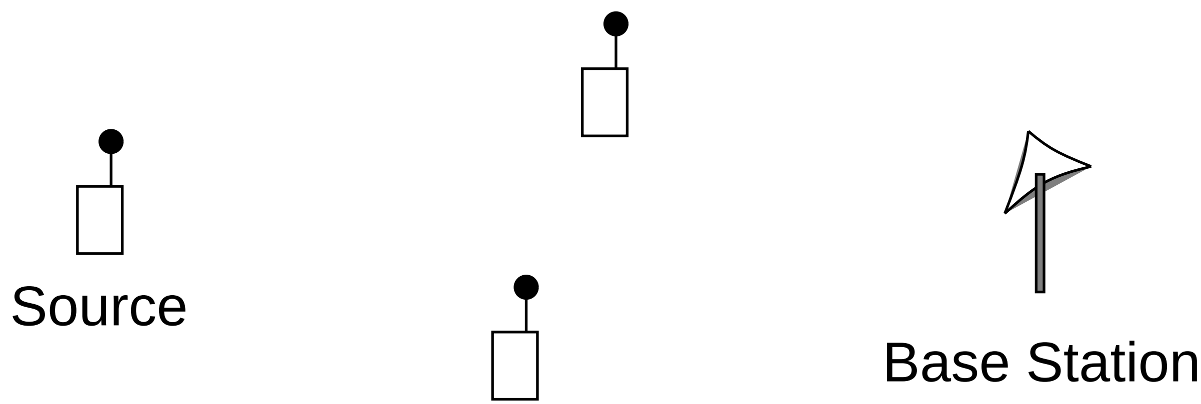
Motivation

Why do we care about relay situations?



Two effects: Source saves power by transmitting to the relays instead of the base station.

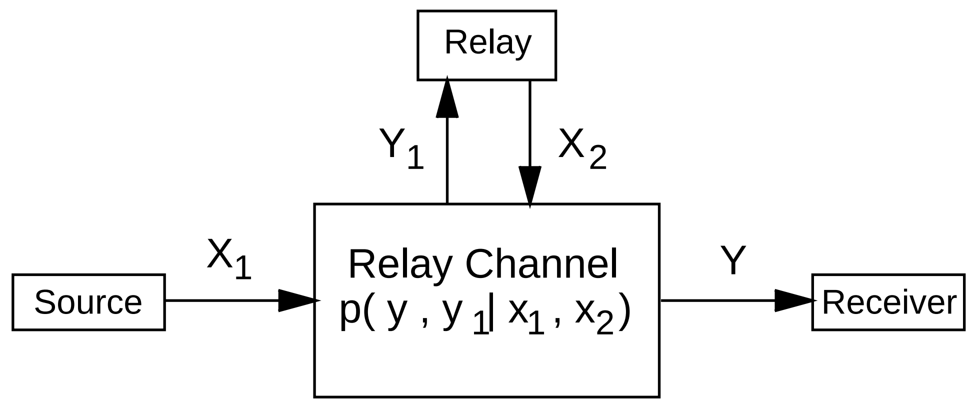
Relays get an effective power boost by cooperating in transmitting to the base station.

Motivation

Why do we care about relay situations?

Deep-space probe has very weak signal-to-noise ratio.
Independent observations are extremely useful.

Relay channel model



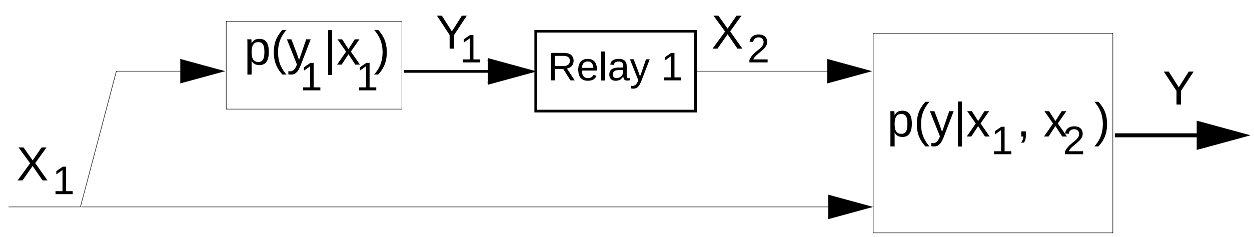
Relay transmission $x_{2,k}$ affects "next" relay observation $\mathbf{Y}_{1,k}$.

