6.962: Week 9 Summary of Discussion

Linear Multiuser Receivers: Effective Interference, Effective Bandwidth and User Capacity

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1 Introduction

In the past two weeks Aaron and Huan focused on the multiple antenna channels over which a single user communicates. This week we will talk about a multiuser detection problem in which K users transmit to a single receiver which demodulates signals of each user. We will examine a number of multiuser receivers in some detail.

1.1 Motivation

Spread-spectrum techniques (just like multiple antenna systems) provide additional *degrees of freedom* through which communication can take place. To exploit these extra degrees of freedom, multiuser receivers such as the linear minimum-mean square error (MMSE) and linear decorrelator take the structure of the interference on a user from other users into account when demodulating a user. While it is rather easier to understand the effect of the total interference that a user experiences, it is harder but important to separate out the impacts of individual interferers. The previous work on performance analysis of multiuser receivers focused on the ability of the receiver to reject worst case interference ([3]). This paper ([1]) gives insight on the impact of individual users on the performance of multiuser receivers.

It turns out that, asymptotically (as number of users and degrees of freedom approaches infinity), decoupling of the interfering effects of each user is possible. Namely, each interferer can be assigned a level of "effective interference" which it provides to every other user.

1.2 System Model

We will consider a symbol-synchronous multi-access spread-spectrum system. There are two identical alternatives we can consider, uplink and downlink, as shown in Fig. 1 and Fig. 2. Note that users are assumed to be synchronous. A sample of the received symbol is

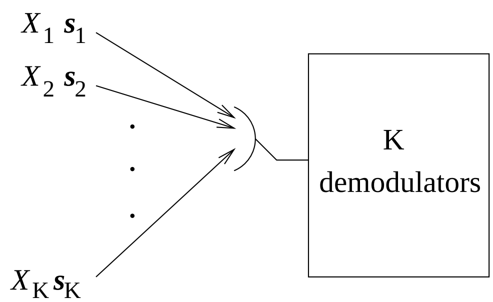

Figure 1: Multiuser, single antenna system (uplink)

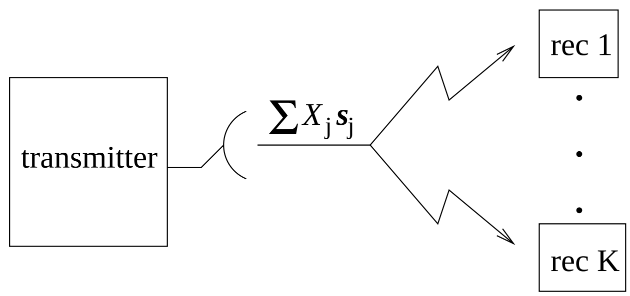

Figure 2: Multiuser, single antenna system (downlink)

represented by the vector $\mathbf{Y}$:

$$\mathbf{Y} = \sum_{i=1}^{K} X_i \mathbf{s}_i + \mathbf{W} \tag{1}$$

where $X_i \in \Re$ is the transmitted signal, $\mathbf{s}_i \in \Re^N$ is the spreading sequence of user i and $\mathbf{W} \sim N(0, \sigma^2 I)$ is the background additive white Gaussian noise (AWGN). The length of the signature sequences, N, is thought of as the number of degrees of freedom. X_i's are independent with $E[X_i] = 0$ and $E[X_i^2] = P_i$ where P_i is the received power of user i. The signature sequences are assumed to be randomly generated and known by all the receivers. In general the multi-access interference is not white, i.e., the covariance matrix of $\sum_i X_i s_i$ is not proportional to the identity matrix.

The paper deals with the problem of extracting good estimates of the symbols of each user as soft decisions to be used by the decoders. The estimates are linear functions of the received vector $\mathbf{Y}$. For instance, for user 1, the linear demodulator $\mathbf{c}_1$ generates the estimate, $\hat{X}_1 = \mathbf{c}_1^T Y$ with signal to interference ratio (SIR[1]):

$$\begin{aligned}
\beta_1 &= \frac{E\left[(X_1 \mathbf{c}_1^T \mathbf{s}_1)^2\right]}{(\mathbf{c}_1^T \mathbf{c}_1)\sigma^2 + \sum_{i=2}^{K} E\left[(X_i \mathbf{c}_1^T \mathbf{s}_i)^2\right]} \\
&= \frac{P_1 (\mathbf{c}_1^T s_1)^2}{(\mathbf{c}_1^T \mathbf{c}_1)\sigma^2 + \sum_{i=2}^{K} P_i (\mathbf{c}_1^T \mathbf{s}_i)^2}
\end{aligned}$$

