## 6.962: Week 1 Summary of Discussion

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Paper: "Network Information Flow" by Ahlswede, Cai, Li, Yeung (IEEE Transactions on Information Theory, July 2000).

# 1 Summary:

The authors address the problem of multicasting information from a single source to multiple destinations over an arbitrary network of nodes interconnected by independent, discrete, noiseless channels. Each of the intermediary nodes is allowed to decode and re-encode the information it receives before sending it along to the next node(s). It is required that every ultimate destination be able to reconstruct the same information at the same rate, with no distortion and with either arbitrarily-small or zero probability of error. The main contribution of this paper is the definition of a particular class of network coding schemes (called alpha-codes) and a coding theorem which gives necessary and sufficient conditions for the physical network to accomodate a given information rate under the decoding requirements. The authors further conjecture that this coding theorem can be extended to more general network coding methods.

In the "necessary" part of the coding theorem, the authors prove that the information rate must not exceed the smallest among all of the maximum flows to each destination (i.e., the smallest among all minimum cuts to each destination) in order to ensure arbitrarily-small or zero probability of error using alpha-codes. This is very similar to the converse of the Max-flow Min-cut Theorem in network flow theory.

The achievability part of the coding theorem is what distinguishes this result from classical network flow problems. Here, the authors use a random coding argument to show that as long as the information rate does not exceed the maximum flow to

any destination, then there exists a network coding scheme which can achieve that rate to each destination asymptotically as the block length tends to infinity, with zero probability of error.

The authors also discuss an extension to the problem with multiple sources and give an example of how this can be much more difficult than the single-source problem. Finally, the authors prove that even if probabilistic coding is allowed, it is not possible to achieve higher rates than by time-sharing general deterministic codes.

# 2 Background:

## 2.1 The Max-flow Min-cut Theorem

Consider a directed graph $G = (V, E)$ consisting of a set of nodes $V$ interconnected by a set of edges $E$. A *network* $N = (s, t, V, E, \mathbf{R})$ is a directed graph $(V, E)$ together with a source $s$, a destination $t$, and a capacity $R_{ij}$ associated with each edge $(i, j) \in E$. A *flow* (or $s-t$ flow) $f$ in $N$ is a vector in $R^E$ (one component $f(i, j)$ for each edge $(i, j) \in E$ such that:

1. $0 \leq f(i, j) \leq R_{ij}$ for all $(i, j) \in E$.

2. $\sum_{i:(i,j)\in E} f(i, j) = \sum_{k:(j,k)\in E} f(j, k)$ for all $j \in V - \{s, t\}$.

The *value* of flow $f$ is $\sum_{k:(s,k)} f(s, k)$. The *maximum flow* from $s$ to $t$ is the $s-t$ flow with maximum value.

An $s-t$ *cut* is a partition $(B, \overline{B})$ of nodes $V$ into sets $B$ and $\overline{B} = V - B$ such that $s \in B$ and $t \in \overline{B}$. The *capacity* of an $s-t$ cut is

$$C(B, \overline{B}) = \sum_{(i,j)\in E: i\in B, j\in \overline{B}} R_{i,j} \tag{1}$$

**Theorem 2.1 (Max-flow, min-cut)** *The value $v$ of any $s-t$ flow is no greater than the capacity $C(B, \overline{B})$ of any $s-t$ cut. Furthermore, the value of the maximum flow equals the capacity of the minimum cut.*

It has been shown that when the edge capacities $R_{ij}$ are all rational numbers, the maximum flow vector can be explicitly constructed using the Ford-Fulkerson algorithm (refer to *Combinatorial Optimization*, by Papadimitrou, 1982 [4]).

## 2.2 Multilevel Diversity Coding

Multilevel diversity coding (MDC), first introduced by Roche [5] and Yeung [8], is a class of problems in multiterminal source coding. A MDC system consists of an information source, a set of encoders, and a set of decoders. Each encoder encodes the same information source, and each decoder has access to a particular subset of encoders (the particular assignment of subsets of encoders to decoders is called the *configuration* of the system). The set of decoders is partitioned into multiple *levels*, where decoders within the same level are subject to the same reconstruction requirements. The problem is to determine the coding rate region for a given configuration of the MDC system subject to certain distortion criteria.

The case of an MDC system with 2 decoding levels was solved by Yeung [8], and the case of 3 decoding levels with a symmetric configuration is solved by Roche, Yeung, and Hau [6].

The network information flow model presented in the current paper pedagogically includes the class of all MDC systems. The actual results obtained for the single-source network problem, however, only apply to the case of a MDC system with a single decoding level, hence adds nothing new to the results of [8].

## 2.3 Other Related Work

In [3], T.S. Han considered the problem of transmitting multiple correlated information sources to a single destination over a general network of discrete (noisy or noiseless) independent channels. The setup is almost identical to the current paper's, in that intermediate nodes are allowed to decode and re-encode information, and the reconstruction assumes no distortion. The only differences are that Han assumes multi-source/single-destination communication over possibly *noisy* channels (as opposed to the single-source/multi-destination communication over noiseless channels in the current paper). The main result in [3] gives a necessary and sufficient condition for a given network to be achievable, in the sense that all the sources can be reconstructed at the destination with arbitrarily small probability of error. Interestingly, the authors of the current paper do not mention this reference, despite its close similarity.

Other references to classical multiterminal source coding can be found in Slepian and Wolf [7], Cover [1], and Cover and Thomas [2].

# References


[1] T. Cover. A proof of the data compression theorem of slepian and wolf for ergodic sources. *IEEE Trans. Info. Theory*, IT-21:226–228, 1975.

[2] T. Cover and J. Thomas. *Elements of Information Theory*, 1991.

[3] T.S. Han. Slepian-wolf-cover theorem for networks of channels. *Information and Control*, 47:67–83, 1980.

[4] C. Papadimitriou. *Combinatorial Optimization: Algorithms and Complexity*, 1982.

[5] J. Roche. Distributed information storage. *Ph.D. dissertation, Stanford University*, March 1992.

[6] J. Roche, R. Yeung, and K. Hau. Symmetrical multilevel diversity coding. *IEEE Trans. Info. Theory*, IT-43:1059–1064, May 1997.

[7] D. Slepian and J.K. Wolf. Noiseless coding of correlated information sources. *IEEE Trans. Info. Theory*, IT-19:471–480, 1973.

[8] R. Yeung. Multilevel diversity coding with distortion. *IEEE Trans. Info. Theory*, IT-41:412–422, March 1995.