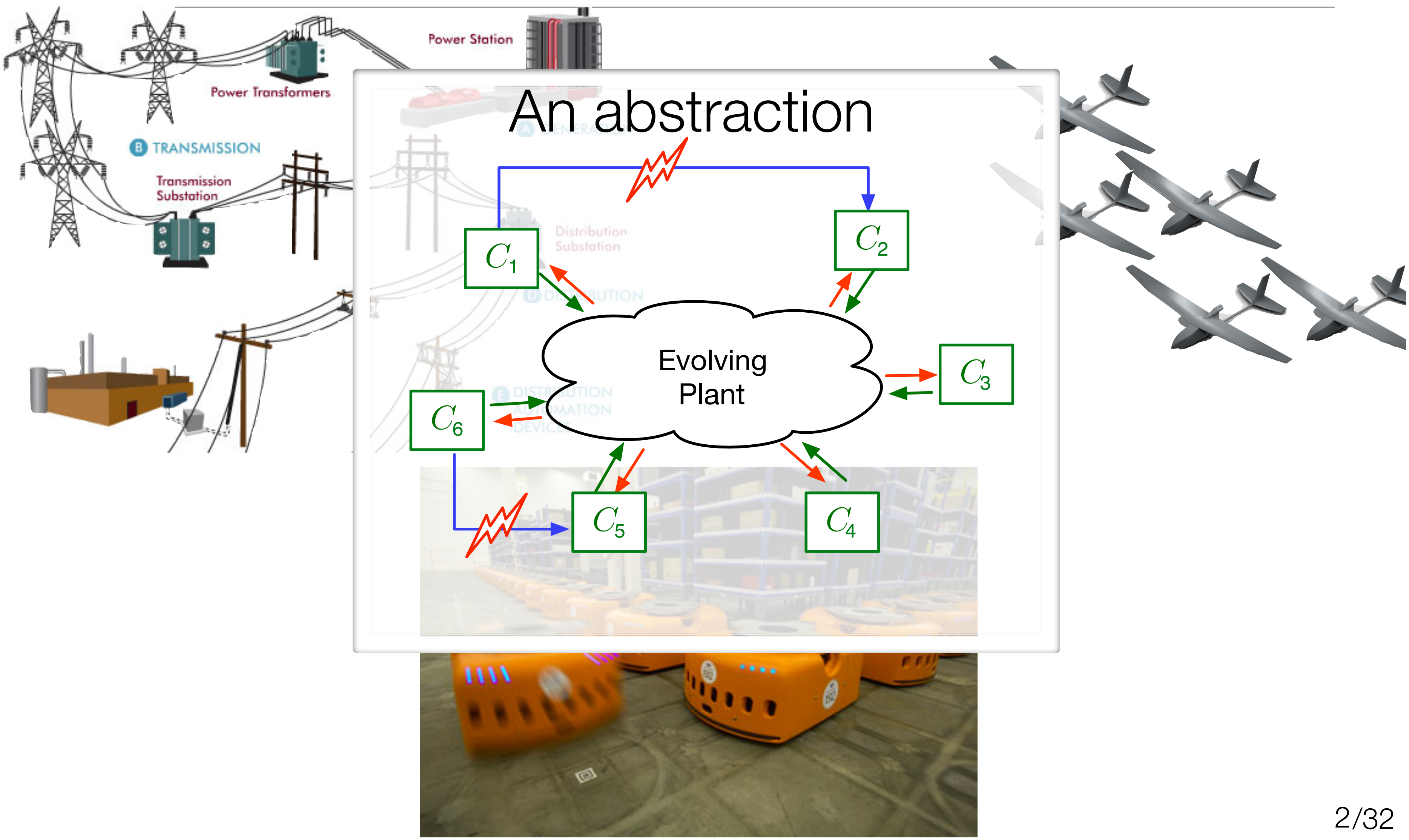
Implicit Communication for Control

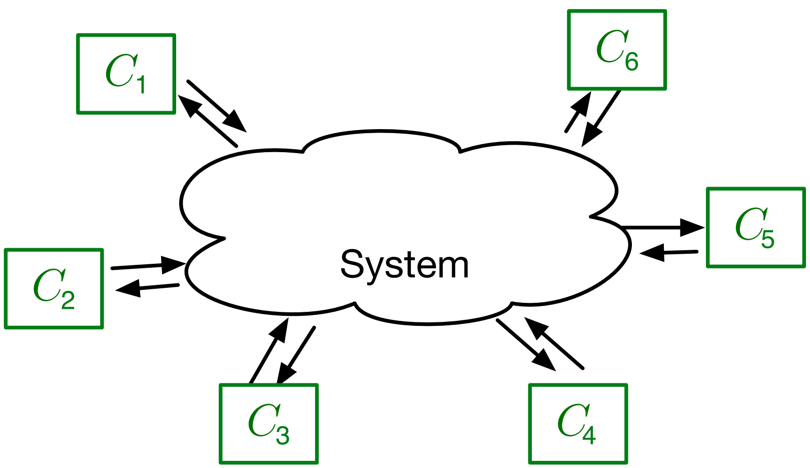
Gireeja Ranade
Works of Pulkit Grover, Anant Sahai, Se Yong Park

