6.962: Week 4 Post-Class Discussion

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Date: October 4, 2000
Topic: Codes on Graphs

1 Applying the Sum-Product Algorithm To Factor Graphs without Probabilistic Factors

An interesting question arose during the presentation, which was: "If a global function has only characteristic functions as its factors (i.e. no probabilistic functions), then how do we interpret the use of the sum-product algorithm in the corresponding factor graph?"

To answer this question, we must remember that what is being passed between nodes are *functions*. As long as we keep that in mind, we can apply the sum-product algorithm to any factor graph, whether or not it has factor nodes representing probabilistic factors.

1.1 A Worked Example

Let C be a binary linear code with parity-check matrix

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}. \tag{1}$$

Then C is the set of all binary 5-tuples $\mathbf{x} \triangleq (x_1, x_2, x_3, x_4, x_5)$ satisfying two simultaneous equations expressed in matrix form as $H\mathbf{x}^T = 0$. Membership in C is completely determined by checking whether each of the two equations

$$\begin{aligned} x_1 \oplus x_2 \oplus x_3 &= 0 \\ x_3 \oplus x_4 \oplus x_5 &= 0 \end{aligned}$$

is satisfied. The global function of interest is the characteristic function

$$g(x_1, x_2, x_3, x_4, x_5) = f_A(x_1, x_2, x_3) f_B(x_3, x_4, x_5) \tag{2}$$

where

$$\begin{aligned} g(x_1, x_2, x_3, x_4, x_5) &= [(x_1, x_2, x_3, x_4, x_5) \in C] \\ f_A(x_1, x_2, x_3) &= [x_1 \oplus x_2 \oplus x_3 = 0] \\ f_B(x_3, x_4, x_5) &= [x_3 \oplus x_4 \oplus x_5 = 0] \end{aligned}$$

and $\oplus$ denotes the sum in $GF(2)$. The corresponding factor graph is shown in Fig. 1.

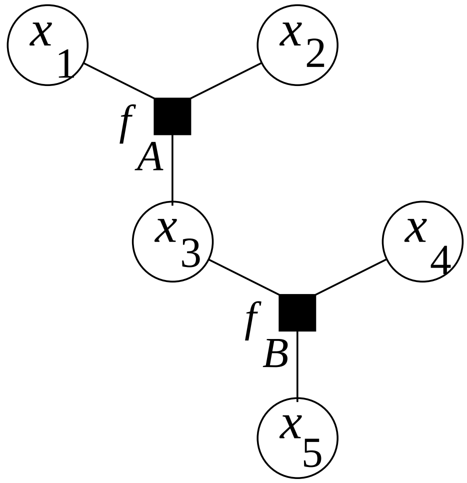

Figure 1: A factor graph for the binary linear code C.

Let $\mu_{x \to f}(x)$ denote the function sent from node x to node f in the operation of the sum-product algorithm, and let $\mu_{f \to x}(x)$ denote the message sent from node f to node x. We now show that the 5 marginal functions of the global function $[(x_1, x_2, x_3, x_4, x_5) \in C]$ can be computed in 4 steps.

Step 1: The initial functions passed from the leaf variable nodes to the factor nodes are functions with value 1 for all possible values of the leaf variables.

$$\mu_{x_1 \to f_A}(x_1) = \begin{cases} 1 & \text{if } x_1 = 0 \\ 1 & \text{if } x_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{x_2 \to f_A}(x_2) = \begin{cases} 1 & \text{if } x_2 = 0 \\ 1 & \text{if } x_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{x_4 \to f_B}(x_4) = \begin{cases} 1 & \text{if } x_4 = 0 \\ 1 & \text{if } x_4 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{x_5 \to f_B}(x_5) = \begin{cases} 1 & \text{if } x_5 = 0 \\ 1 & \text{if } x_5 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Step 2: Functions are then passed from the two factor nodes to the x_3 variable node.

