Interference Alignment and Degrees of Freedom of the K-User Interference Channel

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***Abstract*—For the fully connected K user wireless interference channel where the channel coefficients are time-varying and are drawn from a continuous distribution, the sum capacity is characterized as $C(SNR) = \frac{K}{2}\log(SNR) + o(\log(SNR))$. Thus, the K user time-varying interference channel almost surely has $K/2$ degrees of freedom. Achievability is based on the idea of interference alignment. Examples are also provided of fully connected K user interference channels with constant (not time-varying) coefficients where the capacity is exactly achieved by interference alignment at all SNR values.**

***Index Terms*—Capacity, degrees of freedom, interference alignment, interference channel, multiple-input–multiple-output (MIMO), multiplexing.**

I. Introduction

INFORMATION theorists have pursued capacity characterizations of interference channels for over three decades [1]–[16]. These efforts have produced an extensive array of interesting results that shed light on various aspects of the problem. Recently, a special case of the Han–Kobayashi scheme [3] is shown in [10] to achieve the capacity of the two-user interference channel within one bit. Reference [10] also provides a generalized degrees of freedom characterization that identifies different operational regimes for the two-user interference channel. For optimal wireless *network* design, the natural question is whether the insights from the two-user interference channel generalize to interference channel scenarios with *more than two users*. Unfortunately, for more than two users, even degrees of freedom characterizations are not known. At a coarse level, some of the interference management approaches used in practice and their information theoretic basis may be summarized as follows:

- **Decode**: If interference is strong, then the interfering signal can be decoded along with the desired signal—the tradeoff is that while decoding the interference may improve the rates for the desired signal, the decodability of the interfering signals limits the other users' rates. While less common in practice due to the complexity of multi-user detection, this approach is supported by the capacity results on the "very strong interference" [1], and "strong interference" [3], [4] scenarios in the context of the two-user interference channel. The extension of "strong interference" results to more than two users is not straightforward in general.
- **Treat as Noise**: If interference is weak, then the interfering signal is treated as noise and single user encoding/decoding suffices. This approach has been used in practice for a long time, e.g., for frequency-reuse in cellular systems. However, information theoretic validation for this approach has only recently been obtained through several concurrent works [10], [13], [14], [17]. While treating weak interference as noise may be natural from an engineering standpoint, it is somewhat surprising from an information theoretic perspective that introducing structure into the interference signals is not useful in this regime. This result has been established for more than two users as well.
- **Orthogonalize**: If the strength of interference is comparable to the desired signal, then interference is avoided by orthogonalizing the channel access. This is the basis for time (frequency) division medium access schemes that avoid interference between coexisting wireless systems by dividing spectrum in a cake-cutting fashion. Information-theoretic validation for this approach comes from the capacity pre-log (degrees of freedom) characterizations.[1] Considering only single-antenna nodes, the single user AWGN channel capacity in the absence of interference may be expressed as $\log(\text{SNR}) + o(\log(\text{SNR}))$ so that in the absence of interference the Gaussian channel has 1 degree of freedom. The sum capacity (per user) of the two-user interference channel is known to be $\frac{1}{2}\log(\text{SNR}) + o(\log(\text{SNR}))$ so that each user gets only half the degrees of freedom. It is conjectured in [18] that the sum capacity (per user) for the K user interference channel is $\frac{1}{K}\log(\text{SNR}) + o(\log(\text{SNR}))$. Orthogonal access schemes can be used to divide the 1 degree of freedom among the users such that each user gets a fraction and the sum of these fractions is equal to 1. We refer to this approach as the "cake-cutting" approach.

In this paper, we explore the regime identified with the "orthogonalize" approach above, where all desired and interfering signals are of comparable strength. We show that, for a broad class of wireless networks, even when there are more than two

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[1] If the capacity can be expressed as $C(\text{SNR}) = d\log(\text{SNR}) + o(\log(\text{SNR}))$ then we say the channel has d degrees of freedom (also known as the capacity pre-log or the multiplexing gain).

interfering users, the sum capacity (per user) is $\frac{1}{2}\log(\text{SNR}) + o(\log(\text{SNR}))$—i.e., everyone gets half the cake. The key to this result is an achievable scheme called interference alignment that is especially relevant to the interference channel with more than two users. We begin with the system model.

II. System Model

Consider the K user interference channel, comprised of K transmitters and K receivers. Each node is equipped with only one antenna (multiple antenna nodes are considered later in this paper). The channel output at the kth receiver over the tth time slot is described as follows:

$$Y^{[k]}(t) = H^{[k1]}(t)X^{[1]}(t) + H^{[k2]}(t)X^{[2]}(t) + \cdots + H^{[kK]}(t)X^{[K]}(t) + Z^{[k]}(t)$$

where, $k \in \{1,2,\ldots,K\}$ is the user index, $t \in \mathbb{N}$ is the time slot index, $Y^{[k]}(t)$ is the output signal of the kth receiver, $X^{[k]}(t)$ is the input signal of the kth transmitter, $H^{[kj]}(t)$ is the channel fade coefficient from transmitter j to receiver k over the tth time-slot and $Z^{[k]}(t)$ is the additive white Gaussian noise (AWGN) term at the kth receiver. We assume all noise terms are independent identically distributed (i.i.d.) zero-mean complex Gaussian with unit variance. To avoid degenerate channel conditions (e.g., all channel coefficients are equal or channel coefficients are equal to zero or infinity) we assume that the channel coefficient values are drawn i.i.d. from a continuous distribution and the absolute value of all the channel coefficients is bounded between a nonzero minimum value and a finite maximum value, $0 < H_{\min} \leq |H^{[kj]}| \leq H_{\max} < \infty$. We assume that channel knowledge is causal and globally available, i.e., at time slot t each node knows all channel coefficients $H^{[kj]}(\tau), \forall j,k \in \{1,2,\ldots,K\}, \tau \in \{1,2,\ldots,t\}$.

Remark: For the purpose of this work there is no fundamental distinction between time and frequency dimensions. The channel-use index t in the model described above could equivalently be used to describe time-slots, frequency slots or a time-frequency tuple if coding is performed in both time and frequency. The varying nature of the channel coefficients from one channel-use to another is, however, an important assumption. We also define the term "constant" channel, as the case where all channel coeffcients are fixed $H^{[kj]}(t) = H^{[kj]}, \forall t$.

We assume that transmitters $1,2,\ldots,K$ have independent messages $W_1, W_2, \ldots, W_K$ intended for receivers $1,2,\ldots,K$, respectively. The total power across all transmitters is assumed to be equal to ρ. We indicate the size of the message set by $|W_i(\rho)|$. For codewords spanning t_0 channel uses, the rates $R_i(\rho) = \frac{\log|W_i(\rho)|}{t_0}$ are achievable if the probability of error for all messages can be simultaneously made arbitrarily small by choosing an appropriately large t_0. The capacity region $\mathcal{C}(\rho)$ of the K user interference channel is the set of all *achievable* rate tuples $\mathbf{R}(\rho) = (R_1(\rho), R_2(\rho), \ldots, R_K(\rho))$.

A. Degrees of Freedom

Similar to the degrees of freedom region definition for the multiple-input–multiple-output (MIMO) X channel in [19] we define the degrees of freedom region $\mathcal{D}$ for the K user interference channel as follows:

$$\mathcal{D} = \Bigg\{(d_1, d_2, \ldots, d_K) \in \mathbb{R}_+^K : \forall (w_1, w_2, \ldots, w_K) \in \mathbb{R}_+^K \; w_1 d_1 + w_2 d_2 + \cdots + w_K d_K \leq \limsup_{\rho\to\infty}\left[\sup_{\mathbf{R}(\rho)\in\mathcal{C}(\rho)} [w_1 R_1(\rho) + w_2 R_2(\rho) + \cdots + w_K R_K(\rho)]\frac{1}{\log(\rho)}\right]\Bigg\}. \quad (1)$$

III. Overview of Main Results

The main insight offered in this paper is how the idea of interference alignment can be applied to the K user interference channel to restrict all interference at every receiver to approximately half of the received signal space, leaving the other half interference-free for the desired signal. We present a toy example to illustrate this key concept.

A. Interference Alignment—Toy Example

Consider the *constant* K-user interference channel defined by

$$Y^{[k]}(t) = X^{[k]}(t) + j \sum_{m=1, m\neq k}^{K} X^{[m]}(t) + Z^{[k]}(t) \quad (2)$$

where at the tth channel use, $Y^{[k]}(t), Z^{[k]}(t)$ are the kth receiver's output symbol and zero mean, unit variance, complex circularly symmetric additive white Gaussian noise (respectively) and $X^{[m]}(t)$ is the mth transmitter's input symbol. All direct channel coefficients are equal to 1 while all cross channel (carrying interference) coefficients are equal to $j = \sqrt{-1}$. The channel coefficients are fixed for all channel uses. All symbols are complex and all transmitted signals are subject to a power constraint P, so that $\rho = KP$. In the absence of interference, any user can achieve a capacity $C = \log(1+P)$ and the optimal input distribution is circularly symmetric complex Gaussian.

With all K users present the optimal (sum-capacity achieving) scheme is as follows. Each transmitter sacrifices half the signal space and only sends a *real* Gaussian signal with power P. Each receiver discards the imaginary part of the received signal that contains all the interference and is able to decode the desired signal free from interference at a rate $\frac{1}{2}\log(1 + P/(1/2)) = \frac{1}{2}\log(1+2P)$, where the factor of $1/2$ shows up in the denominator because only the "real" part of the additive noise (which has power $1/2$) is relevant. Thus, the sum rate with interference alignment is $\frac{K}{2}\log(1+2P)$.

Interestingly, the sum capacity of this channel is also $\frac{K}{2}\log(1+2P)$, which means that *for this symmetric channel interference alignment is capacity optimal at any SNR*. The converse argument is as follows. Consider any two users, say users 1 and 2 and eliminate all other users. This cannot hurt the users being considered. Consider any reliable coding scheme for this two user interference channel. Because the coding scheme is reliable by assumption, user 1 can successfully decode his message and subtract it out from the received signal. Now he can add back a phase-shifted version of his signal to reconstruct a new received signal that is statistically equivalent to the received signal of receiver 2. This implies that receiver 1 can decode both messages. Thus, the sum rate achieved by users 1 and 2 cannot be more than the sum-capacity of the two-user multiple access channel to receiver 1. But this multiple-access channel (MAC) has sum capacity $\log(1+2P)$. Similarly, considering any two users we find that their sum rate is bounded above by $\log(1+2P)$. Adding all these bounds together, we find that the outerbound on the sum-rate of all K users in the interference channel is $K/2\log(1+2P)$. Since this is achievable with interference alignment, it is also the capacity of this K-user interference channel. This is true at any SNR (P) value. One particularly interesting aspect of this example is that while the capacity achieving scheme uses Gaussian inputs, they are not circularly symmetric Gaussians. This is remarkable because for Gaussian point to point (MIMO), multiple access, broadcast channels with complex channel coefficients, the inputs (even if they are correlated and have different powers) are individually (element-wise) circularly symmetric Gaussian.

B. Other Examples

Interference alignment examples similar to the ones presented above can also be constructed in other dimensions such as space (beamforming across multiple antennas), time (either through propagation delays or through coding across time-varying channels), frequency (either through doppler-shifts or by coding across multiple-carriers with frequency selective coefficients) and codes (through lattice or multilevel codes that align interference within signal *levels*). Appendix I provides a simple example of interference alignment when each channel has a delay associated with it. As another example, consider two parallel interference channels (for example over two orthogonal carriers). On the first carrier suppose all channel coefficients are equal to 1, while on the second carrier suppose all desired channels are equal to one and the interfering channel coefficients are equal to -1. Then it is easily seen that by spreading the signal over the two carriers with the spreading code $[1\ \ 1]$ all interference is aligned. This example is presented in [20] to establish the result that parallel interference channels are inseparable, i.e., joint coding across parallel channels is necessary to achieve capacity (unlike Gaussian multiple access and broadcast channels where separate coding with optimal power allocation across carriers suffices to achieve capacity). Interference alignment is achieved through lattice codes in the context of many-to-one and one-to-many interference channels in [21] and for certain fully connected interference channels in [22], which also draws an interesting analogy between the propagtion delay example provided in Appendix I and the alignment of signal *levels* through multilevel codes. Quite simply, a multiplication of the transmitted signal X with the channel coefficient (say Q) leads to a decimal-point shift of the $Q-$ary representation (i.e., the base-Q representation) of the transmitted signal value which is similar to a propagation delay in time.

The enabling premise for interference alignment in all the preceding examples is the *relativity of alignment*—i.e., the alignment of signal vector spaces is relative to the observer (the receiver). Two transmitters may appear to be accessing the channel simultaneously to one receiver while they appear to be orthogonal to another receiver. Since each receiver has a different view, there exist scenarios where each receiver, from its own perspective, appears to be privileged relative to others. The goal of interference alignment is to create such scenarios in a wireless network. Specifically, interference alignment refers to a construction of signals in such a manner that they cast overlapping shadows at the receivers where they constitute interference while they remain distinguishable at the receivers where they are desired.

The idea of interference alignment evolved out of the degrees of freedom investigations on the two-user MIMO X channel [19], [23], [24] and the compound broadcast channel [25]. The two-user X channel is a communication system with two transmitters, two receivers, and four independent messages, one from each transmitter to each receiver. Taking advantage of the MAC and the broadcast channel (BC) components contained within the X channel, Maddah-Ali, Motahari, and Khandani proposed an elegant coding scheme (the MMK scheme) in [23] for the two-user MIMO X channel. The MMK scheme naturally combines successive decoding and dirty paper coding, the optimal schemes for the constituent MAC and BC. Interestingly, the MMK scheme achieves $\lfloor \frac{4}{3}M \rfloor$ degrees of freedom on the two-user X channel when all nodes are equipped with M antennas. The key to this result is the implicit interference alignment that is facilitated by the iterative optimization of transmit precoding and receive combining vectors. The first explicit interference alignment scheme is presented in [24] where it is shown that dirty paper coding and successive decoding are not required to achieve the maximum degrees of freedom on the two-user MIMO X channel. The achievability of $\frac{4}{3}M$ degrees of freedom and the converse are established in [19]. Interference alignment is used in [19], [26] to obtain innerbounds on the degrees of freedom region of the MIMO X channel. Interference alignment is also a key ingredient of the degrees of freedom characterization of the compound broadcast channel in [25].