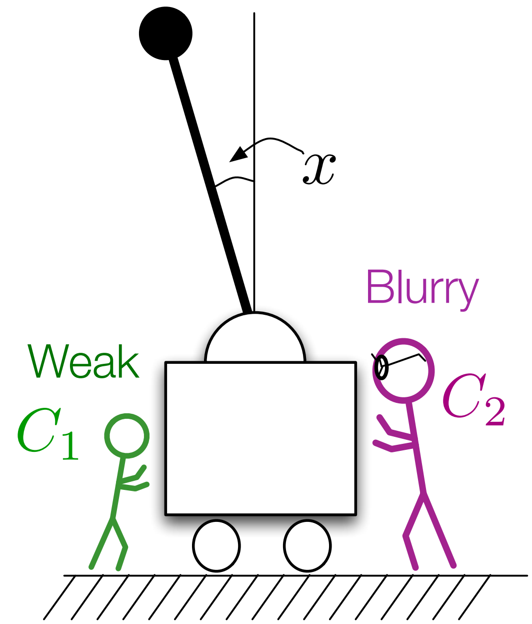
Slides courtesy Pulkit Grover

Decentralized control: “observer-controller” problem



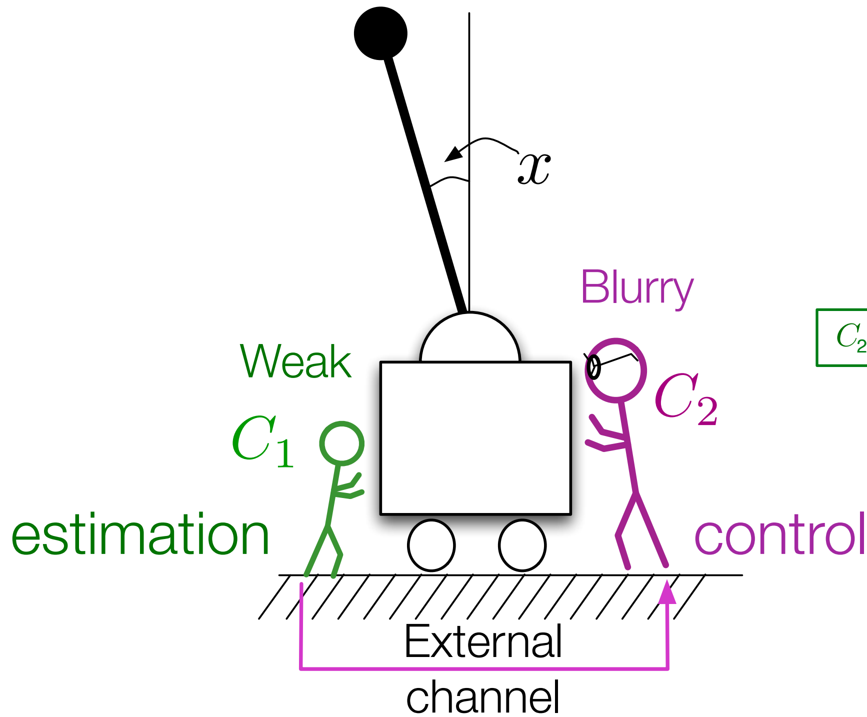
Decentralized control: a simple example

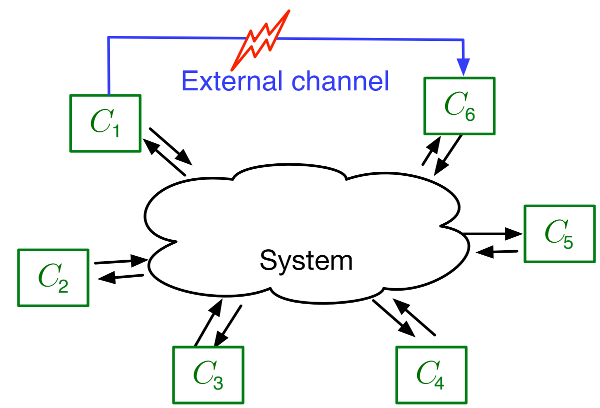




coordination

Coordination via explicit communication



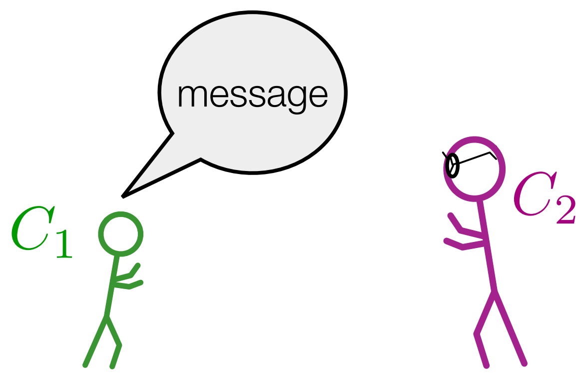


separate estimation and control

point-to-point communication

[Shannon '48]

Coordination via implicit communication?