[1] SIR is defined to be the ratio of the power of the signal to the combined power of the interferers and the AWGN

Type of the receiver depends on the choice of $\mathbf{c}$. Next, we introduce three types of linear receivers: conventional matched filter, MMSE receiver and decorrelator.

2 Linear Multiuser Receivers

2.1 Matched Filter

Let the total interference for the first user be $\mathbf{Z}$:

$$\mathbf{Z} = \sum_{i=2}^{K} X_i \mathbf{s}_i + \mathbf{W} \tag{2}$$

The matched filter receiver of user 1 simply matches the received vector $\mathbf{Y}$ to $\mathbf{s}_1$ by choosing $\mathbf{c}_1 = \mathbf{s}_1$:

$$X_{mf}(\mathbf{Y}) = \frac{\mathbf{s}_1^T \mathbf{Y}}{\mathbf{s}_1^T \mathbf{s}_1}$$

Geometrically, this is taking the projection of the received signal onto the unit vector in the direction of $\mathbf{s}_1$. Thus, matched filter treats the interference terms as noise and does not exploit the structure of them. As we shall see, for higher values of SIR and higher number of users per degrees of freedom the matched filter performs close to optimal receiver (MMSE).

2.2 Decorrelator

The decorrelator was first introduced by Lupas and Verdu [2] Equation 1 can be written in the matrix form as:

$$\mathbf{Y} = S\mathbf{X} + \mathbf{W}$$

where $\mathbf{X} = [X_1 \ \cdots \ X_K]^T$ and $S = [\mathbf{s}_1 \ \cdots \ \mathbf{s}_K]^T$ is the $N \times K$ matrix of signature sequences. Note that the vector of matched filter outputs, $\mathbf{R}$ form a sufficient statistic for inputs $\mathbf{X}$ where,

$$\mathbf{R} = S^T S\mathbf{X} + S^T\mathbf{W}$$

If we filter $\mathbf{R}$ with $(S^T S)^{-1}$ in addition to the matched filter, we get:

$$\mathbf{U} = (S^T S)^{-1}\mathbf{R} = \mathbf{X} + (S^T S)^{-1} S^T \mathbf{W}$$

The overall filter, $(S^T S)^{-1} S^T$ is called the decorrelator. Let us define, $N = (S^T S)^{-1} S^T \mathbf{W}$. The covariance matrix of N, K_N is,

$$K_N = (S^T S)^{-1}\sigma^2$$

The SIR for user i is given by $\frac{P_i}{K_{N,ii}}$. One can see that the correlation between the noise parameters, N_i's is not exploited.

Geometrically, decorrelating is projecting the received signal, $\mathbf{Y}$, onto the subspace $(\text{span}\{(s_j)_{j\neq 1}\})^{\perp}$. For lower values of SIR and lower values of users per degrees of freedom, the decorrelator performs close to optimal receiver (MMSE).

2.3 MMSE Receiver

Let us define the total interference for user 1 as in Eq. 2. The covariance matrix, K_Z, of the interference term is:

$$K_Z = S_1 D_1 S_!^T + \sigma^2 I$$

where S_1 is the $N \times (K \Leftrightarrow 1)$ matrix whose columns are the signature sequences of the interfering users and D_1 is the (diagonal) covariance matrix of $[X_2 \cdots X_K]^T$. With eigenvalue decomposition $K_Z = Q^T \Lambda Q$ where columns of Q are the orthonormal eigenvectors of K_Z and Λ is the diagonal matrix of eigenvalues of K_Z. Note that K_Z is positive definite and the whitening filter is $\Lambda^{-\frac{1}{2}} Q$. Applying this to $\mathbf{Y}$, we get:

$$\Lambda^{-\frac{1}{2}} Q \mathbf{Y} = X_1 \Lambda^{-\frac{1}{2}} Q \mathbf{s}_1 + \Lambda^{-\frac{1}{2}} Q \mathbf{Z}$$