C. Degrees of Freedom of the K User Interference Channel

In this paper we establish that the K user time-varying interference channel defined in Section II has $K/2$ degrees of freedom. Equivalently, at high SNR, every user is (simultaneously and almost surely) able to achieve reliable communication at rates approaching one half of the capacity that he could achieve in the absence of all interference. An interesting implication of this result is that time-varying interference networks are not fundamentally interference-limited.

The result has the same flavor as the toy examples presented earlier in this section. In both cases the conclusion is that *everyone gets half the cake*. While the toy examples represent contrived scenarios where the channel parameters are carefully selected to facilitate interference alignment, the degrees of freedom result is for channels whose coefficients are random, i.e., selected by nature. There is a penalty involved with random channel coefficients, but the penalty is $o(\log(\text{SNR}))$, i.e., it becomes a negligible fraction of the users' rates at high SNR. Indeed, we expect that the rate penalty will increase with the number of users, so that it will take higher and higher SNR to approach half of each user's capacity as the number of users

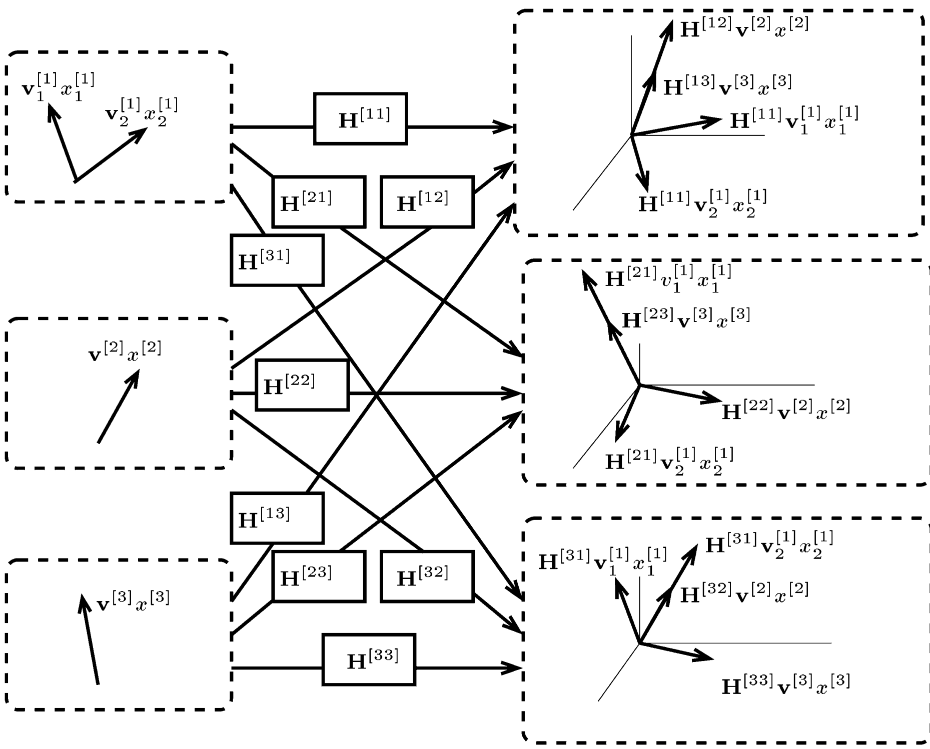

Fig. 1. Interference alignment on the three-user interference channel to achieve $4/3$ degrees of freedom.

increases. The degrees of freedom perspective is too coarse to capture this penalty and therefore does not reveal this competition among users. In this sense, the picture presented by the degrees of freedom result is optimistic.

The degrees of freedom for the constant interference channel (with the exception of certain MIMO scenarios) remains an open problem for more than two users. The interference alignment schemes used in this paper are based on beamforming over multiple symbol extensions of the time-varying channel. These schemes do not exactly achieve the outerbound on the degrees of freedom for a finite symbol extension. Instead, by using longer symbol extensions we are able to approach arbitrarily close to the outerbound. Intuitively, this can be understood as follows. In order to achieve exactly $1/2$ degrees of freedom (per user) over a finite symbol extension, every receiver must be able to partition its observed signal space into two subspaces of equal size, one of which is meant for the desired signals and the other is the "waste basket" for all the interference terms. Moreover, the vector spaces corresponding to the interference contributed by all undesired transmitters must exactly align at every receiver within the waste basket which has the same size as each of the interference signals. It turns out this problem is overconstrained and does not admit a solution. We circumvent this problem by allowing some overflow space (a few extra symbols) for interference terms that do not align perfectly. Fortunately, we find that the size of the overflow space becomes a negligible fraction of the total number of dimensions as we increase the size of the signal space. Thus, for any $\epsilon > 0$ it is possible to align interference to the extent that the achieved degrees of freedom are within an ϵ fraction of the outerbound. The tradeoff is that the smaller the value of ϵ, the larger the number of symbols (time slots) needed to recover a fraction $1-\epsilon$ of the outerbound value per symbol. As an example, consider the $K = 3$ user interference channel. We are able to achieve $3n+1$ degrees of freedom over a $2n+1$ symbol extension of the channel so that the degrees of freedom per symbol equal $\frac{3n+1}{2n+1}$, for any positive integer n. By choosing n large enough we can approach arbitrarily close to the outerbound of $3/2$ degrees of freedom. The case of $n = 1$ is shown in Fig. 1. The figure illustrates how $3n+1 = 4$ degrees of freedom are achieved over a $2n+1 = 3$ symbol extension of the channel with $K = 3$ single antenna users, so that a total of $4/3$ degrees of freedom are achieved per channel use. User 1 achieves 2 degrees of freedom by transmitting two independently coded streams along the beamforming vectors $\mathbf{v}_1^{[1]}, \mathbf{v}_2^{[1]}$ while users 2 and 3 achieve one degree of freedom by sending their independently encoded data streams along the beamforming vectors $\mathbf{v}^{[2]}, \mathbf{v}^{[3]}$, respectively. Let us pick $\mathbf{v}^{[2]}$ be the 3×1 vector of all ones.

$$\mathbf{v}^{[2]} = \mathbf{1}_{3\times 1}.$$

The remaining beamforming vectors are chosen as follows.

- At receiver 1, the interference from transmitters 2 and 3 are perfectly aligned
$$\mathbf{H}^{[12]}\mathbf{v}^{[2]} = \mathbf{H}^{[13]}\mathbf{v}^{[3]} \Rightarrow \mathbf{v}^{[3]} = \left(\mathbf{H}^{[13]}\right)^{-1}\mathbf{H}^{[12]}\mathbf{1}_{3\times 1}.$$
- At receiver 2, the interference from transmitter 3 aligns itself along one of the dimensions of the two-dimensional interference signal from transmitter 1
$$\mathbf{H}^{[23]}\mathbf{v}^{[3]} = \mathbf{H}^{[21]}\mathbf{v}_1^{[1]} \Rightarrow \mathbf{v}_1^{[1]} = \left(\mathbf{H}^{[21]}\right)^{-1}\mathbf{H}^{[23]}\left(\mathbf{H}^{[13]}\right)^{-1}\mathbf{H}^{[12]}\mathbf{1}_{3\times 1}.$$

- Similarly, at receiver 3, the interference from transmitter 2 aligns itself along one of the dimensions of interference from transmitter 1

$$\mathbf{H}^{[32]}\mathbf{v}^{[2]} = \mathbf{H}^{[31]}\mathbf{v}_2^{[1]} \Rightarrow \mathbf{v}_2^{[1]} = \left(\mathbf{H}^{[31]}\right)^{-1}\mathbf{H}^{[32]}\mathbf{1}_{3\times 1}.$$

Remark: Note that noncausal channel knowledge is not required because of the diagonal nature of the channel matrices $\mathbf{H}^{[kj]}$ resulting from symbol extensions over parallel channels.

Remark: Also note that in order to deliver a capacity that grows as $\log(\rho)$, i.e., in order to carry one degree of freedom, it is not necessary for a beamforming vector to be orthogonal to the interference. It suffices if the beamforming vector is linearly independent of the basis vectors of the interference signal space.

Remark: Finally, note that the construction of beamforming vectors for interference alignment is not unique. For example, $\mathbf{v}^{[2]}$ could be any random vector instead of the all ones vector. Moreover, at receiver 2, the interference from transmitter 3, $\mathbf{H}^{[23]}\mathbf{v}^{[3]}$ does not necessarily have to align with one of the beams received from transmitter 1. It only needs to lie within the two-dimensional (2-D) space spanned by the two beams received from transmitter 1.

$$\mathbf{H}^{[23]}\mathbf{v}^{[3]} \in \text{span}\left[\mathbf{H}^{[21]}\mathbf{v}_1^{[1]} \quad \mathbf{H}^{[21]}\mathbf{v}_2^{[1]}\right].$$

Similarly, at receiver 3, we only need

$$\mathbf{H}^{[32]}\mathbf{v}^{[2]} \in \text{span}\left[\mathbf{H}^{[31]}\mathbf{v}_1^{[1]} \quad \mathbf{H}^{[31]}\mathbf{v}_2^{[1]}\right].$$

Since in this work our interest is only in the degrees of freedom we do not consider the optimization of beamforming vectors over these possibilities.

IV. Degrees of Freedom for the K User Interference Channel

The following theorem presents the main result of this section.

Theorem 1: The number of degrees of freedom per user for the K user interference channel (defined in Section II) is $K/2$

$$\max_{\mathcal{D}} d_1 + d_2 + \cdots + d_K = K/2. \quad (3)$$

A. Converse for Theorem 1

The converse argument for the theorem is a simple extension of the outerbounds presented in [18], [27] which are themselves based on Carleial's outerbound [2]. However, because we assume that the channel coefficients are time-varying our model is different from these works which focus on constant channel coefficients. For the sake of completeness we derive the converse in this section.

The converse follows from the following lemma which provides an outerbound on the degrees of freedom region of the K user interference channel.

Lemma 1:

$$\max_{\mathcal{D}} d_i + d_j \leq \limsup_{\rho\to\infty} \sup_{\mathbf{R}(\rho)\in\mathcal{C}(\rho)} \frac{R_i(\rho) + R_j(\rho)}{\log(\rho)} \leq 1, \quad \forall\, i,j \in \{1,2,\ldots,K\}, i \neq j. \quad (4)$$

To obtain the converse result of Theorem 1, simply add all the inequalities from Lemma 1. This gives us

$$\max_{\mathcal{D}} \sum_{i,j\in\{1,2,\ldots,K\}, i\neq j} (d_i + d_j) \leq \sum_{i,j\in\{1,2,\ldots,K\}, i\neq j} 1 \quad (5)$$

$$\Rightarrow \max_{\mathcal{D}} d_1 + d_2 + \cdots + d_K \leq K/2. \quad (6)$$

The Proof of Lemma 1 (for the general case where all nodes have M antennas) is provided in Appendix II. A sketch of the proof is provided here. Without loss of generality, let us focus on case $i = 1, j = 2$. In order to obtain the corresponding outerbound, consider any reliable coding scheme for the K user interference channel. Now, suppose we eliminate messages $W_3, W_4, \ldots, W_K$, i.e., we use a pre-determined sequence of bits known to all the transmitters and receivers in place of these messages so that $R_3 = R_4 = \cdots = R_K = 0$. Then all receivers can subtract out the signals received from transmitters $3, 4, \ldots, K$. This is equivalent to a two user interference channel, where receiver 1 and 2 receive signals only from transmitters 1, 2, and decode messages W_1 and W_2, respectively. Next we argue that this two user interference channel can only have one degree of freedom. This argument proceeds as follows.

Let us provide message W_1 to receiver 2. Because receiver 2 has complete knowledge of all channel coefficients and the message W_1, we can eliminate the channel between transmitter 1 and receiver 2. Because the coding scheme is a reliable coding scheme by assumption, receiver 1 is also capable (with high probability) of decoding W_1, its desired message. In that case, we can also eliminate the channel from transmitter 1 to receiver 1. Then we end up with each receiver seeing only transmitter 2's signal with noise. For each channel use, we make sure that receiver 1 has the better channel by reducing noise variance if necessary. Thus, the signal at receiver 2 is a degraded version of the signal at receiver 1. We argue that if receiver 2 can decode its message W_2, receiver 1 must also be able to decode W_2 with a high probability. Finally, the closing argument is that since receiver 1 (with possibly reduced noise) is able to decode both messages W_1, W_2 for any reliable coding scheme, the rates $R_1(\rho), R_2(\rho)$ must lie in the capacity region of the multiple access channel from transmitters 1, 2 to receiver 1 with reduced noise at the receiver. But since this receiver has only one antenna and reducing the noise variance (by a finite amount that depends only on the channel coefficients and not on the SNR ρ) does not affect the degrees of freedom, the total degrees of freedom cannot be more than 1. This gives us the desired outerbound of (4) for the case $i = 1, j = 2$. □

B. Achievability Proof for Theorem 1 With $K = 3$

The achievability proof is presented next. Since the proof is rather involved, we present first the constructive proof for $K = 3$. The proof for general $K \geq 3$ is then provided in Appendix III.

We show that $(d_1, d_2, d_3) = (\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})$ lies in the degrees of freedom region $\forall n \in \mathbb{N}$. Since the degrees of freedom region is closed, this automatically implies that

$$\max_{(d_1,d_2,d_3)\in\mathcal{D}} d_1 + d_2 + d_3 \geq \sup_n \frac{3n+1}{2n+1} = \frac{3}{2}.$$

This result, in conjunction with the converse argument proves the theorem.

