\!- Y_i$.
- $U(i)$ and S_i are independent.
- For small probability of error:

$$\begin{aligned} n(R-\delta) &\leq I(M; Y^n) \\ &\leq \sum_{i=1}^{n} I(M, Y^{i-1}; Y_i) \\ &\leq \sum_{i=1}^{n} I(U(i); Y_i) \\ &\leq nC_c, \end{aligned}$$

Achievability : Non-causal Side Information

Use dual to binning technique from last week.

- Choose distribution $p(u|s)$ and function $f : \mathcal{U} \times \mathcal{S} \mapsto \mathcal{X}$.
- Codebook generation:
  - For each $m \in \{1, \ldots, 2^{nR}\}$, generate $\boldsymbol{U}(m, 1), \ldots, \boldsymbol{U}(m, 2^{nR_0})$ IID according to $p(u)$.
  - A total of $2^{n(R+R_0)}$ codewords.
- Encoding:
  - Given m and s^n, find $\boldsymbol{u}(m, j)$ jointly typical with s^n.
  - Set $x^n = f(\boldsymbol{u}(m, j), s^n)$.
- Decoding:
  - Find $(\hat{m}, \hat{j})$ such that $\boldsymbol{u}(\hat{m}, \hat{j})$ jointly typical with y^n.

Achievability : Non-causal Side Information

- Encoding failure small if $R_0 > I(U;S)$
- Decoding failure small if $R + R_0 < I(U;Y)$.
  - Need Markov lemma.
- Rate achievable if $R < I(U;Y) - I(U;S)$.
- Intuition:
  - Codebook bin $\approx$ quantizer for state sequence.
  - If $I(U;S) > 0$, then use non-causal feedback non-trivially.

Example : Memory with defects

- $\mathcal{U} = \{u_0, u_1\}$, $f(u_i, s) = i$.
- Joint distribution of S, U and X:

	$u_0, 0$	$u_1, 1$
a	$(1-p)/2$	$(1-p)/2$
b	$(1-\epsilon)p/2$	$\epsilon p/2$
c	$\epsilon p/2$	$(1-\epsilon)p/2$

- $I(U;S) = H(U) - H(U|S) = 1 - (1-p) - ph(\epsilon) = p(1-h(\epsilon))$.
- $I(U;Y) = H(Y) - H(Y|U) = 1 - h(\epsilon)$.
- $C_{nc} = I(U;Y) - I(U;S) = (1-p)(1-h(\epsilon)) > C_c$.
  - Also capacity when state known at decoder.
  - Mistake in summary.

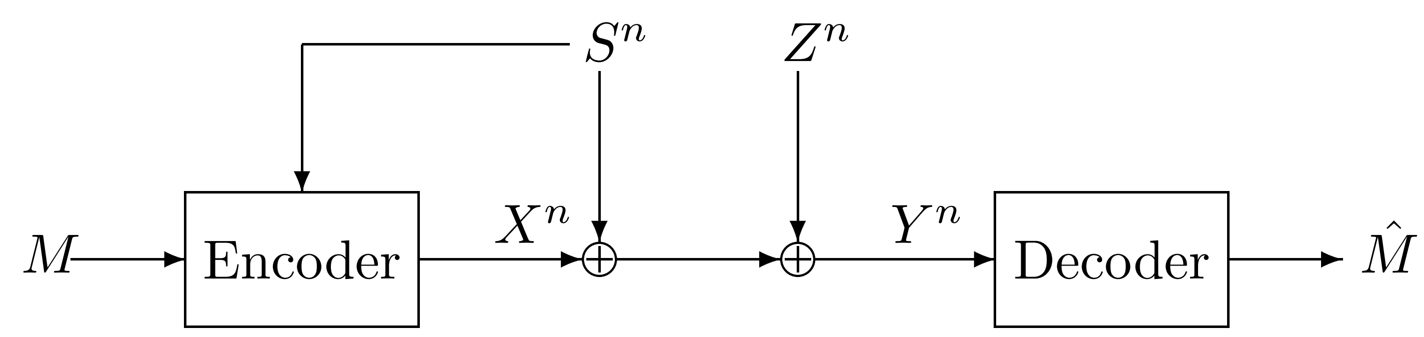
Converse : Non-causal side information

- Let $U(i) = (M, Y_1, \ldots, Y_{i-1}, S_{i+1}, \ldots, S_n)$.
- For small probability of error:

$$\begin{aligned} n(R-\delta) &\leq I(M;Y^n) - I(M;S^n) \\ &\leq \sum_{i=1}^{n} I(U(i);Y_i) - I(U(i);S_i) \\ &\leq nC_{nc} \end{aligned}$$

- Second step: mutual information manipulations.
- Markov chain in causal case not valid here.