The interference is now white and thus a matched filter will give us a scalar sufficient statistic. Namely, we project $\Lambda^{-\frac{1}{2}} Q \mathbf{Y}$ along the direction $\Lambda^{-\frac{1}{2}} Q \mathbf{s}_1$:

$$R = \mathbf{s}_1^T K_Z^{-1} \mathbf{Y} = (\mathbf{s}_1^T K_Z^{-1} \mathbf{s}_1) X_1 + \mathbf{s}_1^T K_Z^{-1} \mathbf{Z}$$

Finally, the MMSE estimate is the linear least squares estimate (LLSE) of X given the observation R:

$$\begin{aligned} X_{mmse}(\mathbf{Y}) &= \frac{\operatorname{cov}(X_1, R)}{\operatorname{var}(R)} R \\ &= \frac{P_1 R}{1 + P_1 R} \\ &= \frac{P_1 \mathbf{s}_1^T K_Z^{-1} \mathbf{Y}}{1 + P_1 \mathbf{s}_1^T K_Z^{-1} \mathbf{Y}} \end{aligned}$$

The signal to interference ratio for user 1 is:

$$\begin{aligned} SIR_1 &= \frac{(\mathbf{s}_1^T K_Z^{-1} \mathbf{s}_1)^2 P_1}{\mathbf{s}_1^T K_Z^{-1} \mathbf{s}_1} \\ &= P_1 \mathbf{s}_1^T K_Z^{-1} \mathbf{s}_1 \end{aligned}$$

which is the maximum achievable by linear multiuser receivers.

3 Main Results

Recall that the spreading sequences are randomly chosen and once picked they are communicated to all the receivers. Namely, the change in the spreading sequences occur at a much slower time scale than the time required to acquire the sequences. The performance of a receiver depends on the initial choice of the sequences and hence is random. The model for the random spreading sequences is as follows. Each $\mathbf{s}_i = \frac{1}{\sqrt{N}} [V_{i1} \ \cdots \ V_{iN}]^T$ for all i, where

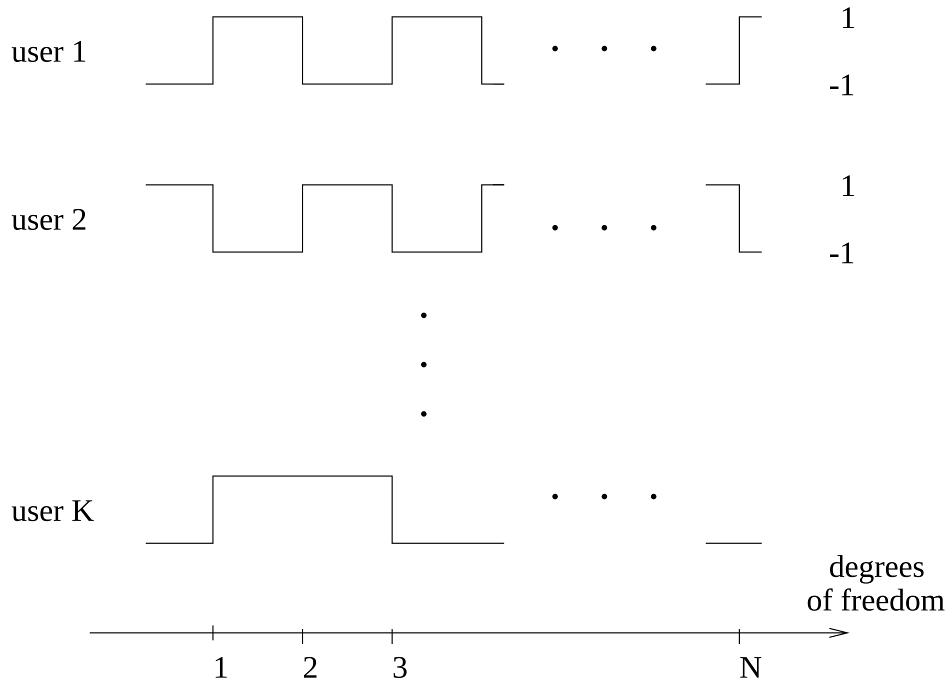

Figure 3: Spreading sequences of size N for K users

V_{ik}'s are iid, zero mean and variance 1. Note that $E\left[\|s_i\|^2\right] = 1$. A simple and typical example of spreading sequences are those composed of -1's and 1's as shown in Fig. 3.