To show that $(\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})$ lies in $\mathcal{D}$, we construct an interference alignment scheme using only $2n+1$ time slots. We collectively denote the $2n+1$ symbols transmitted over $2n+1$ time slots as a supersymbol. We call this the $(2n+1)$ symbol extension of the channel. With the extended channel, the signal vector at the kth user's receiver can be expressed as

$$\bar{\mathbf{Y}}^{[k]}(t) = \bar{\mathbf{H}}^{[k1]}(t)\bar{\mathbf{X}}^{[1]}(t) + \bar{\mathbf{H}}^{[k2]}(t)\bar{\mathbf{X}}^{[2]}(t) + \bar{\mathbf{H}}^{[k3]}(t)\bar{\mathbf{X}}^{[3]}(t) + \bar{\mathbf{Z}}^{[k]}(t), \quad k \in \{1,2,3\}$$

where $\bar{\mathbf{X}}^{[k]}$ is a $(2n+1) \times 1$ column vector representing the $2n+1$ symbol extension of the transmitted symbol $X^{[k]}$, i.e.

$$\bar{\mathbf{X}}^{[k]}(t) \triangleq \begin{bmatrix} X^{[k]}((2n+1)(t-1)+1) \\ X^{[k]}((2n+1)(t-1)+2) \\ \vdots \\ X^{[k]}((2n+1)t) \end{bmatrix}.$$

Similarly $\bar{\mathbf{Y}}^{[k]}$ and $\bar{\mathbf{Z}}^{[k]}$ represent $2n+1$ symbol extensions of the $Y^{[k]}$ and $Z^{[k]}$, respectively. $\bar{\mathbf{H}}^{[kj]}$ is a diagonal $(2n+1) \times (2n+1)$ matrix representing the $2n+1$ symbol extension of the channel as shown in the equation at the bottom of the page.

We show that $(d_1, d_2, d_3) = (n+1, n, n)$ is achievable on this extended channel implying that $(\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})$ lies in the degrees of freedom region of the original channel.

In the extended channel, message W_1 is encoded at transmitter 1 into $n+1$ independent streams $x_m^{[1]}(t), m = 1, 2, \ldots, (n+1)$ sent along vectors $\mathbf{v}_m^{[1]}$ so that $\bar{\mathbf{X}}^{[1]}(t)$ is

$$\bar{\mathbf{X}}^{[1]}(t) = \sum_{m=1}^{n+1} x_m^{[1]}(t)\mathbf{v}_m^{[1]} = \bar{\mathbf{V}}^{[1]}\mathbf{X}^{[1]}(t)$$

where $\mathbf{X}^{[1]}(t)$ is a $(n+1) \times 1$ column vector and $\bar{\mathbf{V}}^{[1]}$ is a $(2n+1) \times (n+1)$-dimensional matrix. Similarly W_2 and W_3 are each encoded into n independent streams by transmitters 2 and 3 as $\mathbf{X}^{[2]}(t)$ and $\mathbf{X}^{[3]}(t)$, respectively.

$$\bar{\mathbf{X}}^{[2]}(t) = \sum_{m=1}^{n} x_m^{[2]}(t)\mathbf{v}_m^{[2]} = \bar{\mathbf{V}}^{[2]}\mathbf{X}^{[2]}(t),$$
$$\bar{\mathbf{X}}^{[3]}(t) = \sum_{m=1}^{n} x_m^{[3]}(t)\mathbf{v}_m^{[3]} = \bar{\mathbf{V}}^{[3]}\mathbf{X}^{[3]}(t).$$

The received signal at the ith receiver can then be written as

$$\bar{\mathbf{Y}}^{[i]}(t) = \bar{\mathbf{H}}^{[i1]}\bar{\mathbf{V}}^{[1]}\mathbf{X}^{[1]}(t) + \bar{\mathbf{H}}^{[i2]}\bar{\mathbf{V}}^{[2]}\mathbf{X}^{[2]}(t) + \bar{\mathbf{H}}^{[i3]}\bar{\mathbf{V}}^{[3]}\mathbf{X}^{[3]}(t) + \bar{\mathbf{Z}}^{[i]}(t).$$

In this achievable scheme, receiver i eliminates interference by zero-forcing all $\bar{\mathbf{V}}^{[j]}, j \neq i$ to decode W_i. At receiver $1, n+1$ desired streams are decoded after zero-forcing the interference to achieve $n+1$ degrees of freedom. To obtain $n+1$ interference free dimensions from a $2n+1$-dimensional received signal vector $\bar{\mathbf{Y}}^{[1]}(t)$, the dimension of the interference should be not more than n. This can be ensured by perfectly aligning the interference from transmitters 2 and 3 as follows:

$$\bar{\mathbf{H}}^{[12]}\bar{\mathbf{V}}^{[2]} = \bar{\mathbf{H}}^{[13]}\bar{\mathbf{V}}^{[3]}. \tag{7}$$

At the same time, receiver 2 zero-forces the interference from $\bar{\mathbf{X}}^{[1]}$ and $\bar{\mathbf{X}}^{[3]}$. To extract n interference-free dimensions from a $2n+1$-dimensional vector, the dimension of the interference has to be not more than $n+1$. For instance

$$\text{rank}\left(\left[\bar{\mathbf{H}}^{[21]}\bar{\mathbf{V}}^{[1]} \quad \bar{\mathbf{H}}^{[23]}\bar{\mathbf{V}}^{[3]}\right]\right) \leq n+1.$$

This can be achieved by choosing $\bar{\mathbf{V}}^{[3]}$ and $\bar{\mathbf{V}}^{[1]}$ so that

$$\bar{\mathbf{H}}^{[23]}\bar{\mathbf{V}}^{[3]} \prec \bar{\mathbf{H}}^{[21]}\bar{\mathbf{V}}^{[1]} \tag{8}$$

where $\mathbf{P} \prec \mathbf{Q}$, means that the set of column vectors of matrix $\mathbf{P}$ is a subset of the set of column vectors of matrix $\mathbf{Q}$. Similarly, to decode W_3 at receiver 3, we wish to choose $\bar{\mathbf{V}}^{[2]}$ and $\bar{\mathbf{V}}^{[1]}$ so that

$$\bar{\mathbf{H}}^{[32]}\bar{\mathbf{V}}^{[2]} \prec \bar{\mathbf{H}}^{[31]}\bar{\mathbf{V}}^{[1]}. \tag{9}$$

Thus, we wish to pick vectors $\bar{\mathbf{V}}^{[1]}$, $\bar{\mathbf{V}}^{[2]}$, and $\bar{\mathbf{V}}^{[3]}$ so that (7), (8), (9) are satisfied. Note that the channel matrices $\bar{\mathbf{H}}^{[ij]}$ have a full rank of $2n+1$ almost surely. Since multiplying by a full rank matrix (or its inverse) does not affect the conditions represented by (7), (8), and (9), they can be equivalently expressed as

$$\mathbf{B} = \mathbf{TC} \tag{10}$$
$$\mathbf{B} \prec \mathbf{A} \tag{11}$$
$$\mathbf{C} \prec \mathbf{A} \tag{12}$$

where

$$\mathbf{A} = \bar{\mathbf{V}}^{[1]} \tag{13}$$
$$\mathbf{B} = (\bar{\mathbf{H}}^{[21]})^{-1}\bar{\mathbf{H}}^{[23]}\bar{\mathbf{V}}^{[3]} \tag{14}$$
$$\mathbf{C} = (\bar{\mathbf{H}}^{[31]})^{-1}\bar{\mathbf{H}}^{[32]}\bar{\mathbf{V}}^{[2]} \tag{15}$$
$$\mathbf{T} = \bar{\mathbf{H}}^{[12]}(\bar{\mathbf{H}}^{[21]})^{-1}\bar{\mathbf{H}}^{[23]}(\bar{\mathbf{H}}^{[32]})^{-1}\bar{\mathbf{H}}^{[31]}(\bar{\mathbf{H}}^{[13]})^{-1}. \tag{16}$$

Note that $\mathbf{A}$ is a $(2n+1) \times (n+1)$ matrix. $\mathbf{B}$ and $\mathbf{C}$ are $(2n+1) \times n$ matrices. Since all channel matrices are invertible, we can choose $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ so that they satisfy (10)–(12) and then

$$\bar{\mathbf{H}}^{[kj]}(t) \triangleq \begin{bmatrix} H^{[kj]}((2n+1)(t-1)+1) & 0 & \cdots & 0 \\ 0 & H^{[kj]}((2n+1)(t-1)+2) & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & H^{[kj]}((2n+1)t) \end{bmatrix}.$$

use (13)–(16) to find $\bar{\mathbf{V}}^{[1]}$,$\bar{\mathbf{V}}^{[2]}$ and $\bar{\mathbf{V}}^{[3]}$. $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are picked as follows. Let $\mathbf{w}$ be the $(2n+1) \times 1$ column vector

$$\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

We now choose $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ as

$$\begin{aligned} \mathbf{A} &= [\mathbf{w} \quad \mathbf{T}\mathbf{w} \quad \mathbf{T}^2\mathbf{w} \quad \dots \quad \mathbf{T}^n\mathbf{w}], \\ \mathbf{B} &= [\mathbf{T}\mathbf{w} \quad \mathbf{T}^2\mathbf{w} \quad \dots \quad \mathbf{T}^n\mathbf{w}], \\ \mathbf{C} &= [\mathbf{w} \quad \mathbf{T}\mathbf{w} \quad \dots \quad \mathbf{T}^{n-1}\mathbf{w}]. \end{aligned}$$

It can be easily verified that $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ satisfy the three equations (10)–(12). Therefore, $\bar{\mathbf{V}}^{[1]}$, $\bar{\mathbf{V}}^{[2]}$ and $\bar{\mathbf{V}}^{[3]}$ satisfy the interference alignment equations in (7), (8) and (9).

Now, consider the received signal vectors at Receiver 1. The desired signal arrives along the $n+1$ vectors $\bar{\mathbf{H}}^{[11]}\bar{\mathbf{V}}^{[1]}$ while the interference arrives along the n vectors $\bar{\mathbf{H}}^{[12]}\bar{\mathbf{V}}^{[2]}$ and the n vectors $\bar{\mathbf{H}}^{[13]}\bar{\mathbf{V}}^{[3]}$. As enforced by (7) the interference vectors are perfectly aligned. Therefore, in order to prove that there are $n+1$ interference free dimensions it suffices to show that the columns of the square, $(2n+1) \times (2n+1)$-dimensional matrix

$$\left[\bar{\mathbf{H}}^{[11]}\bar{\mathbf{V}}^{[1]} \quad \bar{\mathbf{H}}^{[12]}\bar{\mathbf{V}}^{[2]}\right]. \tag{17}$$

are linearly independent almost surely. Multiplying by the full rank matrix $(\bar{\mathbf{H}}^{[11]})^{-1}$ and substituting the values of $\bar{\mathbf{V}}^{[1]}$, $\bar{\mathbf{V}}^{[2]}$, equivalently, we need to show that almost surely (see (18) at the bottom of the page) has linearly independent column vectors where $\mathbf{D} = (\bar{\mathbf{H}}^{[11]})^{-1}\bar{\mathbf{H}}^{[12]}$ is a diagonal matrix. In other words, we need to show $\det(\mathbf{S}) \neq 0$ with probability 1. The proof is obtained by contradiction. If possible, let $\mathbf{S}$ be singular with nonzero probability. For instance, $\Pr(|\mathbf{S}| = 0) > 0$. Further, let the diagonal entries of $\mathbf{T}$ be $\lambda_1, \lambda_2, \dots, \lambda_{2n+1}$ and the diagonal entries of $\mathbf{D}$ be $\kappa_1, \kappa_2 \dots, \kappa_{2n+1}$. Then the equation shown at the bottom of the page is true with nonzero probability.Let C_{ij} indicate the cofactor of the ith row and jth column of $|\mathbf{S}|$. Expanding the determinant along the first row, we get

$$\begin{aligned} |\mathbf{S}| = 0 \Rightarrow & C_{11} + \lambda_1 C_{12} + \dots \lambda_1^n C_{1(n+1)} \\ & + \kappa_1 \left[C_{1(n+2)} + \lambda_1 C_{1(n+3)} + \dots + \lambda_1^{n-1} C_{1(2n+1)}\right] \\ & = 0. \end{aligned}$$

None of "cofactor" terms C_{1j} in the above expansion depend on λ_1 and κ_1. If all values other than κ_1 are given, then the above is a linear equation in κ_1. Now, $|\mathbf{S}| = 0$ implies one of the following two events:

1) κ_1 is a root of the linear equation.
2) All the coefficients forming the linear equation in κ_1 are equal to 0, so that the singularity condition is trivially satisfied for all values of κ_1.

