Linear Multiuser Receivers: Effective Interference, Effective Bandwidth and User Capacity

6.962 Graduate Seminar in Communication

November 9, 2000

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Outline

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4. User capacity under power control
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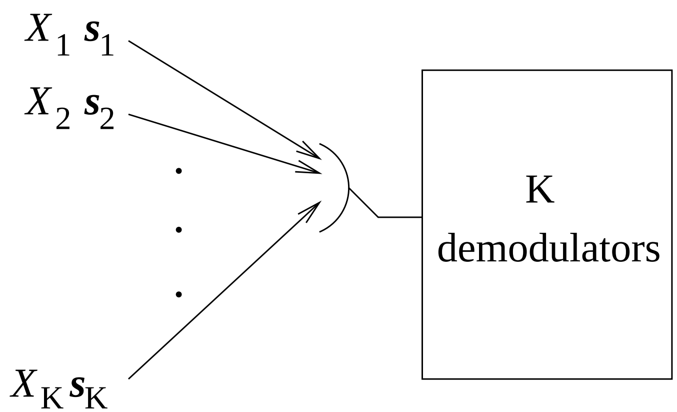
Introduction

Motivation

- High demand for all kinds of applications over wireless
  - Various quality of service (bit rate, probability of error) requirements
  - Can the system accomodate another user with a QoS constraint?
- How to take advantage of the additional degrees of freedom provided by spread-spectrum techniques.
- At the physical layer, signal to interference ratio (SIR) is the key parameter.
- Previous work: Not much insight on how a user affects the system except in the worst case.

System Model

We consider a symbol-synchronous multi-access spread-spectrum system



Received vector, $\mathbf{Y}$:

$$\mathbf{Y} = \sum_{i=1}^{K} X_i \mathbf{s}_i + \mathbf{W}$$

User i: $X_i \in \Re$ is the transmitted signal ($E[X] = 0, E\left[X^2\right] = P_i$, X_i's are iid)

$\mathbf{s}_i \in \Re^N$ is the <u>random</u> spreading sequence and $\mathbf{W} \sim N(0, \sigma^2 I)$

Demodulators: Make a good estimate (soft) on the transmitted symbols.

Model - Continued

- We are interested in the SIR $\leftrightarrow$ rates (bits per symbol). e.g., Gaussian input distribution $\Rightarrow$

$$\frac{1}{2}\log(1+\mathrm{SIR}_i)$$

- Successive cancellation is another possibility

Linear receivers $\rightarrow$ Receiver 1:

$$\hat{X}_1 = \mathbf{c}_1^T\mathbf{Y} = X_1\mathbf{c}_1^T\mathbf{s}_1 + \sum_{i=2}^{K} X_i\mathbf{c}_1^T\mathbf{s}_i + \mathbf{c}_1^T\mathbf{W}$$

$$\begin{aligned}\mathrm{SIR}_1 = \beta_1 \quad &= \quad \frac{E\left[(X_1\mathbf{c}_1^T\mathbf{s}_1)^2\right]}{(\mathbf{c}_1^T\mathbf{c}_1)\sigma^2 + \sum_{i=2}^{K} E\left[(X_i\mathbf{c}_1^T\mathbf{s}_i)^2\right]} \\ &= \quad \frac{P_1(\mathbf{c}_1^T s_1)^2}{(\mathbf{c}_1^T\mathbf{c}_1)\sigma^2 + \sum_{i=2}^{K} P_i(\mathbf{c}_1^T\mathbf{s}_i)^2}\end{aligned}$$

Linear Multiuser Receivers

- Matched filter
  - The filter $\mathbf{c}_i = \mathbf{s}_i$:

$$\hat{X}_{mf,1}(\mathbf{Y}) = \frac{\mathbf{s}_1^T \mathbf{Y}}{\mathbf{s}_1^T \mathbf{s}_1}, \ \mathrm{SIR}_1 = \frac{P_1(\mathbf{s}_1^T s_1)^2}{(\mathbf{s}_1^T \mathbf{s}_1)\sigma^2 + \sum_{i=2}^K P_i(\mathbf{s}_1^T \mathbf{s}_i)^2}$$