Current relay transmission can depend on all past observations:

$$\mathbf{X}_{2,k} = f(\mathbf{Y}_1^{k-1}).$$

Sole purpose of relay is to help get source information to the receiver.

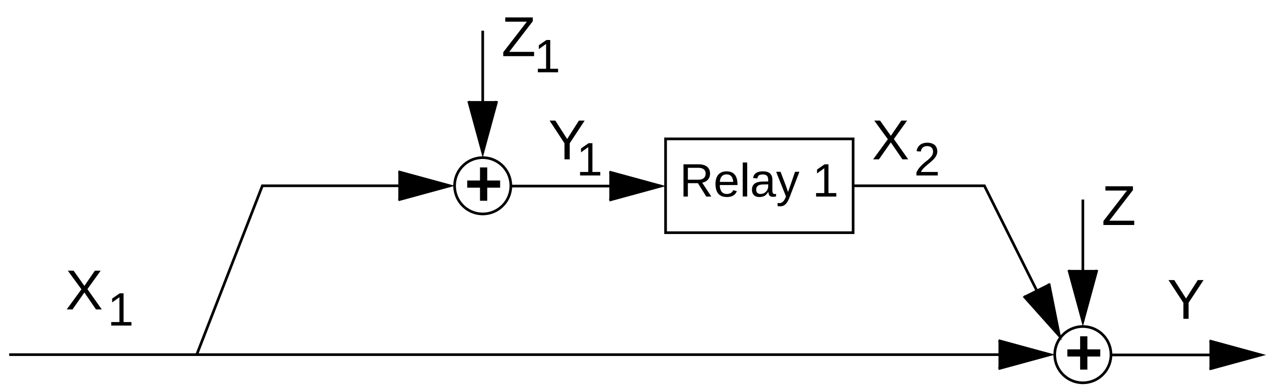
A nice class of relay channels



The input $\mathbf{X}_1$ is broadcast to both relay and receiver.

The receiver sees $\mathbf{Y}$, a noisy combination of source and relay signals.

In the Gaussian case...

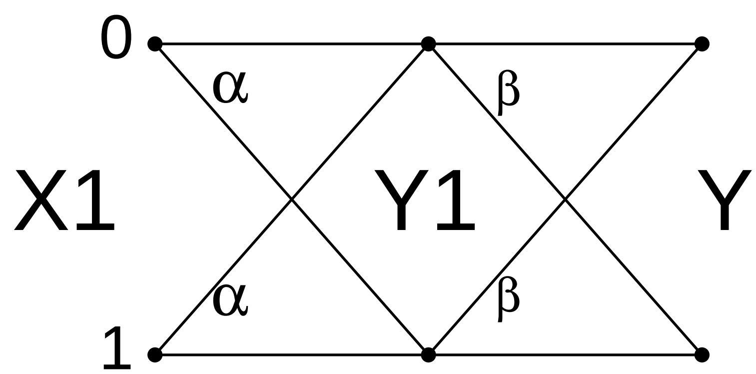


This discrete-time model would require synchronization of the source and relay carrier phases at the receiver.

On degraded relay channels

Physically degraded if $p(y_1, y|x_1, x_2) = p(y_1|x_1, x_2) \cdot p(y|y_1, x_2)$.

$\Rightarrow$ $\mathbf{Y}$ is a noisy version of the *actual outcome* $\mathbf{Y}_1 = y_1$.



This is very unnatural in the Gaussian case considered in the paper.