$$
\begin{aligned}
\mu_{f_A \to x_3}(x_3) &= \sum_{\sim\{x_3\}} [x_1 \oplus x_2 \oplus x_3 = 0]\mu_{x_1 \to f_A}(x_1)\mu_{x_2 \to f_A}(x_2) \\
&= \sum_{\sim\{x_3\}} [x_1 \oplus x_2 \oplus x_3 = 0] \cdot 1 \cdot 1 \\
&= [0 \oplus 0 \oplus x_3 = 0] + [0 \oplus 1 \oplus x_3 = 0] + [1 \oplus 0 \oplus x_3 = 0] + [1 \oplus 1 \oplus x_3 = 0] \\
&= \begin{cases} 2 & \text{if } x_3 = 0 \\ 2 & \text{if } x_3 = 1 \\ 0 & \text{otherwise} \end{cases} \\
\mu_{f_B \to x_3}(x_3) &= \sum_{\sim\{x_3\}} [x_3 \oplus x_4 \oplus x_5 = 0]\mu_{x_4 \to f_B}(x_4)\mu_{x_5 \to f_B}(x_5) \\
&= \sum_{\sim\{x_3\}} [x_3 \oplus x_4 \oplus x_5 = 0] \cdot 1 \cdot 1 \\
&= [x_3 \oplus 0 \oplus 0 = 0] + [x_3 \oplus 0 \oplus 1 = 0] + [x_3 \oplus 1 \oplus 0 = 0] + [x_3 \oplus 1 \oplus 1 = 0] \\
&= \begin{cases} 2 & \text{if } x_3 = 0 \\ 2 & \text{if } x_3 = 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Step 3: The functions arriving at the x_3 variable node are propagated past the x_3 variable node.

$$
\begin{aligned}
\mu_{x_3 \to f_A}(x_3) &= \mu_{f_B \to x_3}(x_3) = \begin{cases} 2 & \text{if } x_3 = 0 \\ 2 & \text{if } x_3 = 1 \\ 0 & \text{otherwise} \end{cases} \\
\mu_{x_3 \to f_B}(x_3) &= \mu_{f_A \to x_3}(x_3) = \begin{cases} 2 & \text{if } x_3 = 0 \\ 2 & \text{if } x_3 = 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Step 4: Functions are sent from the factor nodes to the leaves.