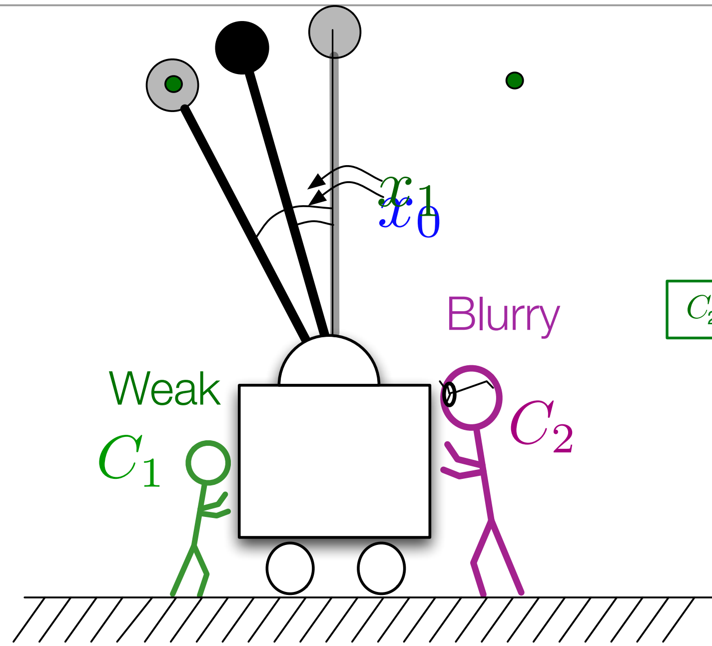


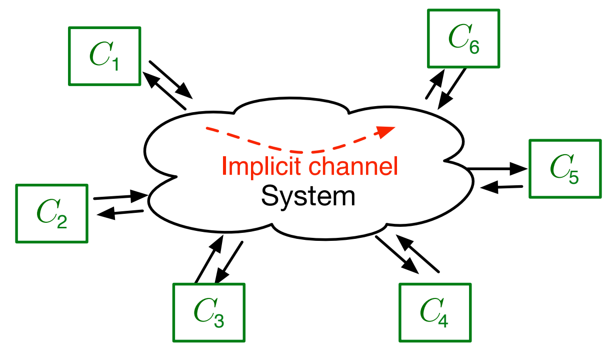
Explicit communication

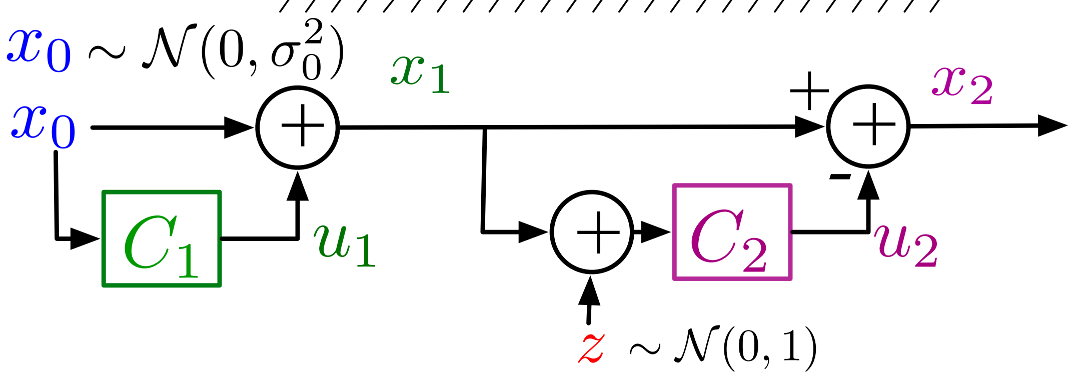
Implicit communication

1. channel: the system itself
2. messages: endogenously generated

Implicit communication: an example



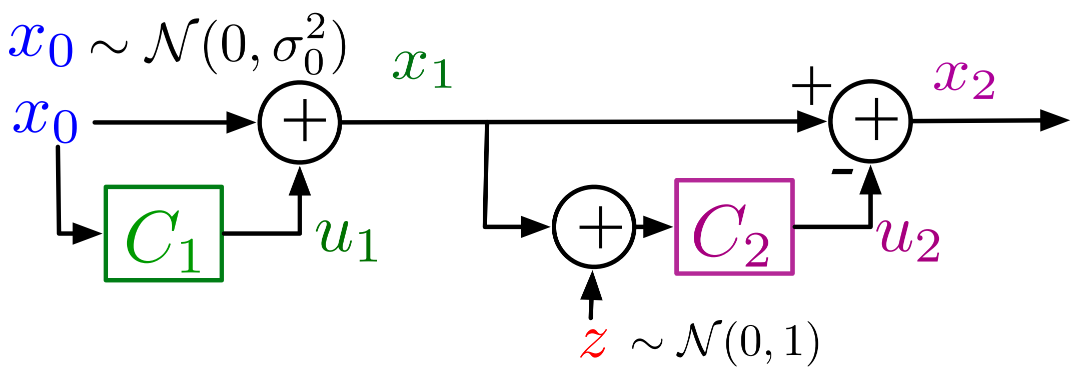