Physically degraded concept is useful for feedback situations:

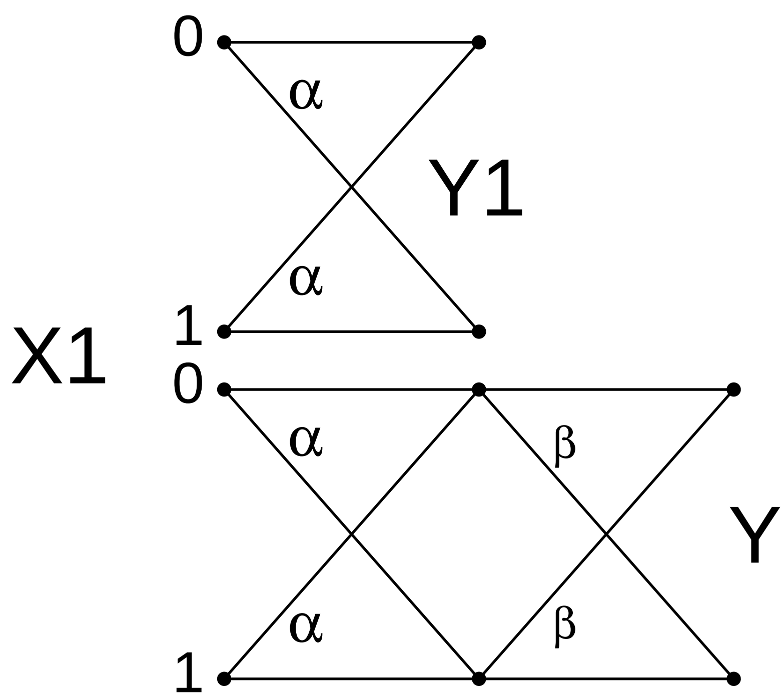
Relay gets $\mathbf{Y}$ fed back perfectly so that $\mathbf{Y}_1$ is really $(\mathbf{Y}_1, \mathbf{Y})$.

More on degraded relay channels

A more natural class is stochastically degraded:

$$p(y|x_1, x_2) = \sum_{y_1} p(y_1|x_1, x_2) \cdot t(y|y_1, x_2)$$

for some transition matrix $t(\cdot)$.



The stochastically degraded class is still an open problem.

A general achievability scheme

Choose $p(x_1, x_2) = p(x_2) \cdot p(x_1|x_2)$.

$$R = \min \ [\ I(\mathbf{X}_1, \mathbf{X}_2 \ ; \mathbf{Y}),\ I(\mathbf{X}_1; \mathbf{Y}_1 |\ \mathbf{X}_2)\ ]$$

There are 2^{nR} messages, w.

Randomly and uniformly partition these into 2^{nR_0} bins, $s(w)$.

There are 2^{nR_0} codewords $x_2(s)$ for sending bin index over $\mathbf{X}_2 \to \mathbf{Y}$ channel, with $\mathbf{X}_1$ being treated as noise.

Achievability approach

Idea:

First stage: Relay learns exact message w_k.

Receiver forms list of possible messages $\mathcal{L}(y_k)$.

Second stage: Relay sends codeword associated with bin of message w_k, $x_2(s(w_k))$.

Receiver finds unique message in $\mathcal{L}(y_k) \cap s(w_k)$.

Code structure:

Generate 2^{nR_0} codewords for the relay $\sim p(x_2)$, indexed $x_2(s)$.

For each such bin codeword, generate 2^{nR} codewords for the source $\sim p(x_1|x_2(s))$.

Steps in the achievability proof

(1) Relay decodes the correct message w_{k+1} if $R < I(X_1; Y_1 | X_2)$.

(2) Receiver decodes the correct bin of message w_k, $s(w_k)$, if $R_0 < I(X_2; Y)$.

(3) w_k is the unique message in $\mathcal{L}(y_k) \cap s(w_k)$ if $R < I(X_1; Y | X_2) + R_0$.

Basically, step (3) works as follows:
For an incorrect message w,

$$\text{Prob}(w \in \mathcal{L}(y_k)) \leq 2^{-n(I(X_1;Y|X_2)-7\epsilon)}.$$

Union bound: $\text{Prob}(\exists\ w \neq w_k,\ w \in \mathcal{L}(y_k) \cap s(w_k))$