The asymptotic structure of interference as $N, K \to \infty$ is analyzed in the paper. Note that if only $N \to \infty$ and K were finite, then a sequence would be orthogonal with all the others with probability 1. Thus, the overall interference would be white. Similarly, if N were finite and K is increased, the aggregate interference would also become increasing white because of averaging. The case where $\frac{K}{N} = \alpha$ (number of users per degrees of freedom) is a positive real number is examined in the paper. In particular, a relation for the SIR of the system is derived for matched filter, decorrelator and the MMSE receiver. It is assumed that the powers of users converge to a fixed distribution $F(P)$.

Matched Filter (*Proposition 3.3*): Let $\beta_{1,MF}^{(N)}$ be the (random) SIR of the conventional matched filter receiver for user 1. As $N, K \to \infty$ with $\frac{K}{N} \to \alpha$,[2]

$$\beta_{1,MF}^{(N)} \to \beta_{1,MF}^{*} = \frac{P_1}{\sigma^2 + \alpha \mathrm{E}_F[P]}$$

with probability 1. Thus, for large N:

$$\beta_{1,MF} \approx \frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{i=2}^{K} P_i} \tag{3}$$

MMSE Receiver (*Theorem 3.1*): Let $\beta_{1,MMSE}^{(N)}$ be the (random) SIR of the MMSE receiver

[2]Note that in the paper there is a typo: α in the denominator is missing.

for user 1. As $N, K \to \infty$ with $\frac{K}{N} \to \alpha$,

$$\beta_{1,MMSE}^{(N)} \to \beta_{1,MMSE}^{*} = \frac{P_1}{\sigma^2 + \alpha \mathrm{E}_F[I(P, P_1, \beta_{1,MMSE}^{*})]}$$

where

$$I(P, P_1, \beta_{1,MMSE}^{*}) = \frac{PP_1}{P_1 + P\beta_{1,MMSE}^{*}}$$

Thus, for large N,

$$\beta_{1,MMSE} \approx \frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{i=2}^{K} I(P_i, P_1, \beta_{1,MMSE})} \tag{4}$$

where, as before, P_i is the received power of user i.

Decorrelator (*Theorem 7.2*): Let $\beta_{1,DEC}^{(N)}$ be the (random) SIR of the decorrelating receiver for user 1. As $N, K \to \infty$ with $\frac{K}{N} \to \alpha$,

$$\beta_{1,DEC}^{(N)} \to \beta_{1,DEC}^{*} = \begin{cases} \frac{P_1(1-\alpha)}{\sigma^2} & \alpha < 1 \\ 0 & \alpha \geqslant 1 \end{cases}$$

- Eq. 3 and 4 yield an interesting interpretation of the effect of each of the interfering users on the SIR of user 1 for matched filter and MMSE receivers. The total interfering effect of other users can be decoupled into a sum of the interference term from each of the users separately. This is not surprising in the matched filter scenario since the receive filter ($\mathbf{s}_1$) of each user is independent of the signature sequences of the other users. In the MMSE receiver, the interference term of a certain user does not depend on the other interfering users except through the attained SIR. This decoupling[3] is rather surprising since the effect of an interferer depends on the MMSE receive filter ($\mathbf{c}_1$) which is indeed a function of the signature sequences and received powers of all users.

- For the matched filter receiver, the interference grows unboundedly as the received power of the interferers increase. For the MMSE receiver and the decorrelator even if the received power of a user gets very large, its interference is bounded. For the decorrelator, this can be justified intuitively since it takes the projection of the received signal onto $(\mathrm{span}\{(s_j)_{j\neq 1}\})^{\perp}$. Thus for stronger interferers, decorrelator does a good job of canceling interference. For the MMSE receiver, the interference from user i is bounded by $\frac{P_1}{\beta_1}$ as $P_i \to \infty$. This is the well known near-far resistance property of MMSE receiver.

