Communication through wireless channel

- Fading
- Diversity: temporal, spatial, frequency

Tarokh et al (1998, 1999): space-time coding

- Concentrate on block codes
- Quasistatic fading (block fading)
- Perfect channel information at receiver

Model parameters

- n: # Tx antennae.
- m: # Rx antennae
- $\alpha_{i,j}$: path gain from Tx i to Rx j

  Assume $\alpha_{i,j}$'s are IID $\sim \mathcal{N}(0, 1/2 \text{ per dim})$.
- l: block length
- c_t^i: Tx signal at time t by Tx antenna i
- r_t^j: Rx signal at time t by Rx antenna j
- η_t^j: AWGN $\sim \mathcal{N}(0, 1/SNR \text{ per dim})$

Model (cont.)

- Codeword $\mathbf{c} = (c_1^1, ..., c_l^1, c_1^2, ..., c_l^2, ..., c_1^n, ..., c_l^n)$
- Receiver obtains

$$r_t^j \;=\; \sum_{i=1}^{n} \alpha_{i,j} c_t^i + \eta_t^j$$

Optimal decision: choose $\mathbf{c}$ that minimizes

$$\sum_{t=1}^{l} \sum_{j=1}^{m} \left| r_t^j - \sum_{i=1}^{n} \alpha_{i,j} c_t^i \right|^2$$

Derivation of performance criteria

- Using $\mathcal{Q}(x) \leq \exp(-x^2/2)$,

$$P(\mathbf{c} \to \mathbf{e} | \alpha_{i,j}) \;\leq\; \exp\left[-\frac{SNR}{2} d^2(\mathbf{c}, \mathbf{e}) \right], \quad \text{where}$$

$$d^2(\mathbf{c}, \mathbf{e}) \;=\; \sum_{j=1}^{m} \sum_{t=1}^{l} \left| \sum_{i=1}^{n} \alpha_{i,j} c_t^i - \sum_{i=1}^{n} \alpha_{i,j} e_t^i \right|^2$$

- $\Omega_j \cong (\alpha_{i,j}, ..., \alpha_{n,j})$,

$A_{i,i'} \cong (c_1^i - e_1^i, ..., c_l^i - e_l^i) \cdot (c_1^{i'} - e_1^{i'}, ..., c_l^i - e_l^{i'})$

$\Rightarrow \; d^2(\mathbf{c}, \mathbf{e}) = \Sigma_{j=1}^{m} \Omega_j A \Omega_j^*$

$$P(\mathbf{c} \to \mathbf{e} | \alpha_{i,j}) \;\leq\; \exp\left[-\frac{SNR}{2} \sum_{j=1}^{m} \Omega_j A \Omega_j^* \right]$$

Derivation of performance criteria (cont.)

$$P(\mathbf{c} \rightarrow \mathbf{e}|\alpha_{i,j}) \;\leq\; \exp\left[-\frac{SNR}{2}\sum_{j=1}^{m}\Omega_j A \Omega_j^*\right]$$

- $\lambda_1, ..., \lambda_n \cong$ eigenvalues of A (possibly 0)

  A is Hermatian $\Rightarrow \lambda_i$'s are real and nonnegative

  $r \cong$ rank of A

- By averaging over $\alpha_{i,j}$'s and with some algebra,

$$P(\mathbf{c} \rightarrow \mathbf{e}) \;\leq\; \left(\prod_{i=1}^{r}\lambda_i\right)^{-m}\left(\frac{SNR}{2}\right)^{-rm}$$

  rm: diversity advantage (diversity order)

  $(\Pi_{i=1}^{r}\lambda_i)^{1/r}$: coding advantage

- Concentrate on diversity order (why?)

Derivation of performance criteria (cont.)

$$B(\mathbf{c},\mathbf{e}) \cong \begin{bmatrix} e_1^1 - c_1^1 & e_2^1 - c_2^1 & \cdots & e_l^1 - c_l^1 \\ e_1^2 - c_1^2 & e_2^2 - c_2^2 & \cdots & e_l^2 - c_l^2 \\ \vdots & \vdots & & \vdots \\ e_1^n - c_1^n & e_2^n - c_2^n & \cdots & e_l^n - c_l^n \end{bmatrix}$$

- $B(\mathbf{c},\mathbf{e})$ is a square root matrix of $A(\mathbf{c},\mathbf{e})$, i.e. $A = BB^*$

  $\text{rank}(B) = \text{rank}(A)$

- The rank criterion:

  For maximum diversity order nm,

  make $B(\mathbf{c},\mathbf{e})$ full rank for all $(\mathbf{c},\mathbf{e})$ pairs.