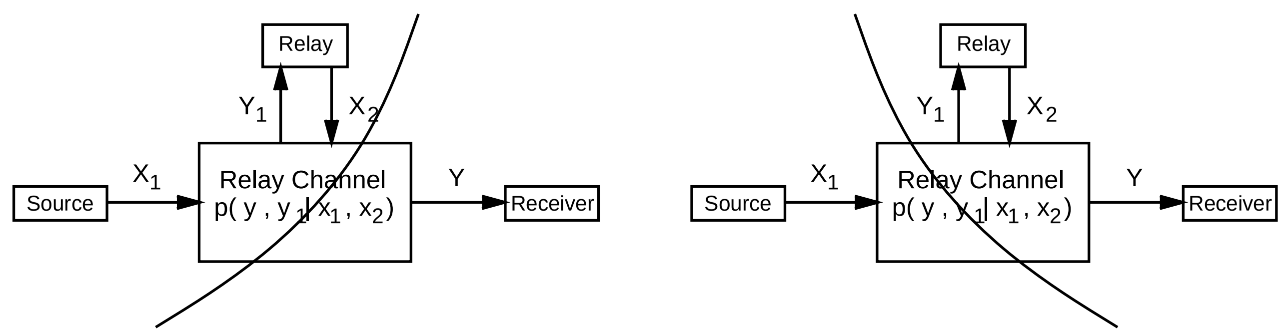
$$\leq (2^{nR}) \cdot (2^{-n(I(X_1;Y|X_2)-7\epsilon)}) \cdot (2^{-nR_0}).$$

A general converse statement

For some $p(x_1, x_2)$,

$$R \leq \min \ [\ I(X_1, X_2 \ ; Y),\ I(X_1; Y_1, Y|\ X_2)\ ]$$

These are basically information-theoretic cut-sets for a memoryless channel.



The derivation follows from basic identities/inequalities, memorylessness of the channel, and $X_{2,k} = f(Y_1^{k-1})$.

Details available upon request!

Physically degraded channel converse

Start with the general converse:

$$R \leq [\ I(X_1, X_2\ ; Y),\ I(X_1; Y_1, Y |\ X_2)\ ].$$

Since $X_1 \to (X_2, Y_1) \to Y$,

$\Rightarrow \quad H(X_1 | X_2, Y_1, Y) = H(X_1 | X_2, Y_1).$

I.e., Y yields no useful information about X_1 once we know Y_1 (and X_2).

So $I(X_1; Y_1, Y |\ X_2) = I(X_1; Y_1 |\ X_2)$, and thus

$$R \leq [\ I(X_1, X_2\ ; Y),\ I(X_1; Y_1 |\ X_2)\ ].$$

Degraded Gaussian channel achievability

$$
\begin{aligned}
X_2 &\sim \mathcal{N}(0, P_2) && \leftarrow \quad \text{relay codebook for current msg} \\
X_{10} &\sim \mathcal{N}(0, \alpha P_1) && \leftarrow \quad \text{source codebook for next msg} \\
X_1 &= X_{10} + \sqrt{\frac{(1-\alpha)P_1}{P_2}}\, X_2 && \leftarrow \quad \text{actual source codeword}
\end{aligned}
$$

Note that $\mathbb{E}\left\{(X_1 + X_2)^2\right\} = \alpha P_1 + (\sqrt{P_2} + \sqrt{(1-\alpha)P_1})^2$.

In the code construction, the source sends a scaled version of the relay codeword (old message) plus an independent codeword (new message).

The common parts coherently combine at the receiver.

Evaluating the achievable rates

For any $\alpha \in [0,1]$ we can achieve

$$R = \min \left[0.5 \log_2 \left(1 + \frac{P_1 + P_2 + 2\sqrt{(1-\alpha)P_1 P_2}}{N_1 + N_2} \right), 0.5 \log_2 \left(1 + \frac{\alpha P_1}{N_1} \right) \right]$$

Converse for degraded Gaussian channel

(1) Start with $nR \leq \sum I(X_{1,i}; Y_{1,i} | X_{2,i})$

Gaussians maximize entropy under second moment constraint, convexity and Jensen's inequality:

$$R \leq 0.5 \log \left(1 + \frac{\frac{1}{n} \sum \mathbb{E}\left\{ \mathrm{var}(X_{1,i} | X_{2,i}) \right\}}{N_1} \right)$$

$$\leq 0.5 \log \left(1 + \frac{P_1 - \frac{1}{n} \sum \mathbb{E}_{X_{2,i}} \left\{ \left(\mathbb{E}_{X_{1,i} | X_{2,i}} \left\{ X_{1,i} | X_{2,i} = x_{2,i} \right\} \right)^2 \right\}}{N_1} \right)$$