  - $\hat{X}_{mf,i}$ is the projection of $\mathbf{Y}$ on $\mathbf{s}_i$.
- Decorrelator
  - In the matrix form, $\mathbf{Y}$ can be written as

$$\mathbf{Y} = S\mathbf{X} + \mathbf{W}$$

    where $\mathbf{X} = [X_1 \ \cdots \ X_K]^T$ and $S = [\mathbf{s}_1 \ \cdots \ \mathbf{s}_K]$
  - matched filter outputs $\mathbf{R}$ form sufficient statistic for $\mathbf{X}$:

$$\mathbf{R} = S^T S\mathbf{X} + S^T\mathbf{W}$$

- Decorrelating filter is $(S^T S)^{-1}$ in addition to matched filter:

$$\mathbf{U} = (S^T S)^{-1}\mathbf{R} = \mathbf{X} + (S^T S)^{-1} S^T \mathbf{W}$$

- Decorrelating receiver for user i is takes the projection of $\mathbf{Y}$ onto $(\text{span}\{(s_j)_{j \neq i}\})^{\perp}$ (Does not exploit correlation between the terms of the interference vector $\Rightarrow$ suboptimal), $\text{SIR}_1 = \frac{P_1}{\Sigma_{ii}}$

• Minimum mean square error (MMSE) receiver

- The total interference for user 1 is:

$$\mathbf{Z} = \sum_{i=2}^{K} X_i \mathbf{s}_i + \mathbf{W}$$

- The covariance matrix of Z is:

$$K_Z = S_1 D_1 S_1^T + \sigma^2 I$$

where S_1 is the $N \times (K \Leftrightarrow 1)$ matrix of signature sequences of interferers and D is the covariance matrix of $[X_2 \cdots X_K]^T$.

- Eigenvalue decomposition $\rightarrow$ $K_Z = Q^T \Lambda Q$. $K_Z > 0$ and the whitening filter for the interference is $\Lambda^{-\frac{1}{2}} Q$:

$$\Lambda^{-\frac{1}{2}} Q\mathbf{Y} = X_1 \Lambda^{-\frac{1}{2}} Q\mathbf{s}_1 + \Lambda^{-\frac{1}{2}} Q\mathbf{Z}$$

- Now that interference is white, apply matched filter to get scalar sufficient statistic for X_1. Project $\Lambda^{-\frac{1}{2}} Q\mathbf{Y}$ along $\Lambda^{-\frac{1}{2}} Q\mathbf{s}_1$:

$$R = \mathbf{s}_1^T K_Z^{-1}\mathbf{Y} = (\mathbf{s}_1^T K_Z^{-1}\mathbf{s}_1) X_1 + \mathbf{s}_1^T K_Z^{-1}\mathbf{Z}$$

- Finally, the MMSE estimate is the linear least squares estimate (LLSE) of X_1 given the observation R:

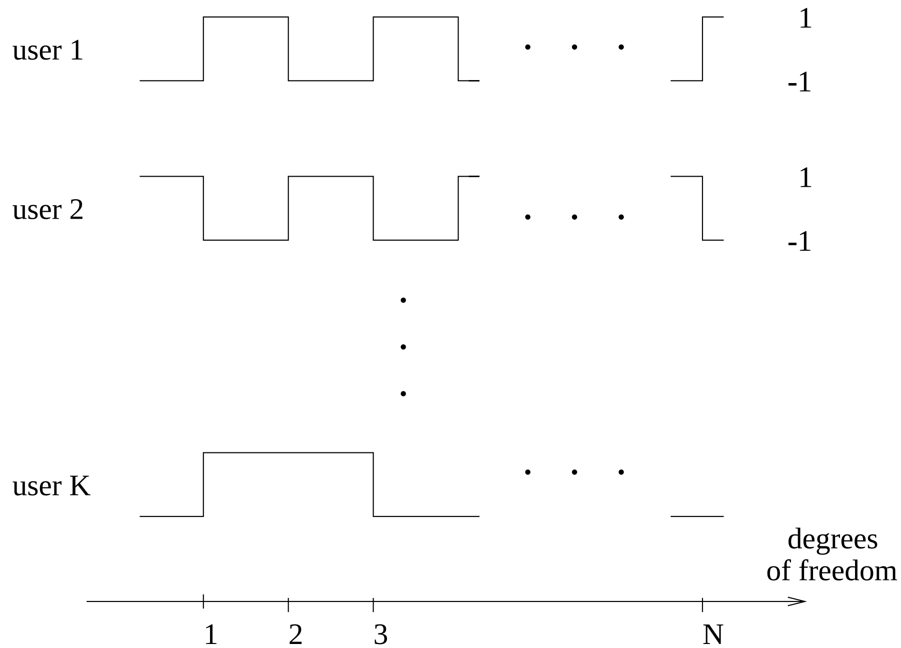
$$\begin{aligned} X_{mmse}(\mathbf{Y}) &= \frac{\text{cov}(X_1, R)}{\text{var}(R)} R = \frac{P_1 R}{1 + P_1 R} \\ &= \frac{P_1 \mathbf{s}_1^T K_Z^{-1}\mathbf{Y}}{1 + P_1 \mathbf{s}_1^T K_Z^{-1}\mathbf{Y}} \end{aligned}$$