Writing on Dirty Paper



- $S_i \sim \mathcal{N}(0, Q)$, $Z_i \sim \mathcal{N}(0, N)$, $\frac{1}{n} \sum X_i^2 \leq P$.
- Costa shows $C_{nc} = \frac{1}{2} \log \left(1 + \frac{P}{N}\right)$.
  - Same as if S^n known to decoder.
  - Dual to Gaussian lossy source coding with side info.

Capacity for Writing on Dirty Paper

- Pick joint distribution on known noise S, input X, and auxiliary random variable U:
  - $X \sim \mathcal{N}(0, P)$, independent of S.
  - $U = X + \alpha S$
- Costa: Compute $I(U; Y) - I(U; S)$ and optimize over α.
- New proof: Choose $\alpha = \frac{P}{P+N}$ and see what happens.
- Important properties:
  1. $X - \alpha(X + Z)$ and $X + Z$ are independent.
  2. $X - \alpha(X + Z)$ and $Y = X + S + Z$ are independent.
  3. X has capacity achieving distribution for AWGN channel.
- Cannot do better than $C(P, N) = \frac{1}{2} \log \left(1 + \frac{P}{N}\right)$.

Writing on Dirty Paper, continued

- Step 1

$$
\begin{aligned}
I(U;Y) - I(U;S) &= \big(h(U) - h(U|Y)\big) - \big(h(U) - h(U|S)\big) \\
&= h(U|S) - h(U|Y)
\end{aligned}
$$

- Step 2

$$
\begin{aligned}
h(U|S) &= h(X + \alpha S|S) \\
&= h(X|S) \\
&= h(X) && X \text{ and } S \text{ independent}
\end{aligned}
$$

Writing on Dirty Paper, continued

- Step 3

$$\begin{aligned}
h(U|Y) &= h(X+\alpha S|Y) \\
&= h\big(X+\alpha(S-Y)|Y\big) \\
&= h\big(X-\alpha(X+Z)|Y\big) \\
&= h\big(X-\alpha(X+Z)\big) && \text{Property 2} \\
&= h\big(X-\alpha(X+Z)|X+Z\big) && \text{Property 1} \\
&= h(X|X+Z)
\end{aligned}$$

- Step 4

$$\begin{aligned}
I(U;Y)-I(U;S) &= h(X)-h(X|X+Z) && \text{Steps 1, 2 \& 3} \\
&= I(X;X+Z) \\
&= C(P,N) && \text{Property 3}
\end{aligned}$$

Extension of "Writing on Dirty Paper"

For any distributions on S and Z, similar result if there exists X such that both

- X is capacity achieving for channel with additive noise Z.
- $X - a(X + Z)$ and $X + Z$ independent for some linear $a(\cdot)$.

In particular,

- S can have any (power-limited) distribution.
- Z can be colored Gaussian.
  - Capacity achieving distribution also Gaussian (waterfilling).

Similar extension given by Erez, Shamai & Zamir '00.

Writing on Dirty Tape

What about C_c for this problem?

- Only definitive result (Erez *et. al.*):

$$\lim_{N \to 0} \lim_{Q \to \infty} C_{nc} - C_c = \frac{1}{2} \log \left(\frac{\pi e}{6} \right)$$

- $\frac{\pi e}{6}$ = ultimate "shaping gain"
  - Asymptotic MSE difference of vector vs. scalar quantization.
- Suggested scheme: Codewords as sequences of scalar quantizers.
  - Version of Quantization Index Modulation (Brian Chen).
- Any ideas for how to find capacity non-asymptotically?