$$= 0.5 \log \left(1 + \frac{\alpha P_1}{N_1} \right).$$

Converse for degraded Gaussian channel

(2) Start with $nR \leq \sum I(X_{1,i}, X_{2,i}; Y_{1,i})$

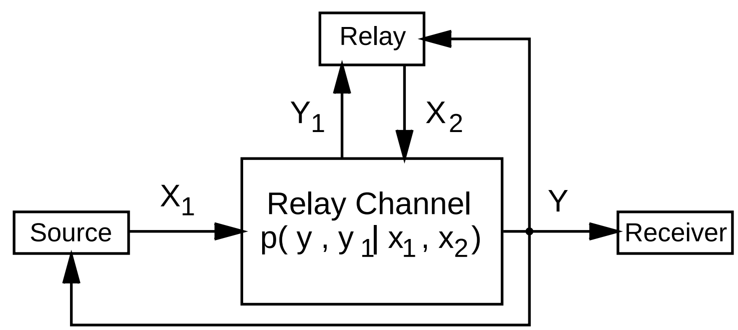
Similarly, with Gaussian entropy and convexity,

$$R \leq 0.5 \log\left(1 + \frac{P_1 + P_2 + \frac{2}{n}\sum \mathbb{E}\{X_{2,i}X_{1,i}\}}{N_1 + N_2}\right).$$

With some careful manipulation, it can be shown that

$$R \leq 0.5 \log\left(1 + \frac{P_1 + P_2 + 2\sqrt{(1-\alpha)P_1P_2}}{N_1 + N_2}\right).$$

General relay channel with feedback



Apply Theorem 1 to achieve

$$\min \ [\ I(X_1, X_2 \ ; Y), \ I(X_1; (Y_1, Y)| \ X_2) \ ]$$

The converse follows exactly as before.

A second achievability scheme (Theorem 6)

Choose $p(x_1), p(x_2)$, and $p(\hat{y}_1|y_1, x_2)$.

Provided $I(X_2;Y) \geq I(Y_1;\widehat{Y}_1|X_2,Y)$, we can achieve

$$R = I(X_1;\widehat{Y}_1, Y|X_2).$$

There are 2^{nR} messages, w.

There are 2^{nR_0} codewords $x_2(s)$ for the noisy channel $\mathbf{X}_2 \to \mathbf{Y}$, with $\mathbf{X}_1$ being treated as noise.

For each of these, there are $2^{n\widehat{R}}$ quantization/rate-distortion codewords for the relay observation.

All $2^{n(\widehat{R}+R_0)}$ quantization codewords are randomly and uniformly partitioned into 2^{nR_0} bins, $s(w)$.

Second achievability approach

<u>Idea:</u>

First stage:

Source transmits new message $x_1(w_k)$.

Relay transmits bin codeword $x_2(s_{k-1})$ (old info).

Relay quantizes $y_{1,k}$ with codebook $\left\{\widehat{Y}_1 | s_{k-1}\right\}$.

Receiver decodes bin codeword $x_2(s_{k-1})$.

Receiver generates list of possible relay repr. vectors $\mathcal{L}(y_k)$ consistent with y_k and $x_2(s_{k-1})$.

Second stage:

Relay sends bin codeword $x_2(s_k)$.

$\mathcal{L}(y_k) \cap s_k$ contains only the correct $\hat{y}_1 | s_{k-1}$.

Receiver decodes message with the pair of observations $(\hat{y}_{1,k}, y_k)$.

Steps in the achievability proof

(1) Relay has enough repr. vectors to cover $Y_1|x_2$ if $\widehat{R} > I(\widehat{Y}_1; Y_1|X_2)$ (source coding with side information — strong typicality required).

(2) Receiver decodes the correct bin codeword if $R_0 < I(X_2; Y)$.

(3) Receiver decodes correct $\hat{y}_{1,k}|x_2(s)$ if $\widehat{R} < I(\widehat{Y}_1; Y|X_2) + R_0$.

(4) Receiver decodes correct message w if $R < I(X_1; \widehat{Y}_1, Y|X_2)$.