- If $N < K$, $\beta_{1,DEC}^{(N)} = 0$. Asymptotically the dimension of the space $\mathrm{span}\{(s_j)_{j\neq 1}\}$ approaches to $\min(N, K)$ with probability 1. The decorrelator projects the received signal onto $(\mathrm{span}\{(s_j)_{j\neq 1}\})^{\perp}$ and the projection will be the 0 vector if $\mathbf{s}_1 \in \mathrm{span}\{(s_j)_{j\neq 1}\}$. Thus the SIR will be 0.

[3]Note that this decoupling does not imply that the interfering effect of the other users on a particular user is additive across users.

- In general, Eq. 4 does not have an explicit solution for $\beta_{1,MMSE}$. But, it can still be very useful in the following sense. Suppose user 1 requires an SIR of β_T and let β_1^* be the (unique) solution to Eq. 4. It can be shown that (Proposition 3.2),

$$\beta_T \leqslant \beta_1^* \;\Leftrightarrow\; \beta_T \leqslant \frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{i=2}^{K} I(P_i, P_1, \beta_T)}$$

  Thus, to check whether the SIR requirement of user 1 can be met by the system, it suffices to check if the second condition above holds. This process does not require the knowledge of the exact solution of Eq. 4. Based on this, we can term $I(P_i, P_1, \beta_T)$ as the effective interference of user i on user 1. This presents a nice way to calculate the user capacity of a system (the maximum number of admissible users per degrees of freedom).

- We already mentioned that if either K or N approaches to infinity, the aggregate interference is white. But if both K and $N \to \infty$ as $\frac{K}{N} = \alpha$ the total interference remains colored. This is shown in Theorem 4.1 which is based on a fundamental property of random matrices ([5]).

- The user capacity of a system can be improved if power control is employed. Assuming all users have identical SIR requirements, two cases are considered in the paper.

  1. What is the minimum power necessary to achieve the target SIR for a given number of users?
  2. What is the maximum number of users per degree of freedom for a given power constraint?

  It turns out that the number of users supportable with a matched filter receiver is inversely proportional to the SIR requirement of the users. This is plausible since with conventional matched filter, the total power of the interference is proportional to the number of users. On the other hand, with an MMSE receiver arbitrarily high values of SIR is supportable with sufficiently high signal power. Indeed, MMSE receiver can support one extra user per degree of freedom over the matched filter. Also, the user capacity is maximized if the power is uniform over the users.

- In perfectly power controlled systems, the notion of effective bandwidths are very useful. If a single resource is shared by multiple users, effective bandwidths give a measure of what portion of the resources is consumed by a given user. Effective bandwidth is an asymptotic quantity and effective bandwidth of a user is a non-decreasing function of its SIR (quality of service to be more general) requirement.

  In the presence of different user types (users with different SIR requirements) one can talk about the user capacity region. The coordinates of a point in the capacity region gives a set of numbers of supportable users of each type. If the problem model is "nice" and the boundary of the capacity region turns out to be linear, the effective bandwidth formulation is even more useful. For example, the user capacity constraint for the MMSE receiver with J classes is given by:

$$\sum_{j=1}^{J} \alpha_j \frac{\beta_j}{1+\beta_j} < 1$$

where β_j is the SIR requirement of type j users. The capacity region is the set of feasible values of $(\alpha_1, \ldots, \alpha_J)$. It seems very reasonable to call $\frac{\beta_j}{1+\beta_j}$ the effective bandwidth of class j users. The linearity of the boundary of the region is a consequence of the asymptotic decoupling of the interference due to other users. This formulation explicitly shows us the the amount of consumption by a certain type of user. A general discussion about effective bandwidths and their applications to queueing systems can be found in [4].

- The model for a synchronous multi-access antenna array system is:

$$\mathbf{Y} = \sum_{m=1}^{K} X_m \mathbf{h}_m + \mathbf{W}$$

where the vector $\mathbf{h}_m$ represents the fading of the m^{th} user at each of the antenna array. Assuming that the channel fading of the users can be measured and tracked perfectly at the receiver, the model is very similar to that of the single antenna spread spectrum system. Thus similar techniques can be used to analyze antenna array systems since the asymptotic limit does not depend on the interpretation of the $\mathbf{s}_i$'s as spreading sequences or as channel fading.

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