- The signal to interference ratio for user 1 is:

$$SIR_1 = \frac{(\mathbf{s}_1^T K_Z^{-1}\mathbf{s}_1)^2 P_1}{\mathbf{s}_1^T K_Z^{-1}\mathbf{s}_1} = P_1 \mathbf{s}_1^T K_Z^{-1}\mathbf{s}_1$$

Performance Under Random Spreading Sequences

- Spreading sequences: $\mathbf{s}_i = \frac{1}{\sqrt{N}}[V_{i1} \cdots V_{iN}]^T$, where V_{ik}'s are iid 0 mean and variance 1 $\Rightarrow E\left[\|\mathbf{s}_i\|^2\right] = 1$. e.g.,



- We are interested in the case $K, N \to \infty$, $\frac{K}{N} = \alpha$. Assume that asymptotically empirical distribution of the powers of users (i.e., $\frac{1}{K}\sum X_i^2 = P_i$) converge to $F(P)$

Matched Filter

Proposition 3.3: Let $\beta_{1,MF}^{(N)}$ *be the (random) SIR of the conventional matched filter receiver for user 1. Then, with probability 1:*

$$\beta_{1,MF}^{(N)} \to \beta_{1,MF}^{*} = \frac{P_1}{\sigma^2 + \alpha \mathrm{E}_F[P]}$$

Sketch of the Proof: By definition,

$$\beta_{1,MF} = \frac{P_1(\mathbf{s}_1^T s_1)^2}{(\mathbf{s}_1^T \mathbf{s}_1)\sigma^2 + \sum_{i=2}^{K} P_i(\mathbf{s}_1^T \mathbf{s}_i)^2}$$

Note that $\mathbf{s}_1^T s_1 \to 1$ w.p. 1 and expanding $\mathbf{s}_1^T s_i$, it was shown that $\mathrm{var}\left[\sum_{i=2}^{K} P_i(\mathbf{s}_1^T \mathbf{s}_i)^2 | P_1, P_2, \ldots\right] = 0$, $\forall$ realizations of P_i's. Thus,

$$\sum_{i=2}^{K} P_i(\mathbf{s}_1^T \mathbf{s}_i)^2 = E\left[\sum_{i=2}^{K} P_i(\mathbf{s}_1^T \mathbf{s}_i)^2\right] = \frac{K}{N}\frac{1}{K}\sum_{i=2}^{K} P_i \to \alpha \mathrm{E}_F[P]$$

with probability 1. Hence, $N \to \infty$, $\mathrm{P}(\sum \mathrm{interferer}_i) = \sum \mathrm{P}(\mathrm{interferer}_i)$

$$\beta_{1,MF} = \frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{i=2}^{K} P_i}$$

MMSE Receiver

Theorem 3.1: Let $\beta_{1,MMSE}^{(N)}$ be the (random) SIR of the MMSE receiver for user 1. Then, with probability 1:

$$\beta_{1,MMSE}^{(N)} \to \beta_{1,MMSE}^{*} = \frac{P_1}{\sigma^2 + \alpha \mathrm{E}_F[I(P, P_1, \beta_{1,MMSE}^{*})]}$$

where

$$I(P, P_1, \beta_{1,MMSE}^{*}) = \frac{P P_1}{P_1 + P \beta_{1,MMSE}^{*}}$$

Notes on the Proof: We will use the following theorem due to Silverstein and Bai about the limiting eigenvalue distribution of large matrices.

Let $A_{n\times m}$ be a $n \times m$ matrix whose $(i,j)^{\mathrm{th}}$ entry is $\frac{X_{ij}}{\sqrt{n}}$ where X_{ij}'s are iid with unit variance. Let T_m be a $m \times m$ diagonal matrix whose entries are real valued random variables. The matrix $A_{n\times m} T_m A_{n\times m}^T$ has real non-negative eigenvalues with empirical distribution $G_n(\lambda)$. Note that $G_n(\lambda)$ is a random variable. As $n, m \to \infty$, $\frac{m}{n} = \alpha$, $G_n(\lambda)$ approaches to a deterministic function, $G(\lambda)$.