$$
\begin{aligned}
\mu_{f_A \to x_1}(x_1) &= \sum_{\sim\{x_1\}} [x_1 \oplus x_2 \oplus x_3 = 0] \mu_{x_2 \to f_A}(x_2) \mu_{x_3 \to f_A}(x_3) \\
&= \sum_{\sim\{x_1\}} [x_1 \oplus x_2 \oplus x_3 = 0] \cdot 1 \cdot 2 \\
&= 2([x_1 \oplus 0 \oplus 0 = 0] + [x_1 \oplus 0 \oplus 1 = 0] + [x_1 \oplus 1 \oplus 0 = 0] + [x_1 \oplus 1 \oplus 1 = 0]) \\
&= \begin{cases} 4 & \text{if } x_1 = 0 \\ 4 & \text{if } x_1 = 1 \\ 0 & \text{otherwise} \end{cases} \\
\mu_{f_A \to x_2}(x_2) &= \sum_{\sim\{x_2\}} [x_1 \oplus x_2 \oplus x_3 = 0] \mu_{x_1 \to f_A}(x_1) \mu_{x_3 \to f_A}(x_3) \\
&= \sum_{\sim\{x_2\}} [x_1 \oplus x_2 \oplus x_3 = 0] \cdot 1 \cdot 2 \\
&= 2([0 \oplus x_2 \oplus 0 = 0] + [0 \oplus x_2 \oplus 1 = 0] + [1 \oplus x_2 \oplus 0 = 0] + [1 \oplus x_2 \oplus 1 = 0]) \\
&= \begin{cases} 4 & \text{if } x_2 = 0 \\ 4 & \text{if } x_2 = 1 \\ 0 & \text{otherwise} \end{cases} \\
\mu_{f_B \to x_4}(x_4) &= \sum_{\sim\{x_4\}} [x_3 \oplus x_4 \oplus x_5 = 0] \mu_{x_3 \to f_B}(x_3) \mu_{x_5 \to f_B}(x_5) \\
&= \sum_{\sim\{x_4\}} [x_3 \oplus x_4 \oplus x_5 = 0] \cdot 2 \cdot 1 \\
&= 2([0 \oplus x_4 \oplus 0 = 0] + [0 \oplus x_4 \oplus 1 = 0] + [1 \oplus x_4 \oplus 0 = 0] + [1 \oplus x_4 \oplus 1 = 0]) \\
&= \begin{cases} 4 & \text{if } x_4 = 0 \\ 4 & \text{if } x_4 = 1 \\ 0 & \text{otherwise} \end{cases} \\
\mu_{f_B \to x_5}(x_5) &= \sum_{\sim\{x_5\}} [x_3 \oplus x_4 \oplus x_5 = 0] \mu_{x_3 \to f_B}(x_3) \mu_{x_4 \to f_B}(x_4) \\
&= \sum_{\sim\{x_5\}} [x_3 \oplus x_4 \oplus x_5 = 0] \cdot 2 \cdot 1 \\
&= 2([0 \oplus 0 \oplus x_5 = 0] + [0 \oplus 1 \oplus x_5 = 0] + [1 \oplus 0 \oplus x_5 = 0] + [1 \oplus 1 \oplus x_5 = 0]) \\
&= \begin{cases} 4 & \text{if } x_5 = 0 \\ 4 & \text{if } x_5 = 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Termination: Marginal functions of the global function are computed.

$$
\begin{aligned}
g_1(x_1) &= \mu_{f_A \to x_1}(x_1) = \begin{cases} 4 & \text{if } x_1 = 0 \\ 4 & \text{if } x_1 = 1 \\ 0 & \text{otherwise} \end{cases} \\
g_2(x_2) &= \mu_{f_A \to x_2}(x_2) = \begin{cases} 4 & \text{if } x_2 = 0 \\ 4 & \text{if } x_2 = 1 \\ 0 & \text{otherwise} \end{cases} \\
g_3(x_3) &= \mu_{f_A \to x_3}(x_3)\mu_{f_B \to x_3}(x_3) = \begin{cases} 4 & \text{if } x_3 = 0 \\ 4 & \text{if } x_3 = 1 \\ 0 & \text{otherwise} \end{cases} \\
g_4(x_4) &= \mu_{f_B \to x_4}(x_4) = \begin{cases} 4 & \text{if } x_4 = 0 \\ 4 & \text{if } x_4 = 1 \\ 0 & \text{otherwise} \end{cases} \\
g_5(x_5) &= \mu_{f_B \to x_5}(x_5) = \begin{cases} 4 & \text{if } x_5 = 0 \\ 4 & \text{if } x_5 = 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

How do we interpret these marginal functions? Remember that the global function is the characteristic function $g(x_1, x_2, x_3, x_4, x_5) = [(x_1, x_2, x_3, x_4, x_5) \in C]$, and a marginal function is

$$g_i(x_i) = \sum_{\sim\{x_i\}} [(x_1, x_2, x_3, x_4, x_5) \in C]. \tag{3}$$

Thus, $g_i(x_i = 0)$ is the number of valid codewords in C with $x_i = 0$, and $g_i(x_i = 1)$ is the number of valid codewords in C with $x_i = 1$. For example, if we want to know how many codewords in C have $x_4 = 0$, the sum-product algorithm tells us that there are $g_4(x_4 = 0) = 4$ such codewords. Indeed, this is true—$\{(0,0,0,0,0), (1,1,0,0,0), (0,0,1,0,1), (1,1,1,0,1)\}$ are the 4 codewords.