Since κ_1 is a random variable drawn from a continuous distribution, the probability of κ_1 taking a value which is equal to the root of this linear equation is zero. Therefore, the second event happens with probability greater than 0, and we can write

$$\begin{aligned} &\Pr(|\mathbf{S}| = 0) > 0 \\ &\quad \Rightarrow \Pr(C_{1(n+2)} + \lambda_1 C_{1(n+3)} + \dots + \lambda_1^{n-1} C_{1(2n+1)} = 0) > 0. \end{aligned}$$

Consider the equation

$$C_{1(n+2)} + \lambda_1 C_{1(n+3)} + \dots + \lambda_1^{n-1} C_{1(2n+1)} = 0.$$

Since the terms C_{1j} do not depend on λ_1, the above equation is a polynomial of degree n in λ_1. Again, as before, there are two possibilities. The first possibility is that λ_1 takes a value equal to one of the n roots of the above equation. Since λ_1 is drawn from a continuous distribution, the probability of this event happening is zero. The second possibility is that all the coefficients of the above polynomial are zero with nonzero probability and we can write

$$\begin{aligned} &\Pr\left(C_{1(n+2)} + \dots + \kappa_1 \lambda_1^n C_{1(2n+1)} = 0\right) \\ &\qquad > 0 \Rightarrow \Pr(C_{1(2n+1)} = 0) > 0. \end{aligned}$$

We have now shown that if the determinant of the $(2n+1) \times (2n+1)$ matrix $\mathbf{S}$ is equal to 0 with nonzero probability, then the determinant of following: $2n \times 2n$ matrix (obtained by stripping off the first row and last column of $\mathbf{S}$) is equal to 0 with nonzero probability as shown in the equation at the bottom of the following page with probability greater than 0. Repeating the above argument and eliminating the first row and last column at each stage, we get

$$\det \begin{bmatrix} 1 & \lambda_{n+1} & \lambda_{n+1}^2 & \dots & \lambda_{n+1}^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{2n+1} & \lambda_{2n+1}^2 & \dots & \lambda_{2n+1}^n \end{bmatrix} = 0$$

with probability greater than 0. But this is a Vandermonde matrix and its determinant

$$\prod_{n+1 \le i < j \le 2n+1} (\lambda_i - \lambda_j)$$

is equal to 0 only if $\lambda_i = \lambda_j$ for some $i \neq j$. Since λ_i are drawn independently from a continuous distribution, they are all distinct almost surely. This implies that $\Pr(|\mathbf{S}| = 0) = 0$.

$$\mathbf{S} \triangleq \left[\mathbf{w} \quad \mathbf{T}\mathbf{w} \quad \mathbf{T}^2\mathbf{w} \quad \dots \quad \mathbf{T}^n\mathbf{w} \quad \mathbf{D}\mathbf{w} \quad \mathbf{D}\mathbf{T}\mathbf{w} \quad \mathbf{D}\mathbf{T}^2\mathbf{w} \quad \dots \quad \mathbf{D}\mathbf{T}^{n-1}\mathbf{w}\right] \tag{18}$$

$$|\mathbf{S}| = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^n & \kappa_1 & \kappa_1\lambda_1 & \dots & \kappa_1\lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^n & \kappa_2 & \kappa_2\lambda_2 & \dots & \kappa_2\lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{2n+1} & \lambda_{2n+1}^2 & \dots & \lambda_{2n+1}^n & \kappa_{2n+1} & \kappa_{2n+1}\lambda_{2n+1} & \dots & \kappa_{2n+1}\lambda_{2n+1}^{n-1} \end{vmatrix} = 0.$$

Thus, the $n+1$ vectors carrying the desired signal at receiver 1 are linearly independent of the n interference vectors which allows the receiver to zero force interference and obtain $n+1$ interference free dimensions, and therefore $n+1$ degrees of freedom for its message.

At receiver 2 the desired signal arrives along the n vectors $\bar{\mathbf{H}}^{[22]}\bar{\mathbf{V}}^{[2]}$ while the interference arrives along the $n+1$ vectors $\bar{\mathbf{H}}^{[21]}\bar{\mathbf{V}}^{[1]}$ and the n vectors $\bar{\mathbf{H}}^{[23]}\bar{\mathbf{V}}^{[3]}$. As enforced by (8) the interference vectors $\bar{\mathbf{H}}^{[23]}\bar{\mathbf{V}}^{[3]}$ are perfectly aligned within the interference vectors $\bar{\mathbf{H}}^{[21]}\bar{\mathbf{V}}^{[1]}$. Therefore, in order to prove that there are n interference free dimensions at receiver 2 it suffices to show that the columns of the square, $(2n+1)\times(2n+1)$-dimensional matrix

$$\left[\bar{\mathbf{H}}^{[22]}\bar{\mathbf{V}}^{[2]} \quad \bar{\mathbf{H}}^{[21]}\bar{\mathbf{V}}^{[1]}\right] \tag{19}$$

are linearly independent almost surely. This proof is quite similar to the proof presented above for receiver 1 and is therefore omitted to avoid repetition. Using the same arguments we can show that both receivers 2 and 3 are able to zero force the $n+1$ interference vectors and obtain n interference free dimensions for their respective desired signals so that they each achieve n degrees of freedom.

Thus we established the achievability of $d_1+d_2+d_3=\frac{3n+1}{2n+1}$ for any n. This scheme, along with the converse automatically imply that

$$\sup_{(d_1,d_2,d_3)\in\mathcal{D}} d_1+d_2+d_3=\frac{3}{2}.$$

C. The Degrees of Freedom Region for the 3 User Interference Channel

Theorem 2: The degrees of freedom region of the three-user interference channel is characterized as follows:

$$\begin{aligned}\mathcal{D}=\{(d_1,d_2,d_3):&\\ d_1+d_2&\le 1\\ d_2+d_3&\le 1\\ d_1+d_3&\le 1\}.\end{aligned} \tag{20}$$

Proof: The converse argument is identical to the converse argument for Theorem 1 and is therefore omitted. We show achievability as follows. Let $\mathcal{D}'$ be the degrees of freedom region of the three-user interference channel. We need to prove that $\mathcal{D}'=\mathcal{D}$. We show that $\mathcal{D}\subset\mathcal{D}'$ which along with the converse proves the stated result.

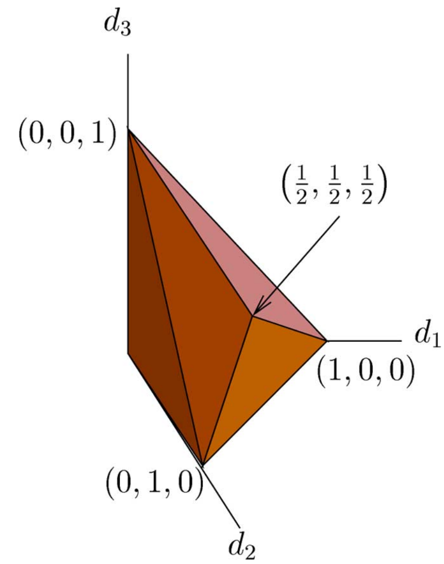

Fig. 2. Degrees of Freedom Region for the three-user interference channel.

The points $K=(0,0,1), L=(0,1,0), J=(1,0,0)$ can be verified to lie in $\mathcal{D}'$ through trivial achievable schemes. Also, Theorem 1 implies that $N=(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ lies in $\mathcal{D}'$ (Note that this is the only point which achieves a total of $\frac{3}{2}$ degrees of freedom and satisfies the inequalities in (20).)f Consider any point $(d_1,d_2,d_3)\in\mathcal{D}$ as defined by the statement of the theorem. The point (d_1,d_2,d_3) can then be shown to lie in a convex region whose corner points are $(0,0,0)$, J, K, L, and N. For instance, (d_1,d_2,d_3) can be expressed as a convex combination of the end points (see Fig. 2)

$$\begin{aligned}(d_1,&d_2,d_3)\\ &=\alpha_1(1,0,0)+\alpha_2(0,1,0)+\alpha_3(0,0,1)\\ &\quad+\alpha_4\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)+\alpha_5(0,0,0)\end{aligned}$$

where the constants α_i are defined as shown in the table at the bottom of the page.

It is easily verified that the values of α_i are nonnegative for all $(d_1,d_2,d_3)\in\mathcal{D}$ and that they add up to one. Thus, all points in $\mathcal{D}$ are convex combinations of achievable points J,K,L,N

$$\det\begin{bmatrix}1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^n & \kappa_2 & \kappa_2\lambda_2 & \dots & \kappa_2\lambda_2^{n-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & \lambda_{2n+1} & \lambda_{2n+1}^2 & \dots & \lambda_{2n+1}^n & \kappa_{2n+1} & \kappa_{2n+1}\lambda_{2n+1} & \dots & \kappa_{2n+1}\lambda_{2n+1}^{n-2}\end{bmatrix}=0$$

	α_1	α_2	α_3	α_4	α_5
$d_1+d_2+d_3\le 1$	d_1	d_2	d_3	0	$1-d_1-d_2-d_3$
$d_1+d_2+d_3>1$	$\frac{d_1-d_2-d_3+1}{2}$	$\frac{d_2-d_1-d_3+1}{2}$	$\frac{d_3-d_1-d_2+1}{2}$	$d_1+d_2+d_3-1$	0

and $(0,0,0)$. Since convex combinations are achievable by time sharing between the end points, this implies that $\mathcal{D} \subset \mathcal{D}'$. Together with the converse, we have $\mathcal{D} = \mathcal{D}'$ and the proof is complete. □

The assumption of time varying channels is intriguing because it is not clear if $K/2$ degrees of freedom will be achieved with constant channels. Therefore, the validity of the Host–Madsen–Nosratinia conjecture [18] remains unknown. On the one hand the number of degrees of freedom is a discontinuous measure as evident from the point to point channel where it represents the rank of the channel matrix. Therefore the constant coefficient case may be of limited significance. On the other hand, the constant channel case may shed light on novel interference alignment schemes such as interference alignment in the signal "level" dimension as demonstrated for certain special cases in [22] and interference alignment through lattice codes as demonstrated for the one-sided interference channel in [21].

Next we discuss the relationship between degrees of freedom and an $\mathcal{O}(1)$ capacity characterization.

D. The $\mathcal{O}(1)$ Capacity Approximation

The degrees of freedom d provide a capacity approximation that is accurate within $o(\log(\rho))$, i.e.

$$C(\rho) = d\log(\rho) + o(\log(\rho)). \tag{21}$$

In general, a capacity approximation within $\mathcal{O}(1)$ is more accurate than an approximation within $o(\log(\rho))$. However, it turns out that in many cases the two are directly related. For example, it is well known that for the full rank MIMO channel with M input antennas and N output antennas, transmit power ρ and i.i.d. zero mean unit variance additive white Gaussian noise (AWGN) at each receiver, the capacity $C(\rho)$ may be expressed as

$$\begin{aligned} C(\rho) &= \min(M,N)\log(1+\rho) + \mathcal{O}(1) \\ &= d\log(1+\rho) + \mathcal{O}(1). \end{aligned} \tag{22}$$

A similar relationship between the degrees of freedom and the $\mathcal{O}(1)$ capacity characterization also holds for the MIMO multiple access channel, the MIMO broadcast channel, and the two-user MIMO interference channel. For the MIMO, MAC, and BC, the outerbound on sum capacity obtained from full cooperation among the distributed nodes is $d\log(1+\rho)+\mathcal{O}(1)$. The innerbound obtained from zero forcing is also $d\log(1+\rho)+\mathcal{O}(1)$ so that we can write $C(\rho) = d\log(1+\rho) + \mathcal{O}(1)$. For the two user MIMO interference channel and the two-user MIMO X channel the outerbound is obtained following an extension of Carleial's outerbound [2] which results in a MIMO MAC channel. The innerbound is obtained from zero forcing. Since both of these bounds are within $\mathcal{O}(1)$ of $d\log(1+\rho)$ we can similarly write $C(\rho) = d\log(1+\rho)+\mathcal{O}(1)$. However, consider the K user interference channel with single antennas at each node. In this case, we have only shown

$$\begin{aligned} (K/2-\epsilon)\log(1+\rho) + \mathcal{O}(1) &\le C(\rho) \\ &\le (K/2)\log(1+\rho) + \mathcal{O}(1), \forall\, \epsilon > 0. \end{aligned} \tag{23}$$

To claim that capacity of the K user interference channel is $(K/2)\log(1+\rho)$ within $\mathcal{O}(1)$ we need to show an innerbound of $(K/2)\log(1+\rho) + \mathcal{O}(1)$. Since our achievable schemes are based on interference alignment and zero forcing, the natural question to ask is whether an interference alignment and zero forcing based scheme can achieve exactly $K/2$ degrees of freedom. The following explanation uses the $K=3$ case to suggest that the answer is negative.

Consider an achievable scheme that uses a M symbol extension of the channel. Now, consider a point $(\alpha_1, \alpha_2, \alpha_3)$ that can be achieved over this extended channel using interference alignment and zero-forcing alone. If possible, let the total degrees of freedom over this extended channel be $3M/2$. For instance, $\alpha_1 + \alpha_2 + \alpha_3 = 3M/2$. It can be argued along the same lines as the converse part of Theorem 1 that (α_i, α_j) is achievable in the two-user interference channel for $\forall (i,j) \in \{(1,2),(2,3),(3,1)\}$. Therefore

$$\begin{aligned} \alpha_1 + \alpha_2 &\le M, \\ \alpha_2 + \alpha_3 &\le M, \\ \alpha_1 + \alpha_3 &\le M. \end{aligned}$$

It can be easily seen that the only point $(\alpha_1, \alpha_2, \alpha_3)$ that satisfies the above inequalities and achieves a total of $3M/2$ degrees of freedom is $(\frac{M}{2}, \frac{M}{2}, \frac{M}{2})$. Therefore, any scheme that achieves a total of $3M/2$ degrees of freedom over the extended channel achieves the point $(\frac{M}{2}, \frac{M}{2}, \frac{M}{2})$.