$$\min \left\{ k^2 \mathbb{E}\left[u_1^2\right] + \mathbb{E}\left[x_2^2\right] \right\}$$

[Witsenhausen '68]

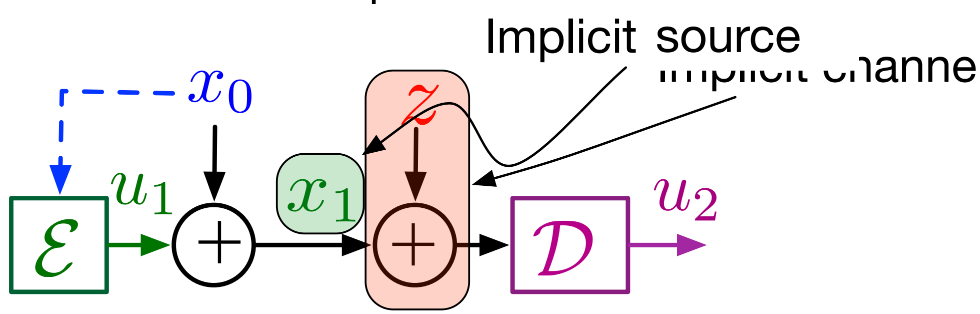
Toy implicit communication problem:
Witsenhausen's counterexample



$$\min \left\{ k^2 \mathbb{E}\left[u_1^2\right] + \mathbb{E}\left[x_2^2\right] \right\}$$

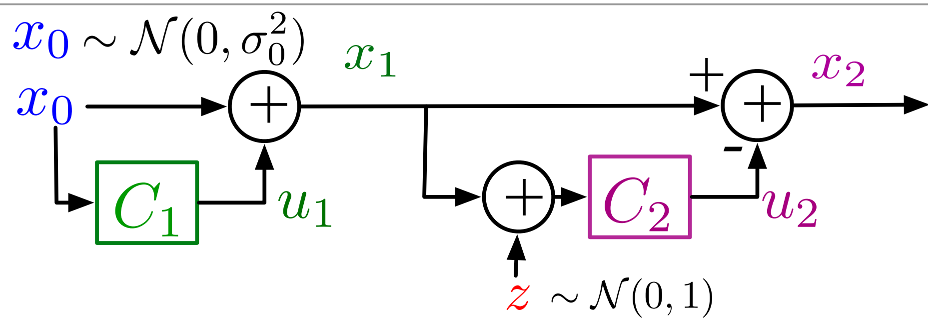
[Witsenhausen '68]