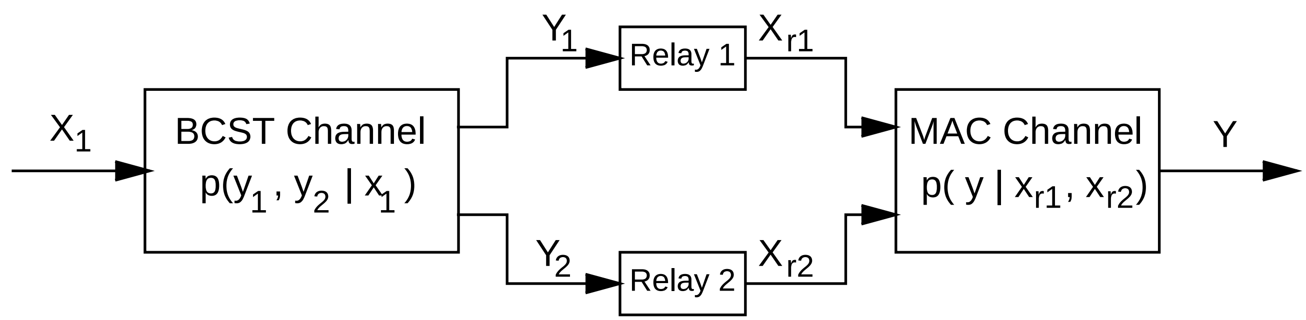
Step (3) works as follows:
For an incorrect representation vector $\hat{y}_1$,

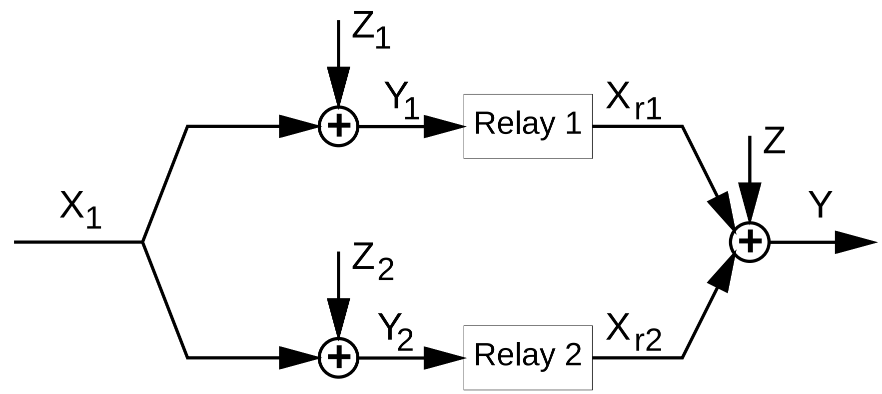
$$\text{Prob}((\hat{y}_1, y) \text{ typical } |x_2) \leq 2^{-n(I(\widehat{Y}_1;Y|X_2)-7\epsilon)}.$$

Union bound: $\text{Prob}(\exists\ \hat{y}_1 \neq \hat{y}_{1,k},\ \hat{y}_1 \in \mathcal{L}(y_k) \cap s(w_k))$

$$\leq (2^{n\widehat{R}}) \cdot (2^{-n(I(\widehat{Y}_1;Y|X_2)-7\epsilon)}) \cdot (2^{-nR_0}).$$

An easier model to consider





Why the double relay network?

Received power drops off rapidly with distance, so a direct path may not be desirable.

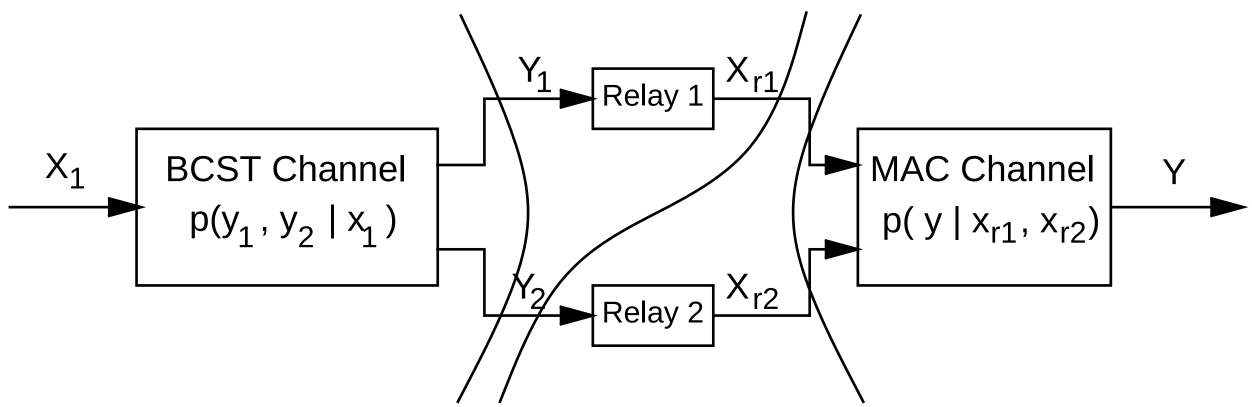
Synchronization in sending relevant information to the receiver.