We assume that the messages W_i are encoded along $M/2$ independent streams similar to the coding scheme in the proof of Theorem 1, i.e.

$$\bar{\mathbf{X}}^{[i]} = \sum_{m=1}^{M/2} x_m^{[i]} \mathbf{v}_m^{[i]} = \bar{\mathbf{V}}^{[i]} \mathbf{X}^{[i]}.$$

Now, at receiver 1, to decode an $M/2$-dimensional signal using zero-forcing, the dimension of the interference has to be at most $M/2$. For instance

$$\operatorname{rank}[\bar{\mathbf{H}}^{[13]}\bar{\mathbf{V}}^{[3]} \quad \bar{\mathbf{H}}^{[12]}\bar{\mathbf{V}}^{[2]}] = M/2. \tag{24}$$

Note that since $\bar{\mathbf{V}}^{[2]}$ has $M/2$ linearly independent column vectors and $\bar{\mathbf{H}}^{[12]}$ is full rank with probability 1, $\operatorname{rank}(\bar{\mathbf{H}}^{[12]}\bar{\mathbf{V}}^{[2]}) = M/2$. Similarly, the dimension of the interference from transmitter 3 is also equal to $M/2$. Therefore, the two vector spaces on the left-hand side of (24) must have full intersection, i.e.,

$$\operatorname{span}\left(\bar{\mathbf{H}}^{[13]}\bar{\mathbf{V}}^{[3]}\right) = \operatorname{span}\left(\bar{\mathbf{H}}^{[12]}\bar{\mathbf{V}}^{[2]}\right) \tag{25}$$

$$\operatorname{span}\left(\bar{\mathbf{H}}^{[23]}\bar{\mathbf{V}}^{[3]}\right) = \operatorname{span}\left(\bar{\mathbf{H}}^{[21]}\bar{\mathbf{V}}^{[1]}\right) \text{ (At receiver 2)} \tag{26}$$

$$\operatorname{span}\left(\bar{\mathbf{H}}^{[32]}\bar{\mathbf{V}}^{[2]}\right) = \operatorname{span}\left(\bar{\mathbf{H}}^{[31]}\bar{\mathbf{V}}^{[1]}\right) \text{ (At receiver 3)} \tag{27}$$

where $\operatorname{span}(\mathbf{A})$ represents the space spanned by the column vectors of matrix $\mathbf{A}$ The above equations imply that

$$\begin{aligned} &\operatorname{span}\left(\bar{\mathbf{H}}^{[13]}\left(\bar{\mathbf{H}}^{[23]}\right)^{-1}\bar{\mathbf{H}}^{[21]}\bar{\mathbf{V}}^{[1]}\right) \\ &= \operatorname{span}\left(\bar{\mathbf{H}}^{[12]}(\bar{\mathbf{H}}^{[32]})^{-1}\bar{\mathbf{H}}^{[31]}\bar{\mathbf{V}}^{[1]}\right) \\ &\Rightarrow \operatorname{span}\left(\bar{\mathbf{V}}^{[1]}\right) = \operatorname{span}(\mathbf{T}\bar{\mathbf{V}}^{[1]}) \end{aligned}$$

where $\mathbf{T} = (\bar{\mathbf{H}}^{[13]})^{-1}\bar{\mathbf{H}}^{[23]}(\bar{\mathbf{H}}^{[21]})^{-1}\bar{\mathbf{H}}^{[12]}(\bar{\mathbf{H}}^{[32]})^{-1}\bar{\mathbf{H}}^{[31]}$. The above equation implies that there exists at least one eigenvector $\mathbf{e}$ of $\mathbf{T}$ in $\operatorname{span}(\bar{\mathbf{V}}^{[1]})$. Note that since all channel matrices are diagonal, the set of eigenvectors of all channel matrices, their inverses and their products are all identical to the set of column vectors of the identity matrix. For instance, vectors of the form $[0\ 0\ \dots\ 1\ \dots\ 0]^T$. Therefore $\mathbf{e}$ is an eigenvector for all channel matrices. Since $\mathbf{e}$ lies in span($\bar{\mathbf{V}}^{[1]}$), (25)–(27) imply that,

$$\mathbf{e} \in \text{span}\left(\bar{\mathbf{H}}^{[ij]}\bar{\mathbf{V}}^{[i]}\right), \forall i,j \in \{1,2,3\}$$
$$\Rightarrow \mathbf{e} \in \text{span}\left(\bar{\mathbf{H}}^{[11]}\bar{\mathbf{V}}^{[1]}\right) \cap \text{span}\left(\bar{\mathbf{H}}^{[12]}\bar{\mathbf{V}}^{[2]}\right).$$

Therefore, at receiver 1, the desired signal $\bar{\mathbf{H}}^{[11]}\bar{\mathbf{V}}^{[1]}$ is *not* linearly independent with the interference $\bar{\mathbf{H}}^{[21]}\bar{\mathbf{V}}^{[2]}$. Therefore, receiver 1 cannot decode W_1 completely by merely zero-forcing the interference signal. Evidently, interference alignment in the manner described above cannot achieve exactly $3/2$ degrees of freedom on the 3 user interference channel with a single antenna at all nodes.

We explore this interesting aspect of the three-user interference channel further in the context of multiple antenna nodes. Our goal is to find out if exactly $3M/2$ degrees of freedom may be achieved with M antennas at each node. As shown by the following theorem, indeed we can achieve exactly $3M/2$ degrees of freedom so that the $\mathcal{O}(1)$ capacity characterization for $M > 1$ is indeed related to the degrees of freedom as $C(\rho) = (3M/2)\log(1+\rho) + \mathcal{O}(1)$.

V. Degrees of Freedom for the Interference Channel With Multiple Antenna Nodes

A. *The Three-User Interference Channel With $M > 1$ Antennas at Each Node and Constant Channel Coefficients*

The three-user MIMO interference channel is interesting because in this case we show that we can achieve exactly $3M/2$ degrees of freedom with *constant* channel matrices, i.e., time-variations are not required. This gives us a lowerbound on sum capacity of $3M/2\log(1+\rho) + \mathcal{O}(1)$. Since the outerbound on sum capacity is also $3M/2\log(1+\rho)+\mathcal{O}(1)$ we have an $\mathcal{O}(1)$ approximation to the capacity of the three-user MIMO interference channel with $M > 1$ antennas at all nodes.

Theorem 3: In a three-user interference channel with $M > 1$ antennas at each transmitter and each receiver and constant coefficients, the sum capacity $C(\rho)$ may be characterized (almost surely) as

$$C(\rho) = (3M/2)\log(1+\rho) + \mathcal{O}(1). \tag{28}$$

The outerbound follows directly from [27] which shows that the two-user interference channel with M antennas at each node and constant channel coefficients has only M degrees of freedom. In the three-user case, we eliminate one message at a time to obtain inequalities $d_1+d_2 \le M, d_2+d_3 \le M, d_1+d_3 \le M$. Adding up all three inequalities we obtain the converse.

The proof is presented in Appendices IV and V.

B. *The K User Interference Channel With Multiple Antenna Nodes*

Theorem 3 in the preceding section shows that $3M/2$ degrees of freedom are achievable on the three-user interference channel with $M > 1$ antennas and constant channel coefficients. It is not known if the result can be extended to $K > 3$ users. However, with time-variations it is easy to find the degrees of freedom for the K user interference channel with M antennas at each node. The result follows directly from Theorem 1 and is presented in the following Corollary.

Corollary 1: The time-varying K user interference channel with M antennas at each node has $KM/2$ degrees of freedom.

Proof: The converse for Corollary 1 is already derived in Appendix 1 in (31).

Achievability of Corollary 1 is also straightforward. Suppose we view each of the M colocated antennas at a node as a separate node. In other words we do not allow joint processing of signals obtained from the co-located antennas. Then, instead of a K user interference channel with M antenna nodes we obtain a KM user interference channel with single antenna nodes. But the result of Theorem 1 establishes that $KM/2$ degrees of freedom are achievable on this interference channel. Thus, we can also achieve $KM/2$ degrees of freedom on the K user interference channel with M antenna nodes. □

Last, let us consider the most general K user interference channel where each node is equipped with possibly different number of antennas. In this case also a lower bound on the degrees of freedom is directly established from the result of Theorem 1. The following Corollary states this result.

Corollary 2: The total degrees of freedom for the K user interference channel where transmitter i has M_i^T antennas and receiver i has M_i^R antennas $\forall i \in \{1,2,\dots,K\}$ is bounded below as

$$d_1 + d_2 + \cdots + d_K \ge \frac{1}{2}\sum_{i=1}^{K} \min\left(M_i^T, M_i^R\right). \tag{29}$$

Thus, no more than half the degrees of freedom are lost on the K user interference channel with multiple antenna nodes.

Proof: The achievability proof is straightforward as, once again, the ith transmitter receiver pair can be replaced with $\min(M_i^T, M_i^R)$ single antenna transmitter and receiver nodes by only allowing distributed processing of signals at each antenna and discarding the remaining antennas. Thus, we obtain an interference channel with $\sum_{i=1}^{K}\min(M_i^T, M_i^R)$ transmitters and receivers, each equipped with only a single antenna. The achievability of $\frac{1}{2}\sum_{i=1}^{K}\min(M_i^T, M_i^R)$ degrees of freedom on this interference channel then follows from the result of Theorem 1. □

Note that Corollary 2 only establishes an innerbound and is not always tight. For example, consider the two-user interference channel where each transmitter has two antennas while each receiver has only 1 antenna. While Corollary 2 only shows achievability of 1 degree of freedom for this channel, it is known that this interference channel has 2 degrees of freedom [27]. However, Corollary 2 is interesting because it shows that interference cannot reduce the degrees of freedom of the interference channel by more than half compared to when each transmitter and receiver is able to operate without interference from other users.

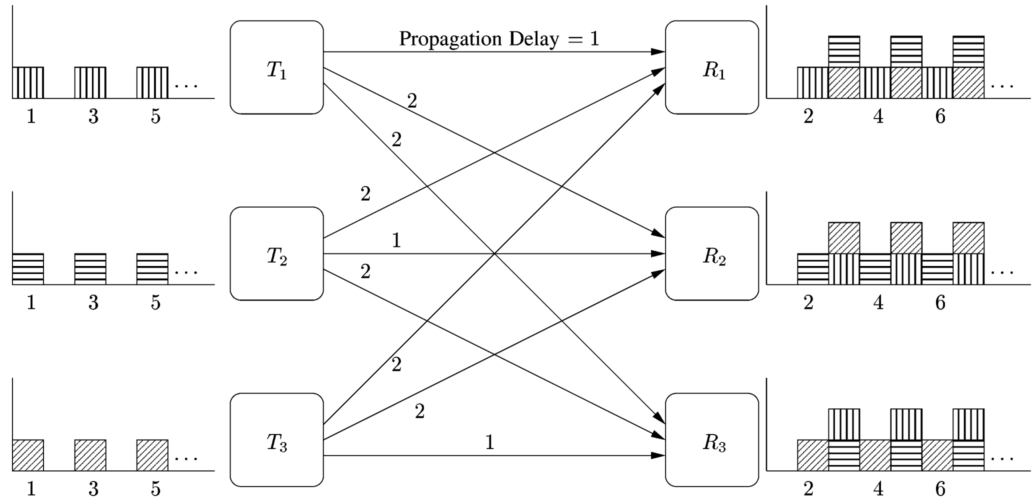

Fig. 3. Interference Alignment: Everyone gets half the cake.

VI. Conclusion

We have shown that with perfect channel knowledge the K user interference channel has (almost surely) $K/2$ degrees of freedom when the channel coefficients are time-varying and are generated from a continuous distribution. The key idea is interference alignment which maximizes the overlap between the signal spaces of all interference signals at each receiver so that the size of the interference-free space is maximized for the desired signal. Due to relativity of alignment, it is possible that signals align at the receivers where they are not desired and remain distinguishable at the receivers where they are desired. Somewhat surprisingly it is shown that all the interference can be concentrated roughly into one half of the signal space at each receiver, leaving the other half available to the desired signal and free of interference. The alignment can be accomplished for any number of users but as the number of users increases a larger signal space is needed for each user to recover nearly half of it.

While this work shows the potential benefits of interference alignment, several challenges must be overcome before these benefits translate into practice. One key issue is the assumption of global channel knowledge. While a node may acquire channel state information for its own channels, it is much harder to learn the channels between other pairs of nodes with which this node is not directly associated. On the other hand, global channel knowledge may not be necessary if there is a feedback channel through which the receivers can guide the transmitters into aligned configurations in real time by applying incremental corrections. Also iterative algorithms based on channel reciprocity may be able to align interference in a distributed fashion [28].

The key insight of this paper is the role of interference alignment in a wireless network. From a capacity perspective the idea of interference alignment reaffirms the need for structured codes in wireless networks, also pointed out by [29]. For the single user point to point Gaussian channel it is well known that the capacity can be achieved through random (Gaussian) codebooks as well as through structured (lattice) codes. There is a growing realization that structured codes, optional for the single user case, may be necessary for approaching the capacity of networks. In an interference network when we design one user's codebook we are also designing the interference/noise that will be seen by other users. Having structure in the interference may therefore be necessary. It is the structure imposed on the transmitted signals that facilitates interference alignment in this work. The intuition from this work is that since random codes will not automatically align themselves, structured codes will be necessary for wireless networks. Indeed interference alignment at the codeword level has been shown to be optimal in the capacity sense in [21] and in the degrees of freedom sense in [22] for some interesting cases. A combination of Han–Kobayashi [3] type achievable schemes and structured codes is a promising avenue in the quest for the capacity of wireless networks.

Appendix I
Example: Interference Alignment via Delay Offsets

Consider the K-user interference network shown in Fig, 3, where there is a propagation delay associated with each channel. In particular, let us assume that the propagation delay is equal to one symbol duration for all desired signal paths and two symbol durations for all paths that carry interference signals. The channel output at receiver $k \in \{1, 2, \dots, K\}$ is defined as

$$Y^{[k]}(n) = \sum_{j \neq k} X^{[j]}(n-2) + X^{[k]}(n-1) + Z^{[k]}(n) \quad (30)$$

where during the nth time slot (symbol duration) transmitter j sends symbol $X^{[j]}(n)$ and $Z^{[k]}(n)$ is i.i.d. zero mean unit variance Gaussian noise (AWGN). All inputs and outputs are complex. The transmit power at each transmitter is $\mathrm{E}[|X^{[k]}|^2] \leq P, \forall k \in \{1, 2, \dots, K\}$. In the absence of interference, each user would achieve a capacity of $C = \log(1+P)$. Now, with all the interferers present, suppose each transmitter transmits only during odd time slots (with power $2P$) and is silent during the even time slots. Let us consider what happens at receiver 1. The symbols sent from its desired transmitter (transmitter 1) are received *free from interference* during the even time slots and all the undesired (interference) transmissions are received simultaneously during the odd time slots. Thus, each user is able to access the channel one-half of the time with no interference from other users. Each user achieves a rate $R = \frac{1}{2}\log(1+2P)$

where the pre-log factor of $1/2$ denotes the degrees of freedom achieved by each user. The sum-rate $\frac{K}{2}\log(1+2P)$ is also found to be the capacity by a converse argument that is nearly identical to the converse for the phase-alignment example.