Implicit communication interpretation :



$$\mathbb{E}\left[u_1^2\right] \leq P \qquad MMSE = \mathbb{E}\left[\left(x_1 - \widehat{x}_1\right)^2\right]$$

A brief history of Witsenhausen's counterexample



$$\min \left\{ k^2 \mathbb{E}\left[u_1^2\right] + \mathbb{E}\left[x_2^2\right] \right\}$$

Linear, Quadratic, Gaussian

Nonlinear strategies can outperform linear [Witsenhausen '68]

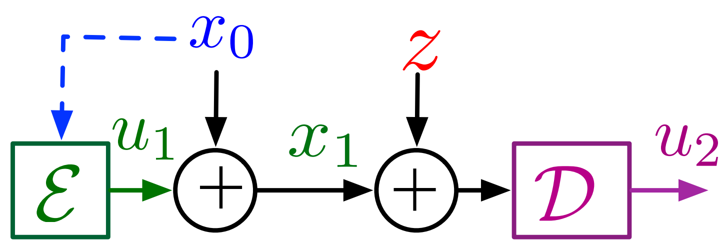
. . . by an unbounded factor [Mitter, Sahai '99]

Finding optimal strategy is NP-hard [Papadimitriou, Tsitsiklis '84]

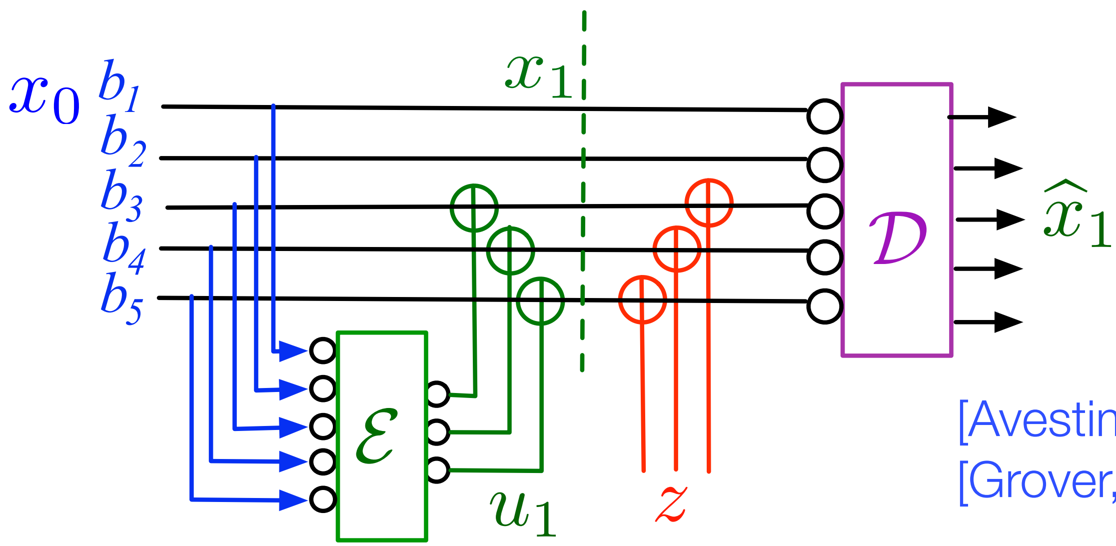
Semi-exhaustive search techniques [Baglietto, Parisini, Zoppoli '01]
[Lee, Lau, Ho '01]
[Lee, Marden, Shamma '09]

Understanding Witsenhausen's counterexample: A deterministic abstraction

Implicit communication interpretation :