Real orthogonal designs

Definition: A real orthogonal design of size n is an $n \times n$ orthogonal matrix whose rows are permutations of real numbers $\pm x_1, ..., \pm x_n$.

$$\text{e.g.} \quad \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix}, \qquad \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & -x_4 & x_3 \\ -x_3 & x_4 & x_1 & -x_2 \\ -x_4 & -x_3 & x_2 & x_1 \end{bmatrix}$$

- WLG, 1st row is $(x_1, ..., x_n)$.
- Hurwitz-Radon theory: real orthogonal designs exist if and only if $n = 2, 4$, or 8.

Space-time block codes from real orthogonal designs

- $\mathcal{A}$: real constellation of size 2^b

  Theorem: If diversity order is nm, Tx rate $\leq b$ bits/s/Hz.

- Encoder takes in blocks of nb bits.

  Encoder picks $s_i, 1 \leq i \leq n$, from $\mathcal{A}$.

  Build an orthogonal design $\mathcal{O}(s_1, ..., s_n)$.

  At time t, n antennae transmit t^{th} row of $\mathcal{O}$.

- Tx rate is b bits/s/Hz.

  Theorem: The diversity order of code from orthogonal design $\mathcal{O}$ is nm.

- But only good for $n = 2, 4,$ or 8.

Generalized real orthogonal designs

Definition: A generalized real orthogonal design of size n is an $p \times n$ matrix $\mathcal{G}$ with entries $0, \pm x_1, ..., \pm x_k$ such that

> $\mathcal{G}^*\mathcal{G} = D$, a $p \times p$ diagonal matrix whose i^{th} diagonal entry is of the form $l_1^i x_1^2 + ... + l_k^i x_k^2$ with $l_1^i = ... = l_k^i$ being strictly positive integers.

WLG, $\mathcal{G}^*\mathcal{G} = I(x_1^2 + ... + x_k^2)$ ($p \times p$ matrix).

$$e.g.\ \mathcal{G}_3 = \begin{bmatrix} x_1 & x_2 & x_3 \\ -x_2 & x_1 & -x_4 \\ -x_3 & x_4 & x_1 \\ -x_4 & x_3 & x_2 \end{bmatrix}, \quad \mathcal{G}_5 = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ -x_2 & x_1 & x_4 & -x_3 & x_6 \\ -x_3 & -x_4 & x_1 & x_2 & x_7 \\ -x_4 & x_3 & -x_2 & x_1 & x_8 \\ -x_5 & -x_6 & -x_7 & -x_8 & x_1 \\ -x_6 & x_5 & -x_8 & x_7 & -x_2 \\ -x_7 & x_8 & x_5 & -x_6 & -x_3 \\ -x_8 & -x_7 & x_6 & x_5 & -x_4 \end{bmatrix}.$$

Space-time block codes from generalized real orthogonal designs

- $\mathcal{A}$: real constellation of size 2^b
- Encoder takes in blocks of kb bits.

  Encoder picks $s_i, 1 \leq t \leq k$, from $\mathcal{A}$.

  Build an orthogonal design $\mathcal{G}(s_1, ..., s_k)$.

  At time t, n antennae transmit t^{th} row of $\mathcal{G}$.
- Tx rate is $kb \leq pb$ bits/s/Hz.

  The rate of the code is k/p.

  Theorem: The diversity order of code from orthogonal design $\mathcal{G}$ is nm.
- But is it good for all n?

Fundamental question of generalized orthogonal design theory

- $A(R, n) \cong \min p$ such that a $p \times n$ generalized orthogonal design with rate $\geq R$ exists.

  If no such p exists, let $A(R, n) = \infty$.

- Finding $A(R, n)$ is the fundamental question.

  Most interesting $A(1, n)$ for efficiency.

  Theorem: $A(R, n) < \infty$ for any R.

Construction of $\mathcal{G}(x_1, ..., x_p)$ with rate 1

- Hurwitz-Radon family of matrices

  Definition: A set of $n \times n$ real matrices $\{B_1, ..., B_k\}$ is called a size k Hurwitz-Radon family of matrices if

$$
\begin{aligned}
&B_i^* B_i = I,\ B_i^* = -B_i, && 1 \le i \le k,\\
&B_i B_j = -B_j B_i, && 1 \le i, j \le k.
\end{aligned}
$$

- Hurwitz-Radon theory: any Hurwitz-Radon family of matrices contains less than $\rho(n) \le n$ matrices.