Some Observations

- The covariance matrix $K_Z = S_1 D_1 S_1^T + \sigma^2 I$ has exactly the desired form.
- The asymptotic eigenvalue distribution, $G(\lambda)$ is not degenerate $\Rightarrow$ in this asymptotic regime, interference is not white.
  - If K were finite and $N \to \infty$, interference would be white w.p. 1.
  - If N were finite and K is increased, interference will be increasingly white.
  - If the interference is white, matched filter and MMSE receiver are identical. Thus, only if $\frac{K}{N}$ is constant, the MMSE receiver outperfroms the matched filter.

Proof - continued

- One last complication left: Just the eigenvalue distribution may not be sufficient for SIR characterization:
  - $K_Z = U^T \Lambda U$ where Λ is diagonal and U is orthogonal for every realization of Z.
  - Recall that

    $$\beta_{1.MMSE} = P_1 \mathbf{s}_1^T K_Z^{-1} \mathbf{s}_1 = P_1 (U\mathbf{s}_1)^T \Lambda^{-1} (U\mathbf{s}_1)$$

    thus the relative position of $\mathbf{s}_1$ wrt. eigenvectors of K_Z also matters.
  - It is shown in Lemma 4.2 that as $N \to \infty$, $\mathbf{s}_1$ is white in any coordinate system and $\|U\mathbf{s}\|$ is constant for any realization of Z.
- Thus, $\beta_{1,MMSE}$ can be characterized using only the eigenvalue distribution K_Z.

Comparison of MF and MMSE Receivers

- The performances of the two receivers for large N:

$$\beta_{1,MF} = \frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{i=2}^{K} P_i}, \ \beta_{1,MMSE} = \frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{i=2}^{K} I(P_i, P_1, \beta_{1,MMSE})}$$

where $I(P, P_1, \beta_{1,MMSE}) = \frac{PP_1}{P_1 + P\beta_{1,MMSE}}$

  - $I(P_i, P_1, \beta_1) < P_i$ since MMSE maximizes SIR
  - $\beta_{1,MF}$ is independent of other signature sequences. $\beta_{1,MF} \to 0$ as $P_i \to \infty$.
  - $I(P_i, P_1, \beta_1) \to \frac{P_1}{\beta_1}$ even as $P_i \to \infty$ (near-far resistance) since MMSE receiver exploits the structure of the interference.
  - Decoupling of interfering effects of other users. Each user contributing an amount called "effective interference"

Further Intuition on MMSE SIR Formula

- The equation

$$\beta = \frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{i=2}^{K} I(P_i, P_1, \beta)}$$

has a unique solution since $\frac{1}{\beta}\frac{P_1}{\sigma^2+\frac{1}{N}\sum_{i=2}^{K} I(P_i,P_1,\beta)}$ is a monotonically decreasing function of β (it cuts 1 at a single point). Let β^* be the unique root of the equation.

- In general it is hard to evaluate β^*. But it can still be very useful:

$$\beta_T \leqslant \beta_1^* \;\Leftrightarrow\; \beta_T \leqslant \frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{i=2}^{K} I(P_i, P_1, \beta_T)}$$

- Thus, to check whether the SIR requirement β_T of a user can be met, check the second condition.

Performance of Decorrelator

- If $\mathbf{W} = \mathbf{0}$ then the input-output relation for decorrelator is as follows:

$$S\mathbf{X} \rightarrow \mathbf{X}$$

- It is clear that decorrelator for user 1, $\mathbf{r}_1$, lies on $(\text{span}\{(s_j)_{j\neq 1}\})^{\perp}$.
- It is further shown that (Proposition 7.1) $\mathbf{r}_1$ is the orthogonal projection of Y along $(\text{span}\{(s_j)_{j\neq 1}\})^{\perp}$. Given $\mathbf{r}_1$,

$$SIR_1 = \frac{P_1(\mathbf{r}_1^T \mathbf{s}_1)}{\sigma^2(\mathbf{r}_1^T \mathbf{r}_1)}$$

- In our asymptotic regime, what happens to $\mathbf{r}_1^T \mathbf{s}_1$?