APPENDIX II
CONVERSE FOR LEMMA 1

We present the proof for the case that all nodes are equipped with M antennas. In this case the statement of the lemma becomes

$$\max_{\mathcal{D}} d_i + d_j \leq \limsup_{\rho\to\infty} \sup_{\mathbf{R}(\rho)\in\mathcal{C}(\rho)} \frac{R_i(\rho)+R_j(\rho)}{\log(\rho)} \leq M, \quad \forall i,j \in \{1,2,\ldots,K\}, i\neq j. \tag{31}$$

We consider the case $i=1, j=2$ and eliminate messages $W_3, W_4, \ldots, W_K$, leaving us with a two-user MIMO interference channel. The following converse holds for both time-varying and constant channel coefficients. The channel input-output equations are written equivalently as:

$$\mathbf{Y}^{[1]}(t) = \mathbf{H}^{[11]}(t)\mathbf{X}^{[1]}(t) + \mathbf{H}^{[12]}(t)\mathbf{X}^{[2]}(t) + \mathbf{Z}^{[1]}(t) \tag{32}$$
$$\mathbf{Y}^{[2]}(t) = \mathbf{H}^{[21]}(t)\mathbf{X}^{[1]}(t) + \mathbf{H}^{[22]}(t)\mathbf{X}^{[2]}(t) + \mathbf{Z}^{[2]}(t). \tag{33}$$

With probability one the channel matrices are invertible. So we can equivalently write

$$\mathbf{Y}^{[1]}(t) = \mathbf{H}^{[11]}(t)\mathbf{X}^{[1]}(t) + \mathbf{H}^{[12]}(t)\mathbf{X}^{[2]}(t) + \mathbf{Z}^{[1]}(t) \tag{34}$$
$$\mathbf{Y}^{[2]'}(t) = \mathbf{H}^{[12]}(t)\left(\mathbf{H}^{[22]}(t)\right)^{-1}\mathbf{H}^{[21]}(t)\mathbf{X}^{[1]}(t) + \mathbf{H}^{[12]}(t)\mathbf{X}^{[2]}(t) + \mathbf{Z}^{[2]'}(t). \tag{35}$$

where

$$\mathbf{Z}^{[1]}(t) \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_M) \tag{36}$$
$$\mathbf{Z}^{[2]'}(t) \sim \mathcal{CN}\left(\mathbf{0}, \mathbf{H}^{[12]}(t)\left(\mathbf{H}^{[22]}(t)\right)^{-1} \times \left(\mathbf{H}^{[22]}(t)\right)^{-\dagger}\left(\mathbf{H}^{[12]}(t)\right)^{\dagger}\right). \tag{37}$$

Since the capacity of the interference channel depends only on the noise marginals, we assume without loss of generality that

$$\mathbf{Z}^{[1]}(t) = \bar{\mathbf{Z}}(t) + \bar{\mathbf{Z}}^{[1]}(t) \tag{38}$$
$$\mathbf{Z}^{[2]'}(t) = \bar{\mathbf{Z}}(t) + \bar{\mathbf{Z}}^{[2]}(t) \tag{39}$$

where

$$\bar{\mathbf{Z}}(t) \sim \mathcal{CN}\left(\mathbf{0}, \alpha(t)\mathbf{I}_M\right) \tag{40}$$
$$\bar{\mathbf{Z}}^{[1]}(t) \sim \mathcal{CN}\left(\mathbf{0}, (1-\alpha(t))\mathbf{I}_M\right) \tag{41}$$
$$\bar{\mathbf{Z}}^{[2]}(t) \sim \mathcal{CN}\left(\mathbf{0}, \mathbf{H}^{[12]}(t)\left(\mathbf{H}^{[22]}(t)\right)^{-1} \times \left(\mathbf{H}^{[22]}(t)\right)^{-\dagger}\left(\mathbf{H}^{[12]}(t)\right)^{\dagger} - \alpha(t)\mathbf{I}_M\right) \tag{42}$$

and

$$\alpha(t) = \min\left(1, \lambda_{\min}\left(\mathbf{H}^{[12]}(t)\left(\mathbf{H}^{[22]}(t)\right)^{-1} \times \left(\mathbf{H}^{[22]}(t)\right)^{-\dagger}\left(\mathbf{H}^{[12]}(t)\right)^{\dagger}\right)\right)$$

is strictly positive with probability 1. Here $\lambda_{\min}(A)$ refers to the smallest eigenvalue of matrix A. $\bar{\mathbf{Z}}(t), \bar{\mathbf{Z}}^{[1]}(t), \bar{\mathbf{Z}}^{[2]}(t)$ are mutually independent and jointly Gaussian.

Consider any reliable coding scheme for this interference channel, spanning N channel uses. We use the notation $[A(t)]_1^N$ to indicate the vector of values taken by variable $A(t)$ for $t=1,2,\ldots,N$. Starting from Fano's inequality, we have

$$\begin{aligned}
R_1(\rho) + R_2(\rho) &\leq \frac{1}{N}I\left(W_1; \left[\mathbf{Y}^{[1]}(t)\right]_1^N\right) + \frac{1}{N}I(W_2; W_1, \left[\mathbf{Y}^{[2]'}(t)\right]_1^N) + \epsilon \\
&= \frac{1}{N}I\left(W_1; \left[\mathbf{Y}^{[1]}(t)\right]_1^N\right) + \frac{1}{N}I\left(W_2; \left[\mathbf{Y}^{[2]'}(t)\right]_1^N \middle| W_1, \left[\mathbf{X}^{[1]}(t)\right]_1^N\right) + \epsilon & (43)\\
&\leq \frac{1}{N}I\left(W_1; \left[\mathbf{Y}^{[1]}(t)\right]_1^N\right) + \frac{1}{N}I\left(W_2; \left[\mathbf{H}^{[12]}(t)\mathbf{X}^{[2]}(t) + \bar{\mathbf{Z}}(t)\right]_1^N \middle| W_1 \left[\mathbf{X}^{[1]}(t)\right]_1^N\right) + \epsilon & (44)\\
&\leq \frac{1}{N}I\left(W_1; \left[\mathbf{H}^{[11]}(t)\mathbf{X}^{[1]}(t) + \mathbf{H}^{[12]}(t)\mathbf{X}^{[2]}(t) + \bar{\mathbf{Z}}(t)\right]_1^N\right) \\
&\quad + \frac{1}{N}I\left(W_2; \left[\mathbf{H}^{[11]}(t)\mathbf{X}^{[1]}(t) + \mathbf{H}^{[12]}(t)\mathbf{X}^{[2]}(t) + \bar{\mathbf{Z}}(t)\right]_1^N \middle| W_1 \left[\mathbf{X}^{[1]}(t)\right]_1^N\right) + \epsilon & (45)\\
&\leq \frac{1}{N}I\left(W_1, W_2; \left[\mathbf{H}^{[11]}(t)\mathbf{X}^{[1]}(t) + \mathbf{H}^{[12]}(t)\mathbf{X}^{[2]}(t) + \bar{\mathbf{Z}}(t)\right]_1^N\right) + \epsilon & (46)\\
&\leq M\log(\rho) + o(\log(\rho)) & (47)
\end{aligned}$$

where the last step follows from the known result that the sum capacity of a multiple access channel with an M antenna receiver can only contribute at most M degrees of freedom. Thus, we have $d_1 + d_2 \leq M$. Similarly, for any $i,j \in \{1,2,\ldots,K\}, i\neq j$ we obtain $d_i + d_j \leq M$. Finally, adding up all the outerbounds in (4), we obtain the converse statement for the degrees of freedom of the K user interference channel with M antennas at each node

$$\max_{\mathcal{D}} d_1 + d_2 + \cdots + d_K \leq MK/2. \tag{48}$$

APPENDIX III
ACHIEVABILITY FOR THEOREM 1 FOR ARBITRARY K

Let $N = (K-1)(K-2) - 1$. We show that $(d_1(n), d_2(n), \ldots, d_K(n))$ lies in the degrees of freedom region of the K user interference channel for any $n \in \mathbb{N}$ where

$$d_1(n) = \frac{(n+1)^N}{(n+1)^N + n^N}$$
$$d_i(n) = \frac{n^N}{(n+1)^N + n^N}, \quad i = 2, 3 \ldots, K.$$

This implies that

$$\max_{(d_1, d_2, \ldots, d_K) \in \mathcal{D}} d_1 + d_2 + \cdots d_K \geq \sup_n \frac{(n+1)^N + (K-1)n^N}{(n+1)^N + n^N} = K/2.$$

We provide an achievable scheme to show that $((n+1)^N, n^N, n^N, \ldots, n^N)$ lies in the degrees of freedom region of an $M_n = (n+1)^N + n^N$ symbol extension of the original channel which automatically implies the desired result. In the extended channel, the signal vector at the kth user's receiver can be expressed as

$$\bar{\mathbf{Y}}^{[k]}(t) = \sum_{j=1}^{K} \bar{\mathbf{H}}^{[kj]} \bar{\mathbf{X}}^{[j]}(t) + \bar{\mathbf{Z}}^{[k]}(t)$$

where $\bar{\mathbf{X}}^{[j]}$ is an $M_n \times 1$ column vector representing the M_n symbol extension of the transmitted symbol $X^{[k]}$, i.e.

$$\bar{\mathbf{X}}^{[j]}(t) \triangleq \begin{bmatrix} X^{[j]}(M_n(t-1)+1) \\ X^{[j]}(M_n(t-1)+2) \\ \vdots \\ X^{[j]}(M_n t)) \end{bmatrix}.$$

Similarly $\bar{\mathbf{Y}}^{[k]}$ and $\bar{\mathbf{Z}}^{[k]}$ represent M_n symbol extensions of the $Y^{[k]}$ and $Z^{[k]}$, respectively. $\bar{\mathbf{H}}^{[kj]}$ is a diagonal $M_n \times M_n$ matrix representing the M_n symbol extension of the channel as shown in the equation at the bottom of the page. Recall that the diagonal elements of $\bar{\mathbf{H}}^{[kj]}$ are drawn independently from a continuous distribution and are therefore distinct with probability 1.

In a manner similar to the $K = 3$ case, message W_1 is encoded at transmitter 1 into $(n+1)^N$ independent streams $x_m^{[1]}(t), m = 1, 2, \ldots, (n+1)^N$ along vectors $\mathbf{v}_m^{[1]}$ so that $\bar{\mathbf{X}}^{[1]}(t)$ is

$$\bar{\mathbf{X}}^{[1]}(t) = \sum_{m=1}^{(n+1)^N} x_m^{[1]}(t) \mathbf{v}_m^{[1]} = \bar{\mathbf{V}}^{[1]} \mathbf{X}^{[1]}(t)$$

where $\mathbf{X}^{[1]}(t)$ is a $(n+1)^N \times 1$ column vector and $\bar{\mathbf{V}}^{[1]}$ is a $M_n \times (n+1)^N$-dimensional matrix. Similarly $W_i, i \neq 1$ is encoded into n^K independent streams by transmitter i as

$$\bar{\mathbf{X}}^{[i]}(t) = \sum_{m=1}^{n^N} x_m^{[i]}(t) \mathbf{v}_m^{[i]} = \bar{\mathbf{V}}^{[i]} \mathbf{X}^{[i]}(t).$$

The received signal at the ith receiver can then be written as

$$\bar{\mathbf{Y}}^{[i]}(t) = \sum_{j=1}^{K} \bar{\mathbf{H}}^{[ij]} \bar{\mathbf{V}}^{[j]} \mathbf{X}^{[j]}(t) + \bar{\mathbf{Z}}^{[i]}(t).$$

All receivers decode the desired signal by zero-forcing the interference vectors. At receiver 1, to obtain $(n+1)^N$ interference free dimensions corresponding to the desired signal from an $M_n = (n+1)^N + n^N$-dimensional received signal vector $\bar{\mathbf{Y}}^{[1]}$, the dimension of the interference should be not more than n^N. This can be ensured by perfectly aligning the interference from transmitters $2, 3, \ldots, K$ as follows:

$$\bar{\mathbf{H}}^{[12]} \bar{\mathbf{V}}^{[2]} = \bar{\mathbf{H}}^{[13]} \bar{\mathbf{V}}^{[3]} = \bar{\mathbf{H}}^{[14]} \bar{\mathbf{V}}^{[4]} = \cdots = \bar{\mathbf{H}}^{[1K]} \bar{\mathbf{V}}^{[K]}. \tag{49}$$

At the same time, receiver 2 zero-forces the interference from $\bar{\mathbf{X}}^{[i]}, i \neq 2$. To extract n^N interference-free dimensions from a $M_n = (n+1)^N + n^N$-dimensional vector, the dimension of the interference has to be not more than $(n+1)^N$.

This can be achieved by choosing $\bar{\mathbf{V}}^{[i]}, i \neq 2$ so that

$$\begin{aligned} \bar{\mathbf{H}}^{[23]} \bar{\mathbf{V}}^{[3]} &\prec \bar{\mathbf{H}}^{[21]} \bar{\mathbf{V}}^{[1]} \\ \bar{\mathbf{H}}^{[24]} \bar{\mathbf{V}}^{[4]} &\prec \bar{\mathbf{H}}^{[21]} \bar{\mathbf{V}}^{[1]} \\ &\vdots \\ \bar{\mathbf{H}}^{[2K]} \bar{\mathbf{V}}^{[K]} &\prec \bar{\mathbf{H}}^{[21]} \bar{\mathbf{V}}^{[1]}. \end{aligned} \tag{50}$$

Notice that the above relations align the interference from $K-2$ transmitters within the interference from transmitter 1 at receiver 2. Similarly, to decode W_i at receiver i when $i \neq 1$ we wish to choose $\bar{\mathbf{V}}^{[i]}$ so that the following $K-2$ relations are satisfied.

$$\bar{\mathbf{H}}^{[ij]} \bar{\mathbf{V}}^{[j]} \prec \bar{\mathbf{H}}^{[i1]} \bar{\mathbf{V}}^{[1]}, j \notin \{1, i\}. \tag{51}$$