  Write $n = 2^a b$, b odd, $a = 4c + d$ with $0 \le d < 4$, $0 \le c$.
  $\rho(n) = 8c + 2^d \le n$

  For any n, by explicit construction, there exists a Hurwitz-Radon family with $\rho(n) - 1$ integer matrices (all entries are 0 or ± 1).

Construction of $\mathcal{G}(x_1, ..., x_p)$ with rate 1 (cont.)

- Choose a Hurwitz-Radon family of $\rho(p) - 1$ integer matrices $\{A_1, A_2, ..., A_{\rho(p)-1}\}$.
  $A_0 \cong I$, $X \cong (x_1, ..., x_p)$

- Construct a $p \times n$ matrix $\mathcal{G}(x_1, ..., x_p)$ by setting the j^{th} column of $\mathcal{G}$ to $A_{j-1}X^*$.

- $\mathcal{G}$ of size n exists for all n.
  Can construct space-time block codes for all n.
  But the block length p can be large?
  (Orthogonal designs are delay optimal at $n = 2, 4$, and 8.)

Complex orthogonal designs

Definition: A complex orthogonal design $\mathcal{O}_c$ of size n is an orthogonal matrix whose rows are permutations of $\pm x_1, ..., \pm x_n$, their conjugates $\pm x_1^*, ..., \pm x_n^*$, or multiples of these indeterminates by $\pm\sqrt{-1}$.

- Complex orthogonal designs exist if and only if $n = 2$.
- Generalized complex orthogonal designs are similarly defined.

  Designs are known to exist for any n only for rate $R \leq 1/2$.

  For rate $R > 1/2$, block codes of rate $3/4$ for $n = 3$ and 4 are shown to exist by explicit construction.

  Designs with rate $R > 1/2$ are still not well understood.

Summary

- Model: block fading, IID path gains, perfect channel info
- Rank criterion for space-time coding
- Real orthogonal designs: $n = 2, 4$, or 8
- Generalized real orthogonal designs
- Block codes from orthogonal designs: rate 1 for any n
- Complex orthogonal designs: rate $\leq 1/2$ for any n

Theorem: If diversity order is nm and $|\mathcal{A}| = 2^b$, Tx rate $\leq b$ bits/s/Hz.

Proof: View each code word $\mathbf{c}$ as a member in $[\mathcal{A}^l]^n$.

$$c_1^1 \cdots c_1^n c_2^1 \cdots c_2^n \cdots c_l^1 \cdots c_l^n \;=\; [(c_1^1 \cdots c_l^1), \cdots, (c_1^n \cdots c_l^n)]$$

$A_{2^{bl}}(n,r) \cong$ max size of code with block length l and Hamming distance r over constellation size 2^{bl}.

Since $B(\mathbf{c},\mathbf{e})$ has rank at least r, at least r rows are nonzero.

Thus, the Hamming distance is at least r for all codewords in $[\mathcal{A}^l]^n$.

Tx rate $\leq \frac{\log_2 A_{2^{bl}}(n,r)}{l}$.

For $r = n$, $A_{2^{bl}}(n,n) = 2^{bl}$ (repetition code). $\square$

Theorem: Diversity order of code from orthogonal design $\mathcal{O}$ is nm.

Proof: The rank criterion requires nonsingularity of

$$B(\tilde{\mathbf{s}}, \mathbf{s}) = \mathcal{O}(\tilde{s}_1, ..., \tilde{s}_n) - \mathcal{O}(s_1, ..., s_n).$$

Note that

$$\mathcal{O}(\tilde{s}_1, ..., \tilde{s}_n) - \mathcal{O}(s_1, ..., s_n) \;=\; \mathcal{O}(\tilde{s}_1 - s_1, ..., \tilde{s}_n - s_n),$$

and

$$\det(\mathcal{O}) \;=\; \det(\mathcal{O}\mathcal{O}^*)^{1/2} = \left[\sum_i x_i^2\right]^{n/2}.$$

Thus

$$\det(\mathcal{O}(\tilde{s}_1 - s_1, ..., \tilde{s}_n - s_n)) \;=\; \left[\sum_i |\tilde{s}_i - s_i|^2\right]^{n/2} \;>\; 0.$$