There no longer seems a need for a two-stage coding procedure, so we can focus on the main issue:

How to design the cooperation in a noisy multi-terminal system?

Double relay network

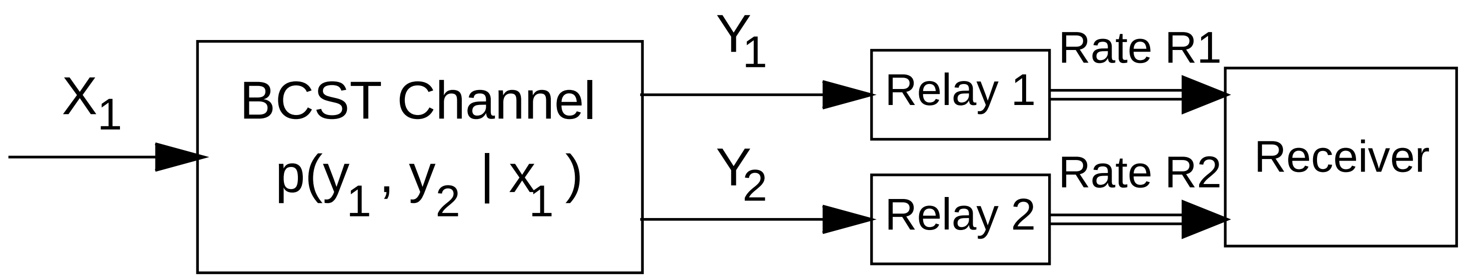
What limits communication is very difficult to understand.
We can still take "cut-sets", but they are loose:



A simple model that must be understood

We can turn an arbitrary MAC channel into a pair of noiseless links by picking a point in the (independent message) capacity region.

Therefore we must first understand what is limiting this simpler situation.



Gaussian double relay network

When the multi-access side is good in a symmetric network, simple amplification at the relays has excellent performance:

$$X_{r1} = k_1 \cdot Y_1 = k_1 \cdot (X_1 + Z_1),$$
$$X_{r2} = k_2 \cdot Y_2 = k_2 \cdot (X_2 + Z_2).$$

We can use single-user techniques / codes with this approach.

Even though neither relay decodes, there is a core component X_1 which coherently combines at the receiver.

Perfect scheme for the deep-space satellite, where the source is extremely weak.

General converse details

$$I(W;Y) \le \sum H(Y_i) - H(Y_i|X_{1,i}, X_{2,i}, W, Y^{i-1}) = \sum I(X_{1,i}, X_{2,i}; Y_i).$$

and

$$I(W;Y) \le I(W;Y_1,Y) = \sum H(W|Y_1^{i-1}, Y^{i-1}) - H(W|Y_1^i, Y^i) \qquad (\dagger)$$

$$X_{2,i} = f(Y_1^{i-1}) \Rightarrow \quad H(W|Y_1^{i-1}, Y^{i-1}) = H(W|Y_1^{i-1}, Y^{i-1}, X_{2,i}).$$

Therefore,

$$\begin{aligned}(\dagger) &\le \sum I(W; Y_{1,i}, Y_i|Y_1^{i-1}, Y^{i-1}, X_{2,i}) \\ &\le \sum H(Y_{1,i}, Y_i|X_{2,i}) - H(Y_{1,i}, Y_i|X_{2,i}, X_{1,i}) \\ &= \sum I(X_{1,i}; Y_{1,i}, Y_i|X_{2,i}).\end{aligned}$$