We now wish to pick vectors $\bar{\mathbf{V}}^{[i]}, i = 1, 2 \ldots, K$ so that (49), (50), and (51) are satisfied. Since channel matrices $\bar{\mathbf{H}}^{[ij]}$ have a full rank of M_n almost surely, (49), (50) and (51) can be equivalently expressed as

$$\bar{\mathbf{V}}^{[j]} = \mathbf{S}^{[j]} \mathbf{B} \quad j = 2, 3, 4 \ldots, K \quad \text{At receiver 1} \tag{52}$$

$$\bar{\mathbf{H}}^{[kj]}(t) \triangleq \begin{bmatrix} H^{[kj]}(M_n(t-1)+1) & 0 & \cdots & 0 \\ 0 & H^{[kj]}(M_n(t-1)+2) & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & H^{[kj]}(M_n t) \end{bmatrix}.$$

$$\left.\begin{array}{r}\mathbf{T}_3^{[2]}\mathbf{B} = \mathbf{B} \prec \bar{\mathbf{V}}^{[1]}\\ \mathbf{T}_4^{[2]}\mathbf{B} \prec \bar{\mathbf{V}}^{[1]}\\ \vdots\\ \mathbf{T}_K^{[2]}\mathbf{B} \prec \bar{\mathbf{V}}^{[1]}\end{array}\right\} \text{At receiver 2} \quad (53)$$

$$\left.\begin{array}{r}\mathbf{T}_2^{[i]}\mathbf{B} \prec \bar{\mathbf{V}}^{[1]}\\ \mathbf{T}_3^{[i]}\mathbf{B} \prec \bar{\mathbf{V}}^{[1]}\\ \vdots\\ \mathbf{T}_{i-1}^{[i]}\mathbf{B} \prec \bar{\mathbf{V}}^{[1]}\\ \mathbf{T}_{i+1}^{[i]}\mathbf{B} \prec \bar{\mathbf{V}}^{[1]}\\ \vdots\\ \mathbf{T}_K^{[i]}\mathbf{B} \prec \bar{\mathbf{V}}^{[1]}\end{array}\right\} \text{At receiver } i \text{where } i = 3,\dots,K \quad (54)$$

where

$$\mathbf{B} = (\bar{\mathbf{H}}^{[21]})^{-1}\bar{\mathbf{H}}^{[23]}\bar{\mathbf{V}}^{[3]} \quad (55)$$

$$\mathbf{S}^{[j]} = \left(\bar{\mathbf{H}}^{[1j]}\right)^{-1}\bar{\mathbf{H}}^{[13]}\left(\bar{\mathbf{H}}^{[23]}\right)^{-1}\bar{\mathbf{H}}^{[21]} \quad j = 2,3,\dots,K \quad (56)$$

$$\mathbf{T}_j^{[i]} = \left(\bar{\mathbf{H}}^{[i1]}\right)^{-1}\bar{\mathbf{H}}^{[ij]}\mathbf{S}^{[j]} \quad i,j = 2,3\dots,K, j \neq i. \quad (57)$$

Note that $\mathbf{T}_3^{[2]} = \mathbf{I}$, the $M_n \times M_n$ identity matrix. We now choose $\bar{\mathbf{V}}^{[1]}$ and $\mathbf{B}$ so that they satisfy the $(K-2)(K-1) = N+1$ relations in (53)–(54) and then use equations in (52) to determine $\bar{\mathbf{V}}^{[2]}, \bar{\mathbf{V}}^{[3]}, \dots, \bar{\mathbf{V}}^{[K]}$. Thus, our goal is to find matrices $\bar{\mathbf{V}}^{[1]}$ and $\mathbf{B}$ so that

$$\mathbf{T}_j^{[i]}\mathbf{B} \prec \bar{\mathbf{V}}^{[1]}$$

for all $i,j = \{2,3,\dots,K\}, i \neq j$.

Let $\mathbf{w}$ be the $M_n \times 1$ column vector

$$\mathbf{w} = \begin{bmatrix}1\\1\\ \vdots\\1\end{bmatrix}.$$

We need to choose $n^{(K-1)(K-2)-1} = n^N$ column vectors for $\mathbf{B}$. The sets of column vectors of $\mathbf{B}$ and $\bar{\mathbf{V}}^{[1]}$ are chosen to be equal to the sets B and $\bar{V}^{[1]}$ where

$$B = \left\{\left(\prod_{m,k\in\{2,3,\dots,K\}, m\neq k, (m,k)\neq(2,3)}\left(\mathbf{T}_k^{[m]}\right)^{\alpha_{mk}}\right)\mathbf{w} : \forall\, \alpha_{mk} \in \{0,1,2\dots,n-1\}\right\}$$

$$\bar{V}^{[1]} = \left\{\left(\prod_{m,k\in\{2,3,\dots,K\}, m\neq k, (m,k)\neq(2,3)}(\mathbf{T}_k^{[m]})^{\alpha_{mk}}\right)\mathbf{w} : \forall\, \alpha_{mk} \in \{0,1,2\dots,n\}\right\}.$$

For example, if $K = 3$ we get $N = 1$. $\mathbf{B}$ and $\bar{\mathbf{V}}^{[1]}$ are chosen as

$$\mathbf{B} = \begin{bmatrix}\mathbf{w} & \mathbf{T}_2^{[3]}\mathbf{w} & \dots & (\mathbf{T}_2^{[3]})^{n-1}\mathbf{w}\end{bmatrix}$$
$$\bar{\mathbf{V}}^{[1]} = \begin{bmatrix}\mathbf{w} & \mathbf{T}_2^{[3]}\mathbf{w} & \dots & (\mathbf{T}_2^{[3]})^{n}\mathbf{w}\end{bmatrix}.$$

To clarify the notation further, consider the case where $K = 4$. Assuming $n = 1$, B consists of exactly one element, i.e., $B = \{\mathbf{w}\}$. The set $\bar{V}^{[1]}$ consists of all $2^N = 2^5 = 32$ column vectors of the form $(\mathbf{T}_4^{[2]})^{\alpha_{24}}(\mathbf{T}_2^{[3]})^{\alpha_{32}}(\mathbf{T}_4^{[3]})^{\alpha_{24}}(\mathbf{T}_3^{[4]})^{\alpha_{43}}(\mathbf{T}_2^{[4]})^{\alpha_{42}}\mathbf{w}$ where all $\alpha_{24}, \alpha_{32}, \alpha_{34}, \alpha_{42}, \alpha_{43}$ take values 0, 1. B and $\bar{V}^{[1]}$ can be verified to have n^N and $(n+1)^N$ elements, respectively.

$\bar{\mathbf{V}}^{[i]}, i = 2,3\dots,K$ are chosen using (52). Clearly, for $(i,j) = (2,3)$,

$$\mathbf{T}_j^{[i]}\mathbf{B} = \mathbf{B} \prec \bar{\mathbf{V}}^{[1]}.$$

Now, for $i \neq j, i,j = 2,\dots,K, (i,j) \neq (2,3)$

$$\mathbf{T}_j^{[i]}B = \left\{\left(\prod_{m,k\in\{2,3,\dots,N\}, m\neq k, (m,k)\neq(2,3)}\left(\mathbf{T}_k^{[m]}\right)^{\alpha_{mk}}\right)\mathbf{w} : \forall\,(m,k)\neq(i,j), \alpha_{mk} \in \{0,1,2\dots,n-1\}, \alpha_{ij} \in \{1,2,\dots,n\}\right\}$$
$$\Rightarrow \mathbf{T}_j^{[i]}B \in \bar{V}^{[1]}$$
$$\Rightarrow \mathbf{T}_j^{[i]}\mathbf{B} \prec \bar{\mathbf{V}}^{[1]}.$$

Thus, the interference alignment conditions (52)–(54) are satisfied.

Through interference alignment, we have now ensured that the dimension of the interference is small enough. We now need to verify that the components of the desired signal are linearly independent of the components of the interference so that the signal stream can be completely decoded by zero-forcing the interference. Consider the received signal vectors at receiver 1. The desired signal arrives along the $(n+1)^N$ vectors $\bar{\mathbf{H}}^{[11]}\bar{\mathbf{V}}^{[1]}$. As enforced by (52), the interference vectors from transmitters $3,4,\dots,K$ are perfectly aligned with the interference from transmitter 2 and therefore, all interference arrives along the n^N vectors $\bar{\mathbf{H}}^{[12]}\bar{\mathbf{V}}^{[2]}$. In order to prove that there are $(n+1)^N$ interference free dimensions it suffices to show that the columns of the square, $M_n \times M_n$-dimensional matrix

$$\begin{bmatrix}\bar{\mathbf{H}}^{[11]}\bar{\mathbf{V}}^{[1]} & \bar{\mathbf{H}}^{[12]}\bar{\mathbf{V}}^{[2]}\end{bmatrix} \quad (58)$$

are linearly independent almost surely. Multiplying the above $M_n \times M_n$ matrix with $(\bar{\mathbf{H}}^{[11]})^{-1}$ and substituting for $\bar{\mathbf{V}}^{[1]}$ and $\bar{\mathbf{V}}^{[2]}$, we get a matrix whose lth row has entries of the forms

$$\prod_{(m,k)\in\{2,3\dots,K\}, m\neq k, (m,k)\neq(2,3)}\left(\lambda_l^{mk}\right)^{\alpha_{mk}}$$

and

$$d_l \prod_{(m,k)\in\{2,3\dots,K\}, m\neq k, (m,k)\neq(2,3)}\left(\lambda_l^{mk}\right)^{\beta_{mk}}$$

where $\alpha_{mk} \in \{0,1,\dots,n-1\}$ and $\beta_{mk} \in \{0,1,\dots,n\}$ and λ_l^{mk}, d_l are drawn independently from a continuous distribution. The same iterative argument as in Section IV-B can be used. For instance, expanding the corresponding determinant along the first row, the linear independence condition boils down to one of the following occurring with nonzero probability:

1) d_1 being equal to one of the roots of a linear equation;
2) the coefficients of the above mentioned linear equation being equal to zero.

Thus the iterative argument can be extended here, stripping the last row and last column at each iteration and the linear independence condition can be shown to be equivalent to the linear independence of a $n^N \times n^N$ matrix whose rows are of the form $\prod_{(m,k)\in\{2,3\dots,K\},m\neq k}(\lambda_l^{mk})^{\alpha_{mk}}$ where $\alpha_{pq} \in \{0,1,\dots,n-1\}$. Note that this matrix is a more general version of the Vandermonde matrix obtained in Section IV-B. So the argument for the $K=3$ case does not extend here. However, the iterative procedure which eliminated the last row and the last column at each iteration, can be continued. For example, expanding the determinant along the first row, the singularity condition simplifies to one of the following:

1) λ_l^{mk} being equal to one of the roots of a finite degree polynomial;
2) the coefficients of the above mentioned polynomial being equal to zero

Since the probability of condition 1 occurring is 0, condition 2 must occur with nonzero probability. Condition 2 leads to a polynomial in another random variable λ_l^{pq} and thus the iterative procedure can be continued until the linear independence condition is shown to be equivalent almost surely to a 1×1 matrix being equal to 0. Assuming, without loss of generality, that we placed the $\mathbf{w}$ in the first row (this corresponds to the term $\alpha_{mk}=0, \forall\,(m,k)$), the linear independence condition boils down to the condition that $1=0$ with nonzero probability—an obvious contradiction. Thus, the matrix

$$\left[\bar{\mathbf{H}}^{[11]}\bar{\mathbf{V}}^{[1]} \quad \bar{\mathbf{H}}^{[12]}\bar{\mathbf{V}}^{[2]}\right]$$

can be shown to be nonsingular with probability 1.

Similarly, the desired signal can be chosen to be linearly independent of the interference at all other receivers almost surely. Thus $\left(\frac{(n+1)^N}{(n+1)^N+n^N}, \frac{n^N}{(n+1)^N+n^N}, \dots, \frac{n^N}{(n+1)^N+n^N}\right)$ lies in the degrees of freedom region of the K user interference channel and therefore, the K user interference channel has $K/2$ degrees of freedom.

Appendix IV
Proof of Theorem 3 for M Even

Proof: To prove achievability we first consider the case when M is even. Through an achievable scheme, we show that there are $M/2$ noninterfering paths between transmitter i and receiver i for each $i=1,2,3$ resulting in a total of $3M/2$ paths in the network.

Transmitter i transmits message W_i for receiver i using $M/2$ independently encoded streams over vectors $\mathbf{v}^{[i]}$, i.e.,

$$\mathbf{X}^{[i]}(t) = \sum_{m=1}^{M/2} x_m^{[i]}(t)\mathbf{v}_m^{[i]} = \mathbf{V}^{[i]}\mathbf{X}^i(t), i=1,2,3.$$

The signal received at receiver i can be written as

$$\mathbf{Y}^{[i]}(t) = \mathbf{H}^{[i1]}\mathbf{V}^{[1]}\mathbf{X}^1(t) + \mathbf{H}^{[i2]}\mathbf{V}^{[2]}\mathbf{X}^2(t) + \mathbf{H}^{[i3]}\mathbf{V}^{[3]}\mathbf{X}^3(t) + \mathbf{Z}_i(t).$$

All receivers cancel the interference by zero-forcing and then decode the desired message. To decode the $M/2$ streams along the column vectors of $\mathbf{V}^{[i]}$ from the M components of the received vector, the dimension of the interference has to be less than or equal to $M/2$. The following three interference alignment equations ensure that the dimension of the interference is equal to $M/2$ at all the receivers.

$$\text{span}(\mathbf{H}^{[12]}\mathbf{V}^{[2]}) = \text{span}(\mathbf{H}^{[13]}\mathbf{V}^{[3]}) \quad (59)$$
$$\mathbf{H}^{[21]}\mathbf{V}^{[1]} = \mathbf{H}^{[23]}\mathbf{V}^{[3]} \quad (60)$$
$$\mathbf{H}^{[31]}\mathbf{V}^{[1]} = \mathbf{H}^{[32]}\mathbf{V}^{[2]} \quad (61)$$

where $\text{span}(\mathbf{A})$ represents the vector space spanned by the column vectors of matrix $\mathbf{A}$ We now wish to choose $\mathbf{V}^{[i]}, i=1,2,3$ so that the above equations are satisfied. Since $\mathbf{H}^{[ij]}, i,j\in\{1,2,3\}$ have a full rank of M almost surely, the above equations can be equivalently represented as

$$\text{span}(\mathbf{V}^{[1]}) = \text{span}(\mathbf{E}\mathbf{V}^{[1]}) \quad (62)$$
$$\mathbf{V}^{[2]} = \mathbf{F}\mathbf{V}^{[1]} \quad (63)$$
$$\mathbf{V}^{[3]} = \mathbf{G}\mathbf{V}^{[1]} \quad (64)$$

where

$$\mathbf{E} = (\mathbf{H}^{[31]})^{-1}\mathbf{H}^{[32]}(\mathbf{H}^{[12]})^{-1}\mathbf{H}^{[13]}(\mathbf{H}^{[23]})^{-1}\mathbf{H}^{[21]}$$
$$\mathbf{F} = (\mathbf{H}^{[32]})^{-1}\mathbf{H}^{[31]}$$
$$\mathbf{G} = (\mathbf{H}^{[23]})^{-1}\mathbf{H}^{[21]}.$$

Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_M$ be the M eigenvectors of $\mathbf{E}$. Then we set $\mathbf{V}_1$ to be

$$\mathbf{V}^{[1]} = [\mathbf{e}_1 \quad \dots \quad \mathbf{e}_{(M/2)}].$$

Then $\mathbf{V}^{[2]}$ and $\mathbf{V}^{[3]}$ are found using (62)–(64). Clearly, $\mathbf{V}^{[i]}, i=1,2,3$ satisfy the desired interference alignment (59)–(61). Now, to decode the message using zero-forcing, we need the desired signal to be linearly independent of the interference at the receivers. For example, at receiver 1, we need the columns of $\mathbf{H}^{[11]}\mathbf{V}^{[1]}$ to be linearly independendent with the columns of $\mathbf{H}^{[21]}\mathbf{V}^{[2]}$ almost surely. i.e., we need the matrix below to be of full rank almost surely

$$\left[\mathbf{H}^{[11]}\mathbf{V}^{[1]} \quad \mathbf{H}^{[12]}\mathbf{V}^{[2]}\right].$$

Substituting values for $\mathbf{V}^{[1]}$ and $\mathbf{V}^{[2]}$ in the above matrix, and multiplying by full rank matrix $(\mathbf{H}^{[11]})^{-1}$, the linear independence condition is equivalent to the condition that the column vectors of

$$\left[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \dots \mathbf{e}_{(M/2)} \quad \mathbf{K}\mathbf{e}_1 \quad \dots \mathbf{K}\mathbf{e}_{(M/2)}\right]$$

are linearly independent almost surely, where $\mathbf{K} = (\mathbf{H}^{[11]})^{-1}\mathbf{H}^{[12]}\mathbf{F}$.