Random SIR of the Decorrelator

- A theorem by Bai and Lin show that the smallest eigenvalue of the random matrix $S_1^T S_1$ converges almost surely to a strictly positive number.
  - Hence S_1 is almost surely of full rank, $\min(K-1, N)$
  - Note that $\dim\left(\text{span}\{(s_j)_{j\neq 1}\}\right)^{\perp} = N - \text{rank}(S_1)$
  - If $K > N \Rightarrow \mathbf{r}_1^T \mathbf{s}_1 \rightarrow 0$. If $K < N \Rightarrow \mathbf{r}_1^T \mathbf{s}_1 \rightarrow \left(1 - \frac{K}{N}\right) \mathbf{r}_1^T \mathbf{r}_1$
- Thus,

$$\beta_{1,DEC}^{(N)} \rightarrow \beta_{1,DEC}^{*} = \begin{cases} \frac{P_1(1-\alpha)}{\sigma^2} & \alpha < 1 \\ 0 & \alpha \geqslant 1 \end{cases}$$

where $\alpha = \frac{K}{N}$

User Capacity under Power Control

- If the power levels of the users can be controlled, interference level can be controlled and
  - The user capacity can be increased (What is the maximum number of users per degree os freedom?) given a maximum power constraint
  - The power consumption per user can be reduced given a constant number of users

Matched Filter

- The asymptotic SIR relation for matched filter is:

$$\beta_{1,MF} = \frac{P_1}{\sigma^2 + \frac{1}{N}\sum_{i=2}^{K} P_i}$$

- To meet the SIR requirement β^*, set all the received power levels to:

$$P_{mf}(\beta^*) = \frac{\beta^* \sigma^2}{1 \Leftrightarrow \alpha\beta^*}$$

- Without a power constraint
  - User capacity is bounded by $\alpha_{\max} < \frac{1}{\beta^*}$ users / degree of freedom
  - As $\alpha \to \frac{1}{\beta^*}$, power requirement $\to \infty$
- With power constraint, $P_{\max}$, set $P_i = P_{\max}, \ \forall i$ to maximize α:

$$\alpha_{\max} = \frac{1}{\beta^*} \Leftrightarrow \frac{\sigma^2}{P_{\max}} \text{users/degree of freedom}$$

MMSE Receiver

Suppose the system supports an SIR of β^* for all users. Thus $\forall i$

$$\frac{P_i}{\sigma^2 + \frac{1}{N}\sum_{j \neq i} I(P_j, P_i, \beta^*)} \geqslant \beta^*$$

Let $P^* = \inf_i P_i$, the power of the weakest user, k. But still,

$$\frac{P^*}{\sigma^2 + \frac{1}{N}\sum_{j \neq k} I(P_j, P^*, \beta^*)} \geqslant \beta^*$$

Since I only decreases as the interferer powers decrease, we have

$$\frac{P^*}{\sigma^2 + \frac{1}{N}\sum_{j \neq k} I(P^*, P^*, \beta^*)} = \frac{P^*}{\sigma^2 + \alpha I(P^*, P^*, \beta^*)} \geqslant \beta^*$$

With $I(P^*, P^*, \beta^*) = \frac{P^*}{1+\beta^*}$, we have:

$$\alpha \leqslant \frac{1+\beta^*}{\beta^*} - (1+\beta^*)\frac{\sigma^2}{P^*}$$

MMSE Receiver - continued

- We can summarize this result as follows:

  1. Given a maximum power constrint P^* and an SIR requirement β^*, the maximum achievable rate $(P_i = P^*, \forall i)$:

  $$\alpha \leqslant \frac{1+\beta^*}{\beta^*} \Leftrightarrow (1+\beta^*)\frac{\sigma^2}{P^*}$$

  2. Without the power constraint, the bound relaxes to:

  $$\alpha < \frac{1+\beta^*}{\beta^*} = 1 + \frac{1}{\beta^*}$$

  3. The minimal solution is the degenerate power assignment. Otherwise, one can decrease the power levels of every other user to that of the weakest user and still meet β^*

- Given an SIR, β^*, the minimum power necessary at rate α is:

$$P_{mmse}(\beta^*) = \frac{\beta^* \sigma^2}{1 \Leftrightarrow \alpha \frac{1+\beta^*}{\beta^*}}$$