This is easily seen to be true because $\mathbf{K}$ is a random (full rank) linear transformation. To get an intuitive understanding of the linear independence condition, consider the case of $M=2$. Let $\mathcal{L}$ represent the line along which lies the first eigenvector

$$\begin{bmatrix} \mathbf{e}_1 & 0 & \mathbf{e}_3 & \dots & 0 & \mathbf{e}_M & \mathbf{K}\mathbf{e}_1 & 0 & \mathbf{K}\mathbf{e}_3 & \dots & 0 & \mathbf{K}\mathbf{e}_M \\ 0 & \mathbf{e}_2 & 0 & \dots & \mathbf{e}_{M-1} & \mathbf{e}_M & 0 & \mathbf{K}\mathbf{e}_2 & 0 & \dots & \mathbf{K}\mathbf{e}_{M-1} & \mathbf{K}\mathbf{e}_M. \end{bmatrix} \quad (72)$$

of the random 2×2 matrix $\mathbf{E}$. The probability of a random rotation (and scaling) $\mathbf{K}$ of $\mathcal{L}$ being collinear with $\mathcal{L}$ is zero. Using a similar argument, we can show that matrices

$$\left[\mathbf{H}^{[22]}\mathbf{V}^{[2]} \quad \mathbf{H}^{[21]}\mathbf{V}^{[1]}\right]$$

and

$$\left[\mathbf{H}^{[33]}\mathbf{V}^{[3]} \quad \mathbf{H}^{[31]}\mathbf{V}^{[1]}\right]$$

have a full rank of M almost surely and therefore receivers 2 and 3 can decode the $M/2$ streams of $\mathbf{V}^{[2]}$ and $\mathbf{V}^{[3]}$ using zero-forcing. Thus, a total $3M/2$ interference free transmissions per channel-use are achievable with probability 1 and the proof is complete. □

Appendix V
Proof of Theorem 3 for M Odd

Proof: Consider a two time-slot symbol extension of the channel, with the same chanel coefficients over the two symbols. It can be expressed as

$$\bar{\mathbf{Y}}^{[k]} = \bar{\mathbf{H}}^{[k1]}\bar{\mathbf{X}}^{[1]} + \bar{\mathbf{H}}^{[k2]}\bar{\mathbf{X}}^{[2]} + \bar{\mathbf{H}}^{[k3]}\mathbf{X}^{[3]} + \bar{\mathbf{Z}}^{[k]}$$

where $\bar{\mathbf{X}}^{[i]}$ is a $2M \times 1$ vector that represents the two symbol extension of the transmitted $M \times 1$ symbol symbol $\mathbf{X}^{[k]}$, i.e.

$$\bar{\mathbf{X}}^{[k]}(t) \triangleq \begin{bmatrix} \mathbf{X}^{[k]}(1, 2t+1) \\ \mathbf{X}^{[k]}(1, 2t+2) \end{bmatrix}$$

where $\mathbf{X}^{[k]}(t)$ is an $M \times 1$ vector representing the vector transmitted at time slot t by transmitter k. Similarly $\bar{\mathbf{Y}}^{[k]}$ and $\bar{\mathbf{Z}}^{[k]}$ represent the two symbol extensions of the received symbol $\mathbf{Y}^{[k]}$ and the noise vector $\mathbf{Z}^{[k]}$ respectively at receiver i. $\bar{\mathbf{H}}^{[ij]}$ is a $2M \times 2M$ block diagonal matrix representing the extension of the channel

$$\bar{\mathbf{H}}^{[ij]} \triangleq \begin{bmatrix} \mathbf{H}^{[ij]}(1) & 0 \\ 0 & \mathbf{H}^{[ij]}(1) \end{bmatrix}.$$

We will now show (M, M, M) lies in the degrees of freedom region of this extended channel channel with an achievable scheme, implying that that a total of $3M/2$ degrees of freedom are achievable over the original channel. Transmitter k transmits message W_i for receiver i using M independently encoded streams over vectors $\mathbf{v}^{[k]}$, i.e.

$$\bar{\mathbf{X}}^{[k]} = \sum_{m=1}^{M} x_m^{[k]}\mathbf{v}_m^{[k]} = \bar{\mathbf{V}}^{[k]}\mathbf{X}^{[k]}$$

where $\bar{\mathbf{V}}^{[k]}$ is a $2M \times M$ matrix and $\bar{\mathbf{X}}[k]$ is a $M \times 1$ vector representing M independent streams. The following three interference alignment equations ensure that the dimension of the interference is equal to M at receivers 1, 2, and 3

$$\text{rank}[\bar{\mathbf{H}}^{[21]}\bar{\mathbf{V}}^{[2]}] = \text{rank}[\bar{\mathbf{H}}^{[31]}\bar{\mathbf{V}}^{[3]}] \quad (65)$$

$$\bar{\mathbf{H}}^{[12]}\bar{\mathbf{V}}^{[1]} = \bar{\mathbf{H}}^{[32]}\bar{\mathbf{V}}^{[3]} \quad (66)$$

$$\bar{\mathbf{H}}^{[13]}\bar{\mathbf{V}}^{[1]} = \bar{\mathbf{H}}^{[23]}\bar{\mathbf{V}}^{[2]}. \quad (67)$$

The above equations imply that

$$\text{span}(\bar{\mathbf{V}}^{[1]}) = \text{span}(\bar{\mathbf{E}}\bar{\mathbf{V}}^{[1]}) \quad (68)$$

$$\bar{\mathbf{V}}^{[2]} = \bar{\mathbf{F}}\bar{\mathbf{V}}^{[1]} \quad (69)$$

$$\bar{\mathbf{V}}^{[3]} = \bar{\mathbf{G}}\bar{\mathbf{V}}^{[1]} \quad (70)$$

where

$$\mathbf{E} = (\mathbf{H}^{[13]})^{-1}\mathbf{H}^{[23]}(\mathbf{H}^{[21]})^{-1}\mathbf{H}^{[31]}(\mathbf{H}^{[32]})^{-1}\mathbf{H}^{[12]}$$
$$\mathbf{F} = (\mathbf{H}^{[13]})^{-1}\mathbf{H}^{[23]}$$
$$\mathbf{G} = (\mathbf{H}^{[12]})^{-1}\mathbf{H}^{[32]}$$

and $\bar{\mathbf{E}}, \bar{\mathbf{F}}$, and $\bar{\mathbf{G}}$ are $2M \times 2M$ block-diagonal matrices representing the $2M$ symbol extension of $\mathbf{E}, \mathbf{F}$ and $\mathbf{G}$, respectively. Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_M$, be the eigen vectors of $\mathbf{E}$. Then, we pick $\bar{\mathbf{V}}^{[1]}$ to be

$$\bar{\mathbf{V}}^{[1]} = \begin{bmatrix} \mathbf{e}_1 & 0 & \mathbf{e}_3 & \dots & 0 & \mathbf{e}_M \\ 0 & \mathbf{e}_2 & 0 & \dots & \mathbf{e}_{M-1} & \mathbf{e}_M \end{bmatrix}. \quad (71)$$

As in the even M case, $\bar{\mathbf{V}}^{[2]}$ and $\bar{\mathbf{V}}^{[3]}$ are then determined by using (68)–(70).

Now, we need the desired signal to be linearly independent of the interference at all the receivers. At receiver 1, the desired linear independence condition boils down to

$$\text{span}(\bar{\mathbf{V}}^{[1]}) \cap \text{span}(\bar{\mathbf{K}}\bar{\mathbf{V}}^{[1]}) = \{0\}$$

where $\mathbf{K} = (\mathbf{H}^{11})^{-1}\mathbf{H}^{[21]}(\mathbf{F})^{-1}$ and $\bar{\mathbf{K}}$ is the two-symbol diagonal extension of $\mathbf{K}$. Notice that $\mathbf{K}$ is an $M \times M$ matrix. The linear independence condition is equivalent to saying that all the columns of the following $2M \times 2M$ matrix are independent as shown in (72) at the top of the page. We now argue that the probability of the columns of the above matrix being linearly dependent is zero. Let $\mathbf{c}_i, i = 1, 2 \dots, 2M$ denote the columns of the above matrix. Suppose the columns $\mathbf{c}_i$ are linearly dependent, then

$$\exists \alpha_i \quad \text{s.t} \quad \sum_{i=1}^{2M} \alpha_i \mathbf{c}_i = 0.$$

Let

$$\mathbf{P} = \{\mathbf{e}_1, \mathbf{e}_3 \dots \mathbf{e}_{M-2}, \mathbf{K}\mathbf{e}_1, \dots, \mathbf{K}\mathbf{e}_{M-2}\}$$
$$\mathbf{Q} = \{\mathbf{e}_2, \mathbf{e}_4, \dots, \mathbf{e}_{M-1}, \mathbf{K}\mathbf{e}_2, \dots, \mathbf{K}\mathbf{e}_{M-1}\}.$$

Now, there are two possibilities

1) $\alpha_M = \alpha_{2M} = 0$. This implies that either one of the following sets of vectors is linearly dependent. Note that both sets are can be expressed as the union of
   a) A set of $\lfloor (M/2) \rfloor$ eigen vectors of $\mathbf{E}$
   b) A random transformation $\mathbf{K}$ of this set.

An argument along the same lines as the even M case leads to the conclusion that the probability of the union of the two sets listed above being linearly dependent in a M-dimensional space is zero.

2) $\alpha_{2M} \neq 0$ or $\alpha_M \neq 0$. This implies that

$$\alpha_M \mathbf{e}_M + \alpha_{2M}\mathbf{K}\mathbf{e}_M \in \text{span}(\mathbf{P}) \cap \text{span}(\mathbf{Q})$$
$$\Rightarrow \text{span}(\{\mathbf{K}\mathbf{e}_M, \mathbf{e}_M\}) \cap \text{span}(\mathbf{P}) \cap \text{span}(\mathbf{Q}) \neq \{0\}.$$

Also,

$$\begin{aligned}&\text{rank}(\text{span}(\mathbf{P}) \cup \text{span}(\mathbf{Q}))\\&\quad = \text{rank}(\mathbf{P}) + \text{rank}(\mathbf{Q}) - \text{rank}(\mathbf{P} \cap \mathbf{Q})\\&\Rightarrow \text{rank}(\mathbf{P} \cap \mathbf{Q})\\&\quad = 2M - 2 - \text{rank}(\text{span}(\mathbf{P}) \cup \text{span}(\mathbf{Q})).\end{aligned}$$

Note that $\mathbf{P}$ and $\mathbf{Q}$ are $M-1$-dimensional spaces. (The case where their dimensions are less than $M-1$ is handled in the first part). Also, $\mathbf{P}$ and $\mathbf{Q}$ are drawn from completely different set of vectors. Therefore, the union of $\mathbf{P}, \mathbf{Q}$ has a rank of M almost surely. Equivalently, $\text{span}(\mathbf{P}) \cap \text{span}(\mathbf{Q})$ has a dimension of $M-2$ almost surely. Since the set $\{\mathbf{e}_M, \mathbf{K}\mathbf{e}_M\}$ is drawn from an eigen vector $\mathbf{e}_M$ that does not exist in either $\mathbf{P}$ or $\mathbf{Q}$, the probability of the 2-D space $\text{span}(\{\mathbf{e}, \mathbf{K}\mathbf{e}_M\})$ intersecting with the $M-2$-dimensional space $\mathbf{P} \cap \mathbf{Q}$ is zero. For example, if $M = 3$, let L indicate the line formed by the intersection of the two planes $\text{span}(\{\mathbf{e}_1, \mathbf{K}\mathbf{e}_1\})$ and $\text{span}(\{\mathbf{e}_2, \mathbf{K}\mathbf{e}_2\})$. The probability that line L lies in the plane formed by $\text{span}(\{\mathbf{e}_3, \mathbf{K}\mathbf{e}_3\})$. Thus, the probability that the desired signal lies in the span of the interference is zero at receiver 1. Similarly, it can be argued that the desired signal is independent of the interference at receivers 2 and 3 almost surely. Therefore, (M, M, M) is achievable over the two-symbol extended channel. Thus $3M/2$ degrees of freedom are achievable over the 3 user interference channel with M antenna at each transmitting and receiving node. □

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