Remarks on Power Control and User Capacity

- With the conventional receiver, arbitrarily high β^* is not achievable: $\beta^* \uparrow \frac{1}{\alpha} \Rightarrow$ saturation ($P^* \to \infty$).
- MMSE receiver supports an extra user per degree of freedom ($\alpha_{\max} = 1 + \frac{1}{\beta^*}$).
  - At $\alpha = 1$ user/degree of freedom, arbitrarily high β^* can be achieved with $P_{mmse} = \mathrm{O}(\beta^{*2})$.
  - At $\alpha < 1$ users/degree of freedom, arbitrarily high β^* can be achieved with $P_{mmse} = \mathrm{O}(\beta^*)$.
- With a similar analysis for the decorrelator,
  - We get $\alpha_{\max} < 1$. The bound is independent of β^* (makes sense because projection along $(\mathrm{span}\{(s_j)_{j\neq 1}\})^\perp$) is independent of interferer powers.
  - At $\alpha < 1$ users/degree of freedom, arbitrarily high β^* can be achieved with $P_{dec} = \mathrm{O}(\beta^*)$.

Multiple Classes and Effective Bandwidths

- Conservation laws
  - Work conservation $\Rightarrow$ total resource consumption is constant. e.g., queueing systems
  - Effective bandwidth of a user is a measure of what portion of the total resource that user consumes.
  - $\sum$ effective bandwidths is constant
- Matched filter with J multiple classes each with $K_j = \alpha_j N$ users
  - Suppose class j users require β_j. The minimum power required by class j users is:

$$P_{mf}(j) = \frac{\beta_j \sigma^2}{1 \Leftrightarrow \sum_{k=1}^{J} \alpha_k \beta_k}$$

  - Thus, $\sum_{j=1}^{J} \alpha_j \beta_j < 1$ gives us the feasible rate region.

Multiple Classes - continued

- Using a similar approach for the MMSE receiver,
  - The minimum power necessary to meet the SIR requirement β_j, $\forall j$ is

$$P_{mmse}(j) = \frac{\beta_j \sigma^2}{1 - \sum_{k=1}^{K} \alpha_k \frac{\beta_k}{1+\beta_k}}$$

  - Thus, $\sum_{j=1}^{J} \alpha_j \frac{\beta_j}{1+\beta_j} < 1$ gives us the feasible rate region.
- Similarly for the decorrelator, the feasible rate region is specified by $\sum_{j=1}^{J} \alpha_j < 1$

Remarks on Effective Bandwidths

- The necessary power to meet β_j is constant over users of type j (otherwise decrease the powers down to that of the weakest type j user without violating the SIR constraint).
- Define $e_{mf,j} = \beta_j,\ e_{mmse,j} = \frac{\beta_j}{1+\beta_j},\ e_{dec,j} = 1$. Feasible rate regions are defined as:

$$\sum_j e_{\cdot,j}\alpha_j < 1$$

  - It seems reasonable to define $e_{\cdot,j}$ degrees of freedom/user as the effective bandwidth of type j users (α_j is inversely proportional to $e_{\cdot,j}$).
  - Boundary of the feasible rate region is linear (convex polyhedral) in all receivers. Makes sense in MF. In MMSE, a consequence of asymptotic decoupling of interference due to other users).
  - Performance with power control: low SIR $\Rightarrow$ MMSE $\sim$ MF; high SIR $\Rightarrow$ MMSE $\sim$ decorrelator

Antenna Diversity

- The results can be applied to a multiple-antenna scenario:
  - The model for a synchronous multi-access antenna-array system:

$$\mathbf{Y} = \sum_{m=1}^{K} X_m \mathbf{h}_m + \mathbf{W}$$

  where X_m is the symbol of m^{th} user
  - Suppose fading vectors, $\mathbf{h}_m$, are slowly varying and iid.
  - We have exactly the same model:
    * Asymptotically interfering users contribute additively to effective interference.
    * User capacity is characterized by sharing the N degrees of freedom among users according to their effective bandwidths.

Final Remarks

- Asymptotically, a decoupling of interfering effects is indeed possible for linear receivers
  - Each user can be ascribed a level of effective interference that it provides to every other user.
  - Under power control, the user capacity region has a linear boundary and each user can be assigned an effective bandwidth representing its resource consumption which is a non-decreasing function of their SIR requirement.

Discussion: TDMA or FDMA systems have a user capacity of 1 user/degree of freedom which is indeed equal to that of the decorrelator and almost identical to MMSE at high SIR. They are much simpler to implement (matched filters $\equiv$ MMSE receiver). What does this imply about direct sequence CDMA systems?