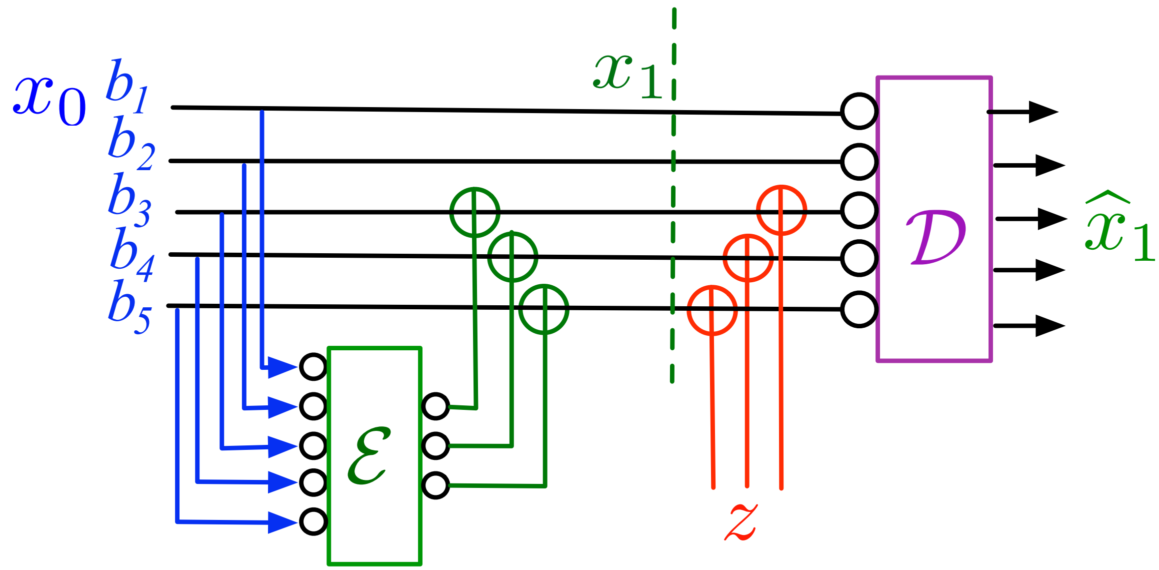


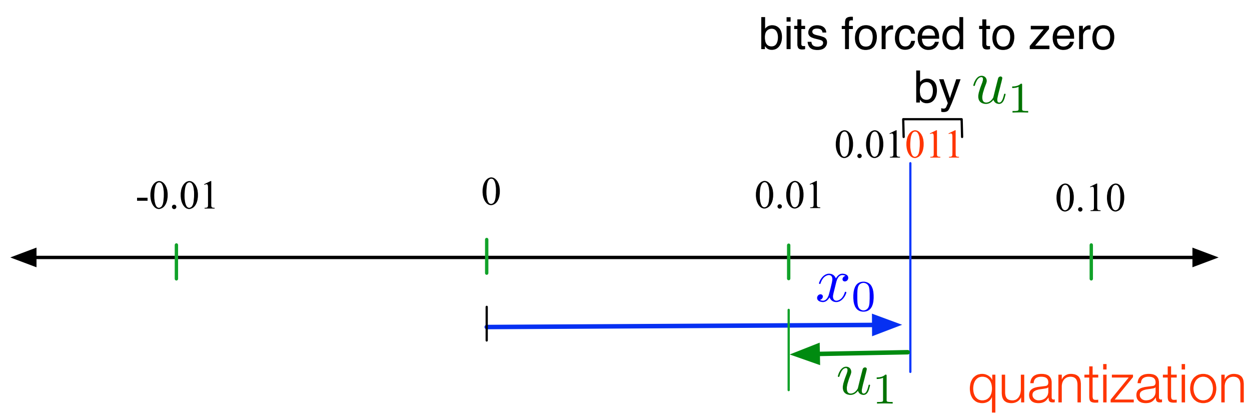
$$\mathbb{E}\left[u_1^2\right] \leq P \qquad MMSE = \mathbb{E}\left[\left(x_1 - \widehat{x}_1\right)^2\right]$$



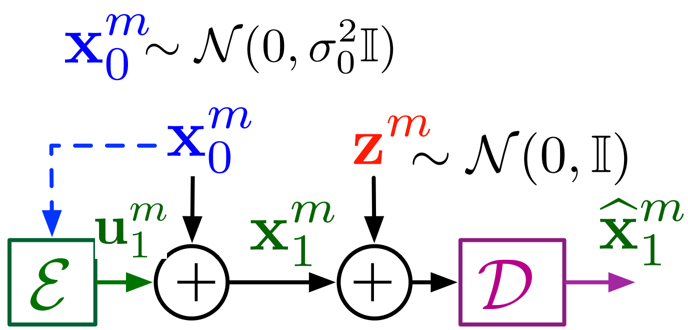
[Avestimehr, Diggavi, Tse '08]
[Grover, Sahai '09]

Strategies for the deterministic abstraction





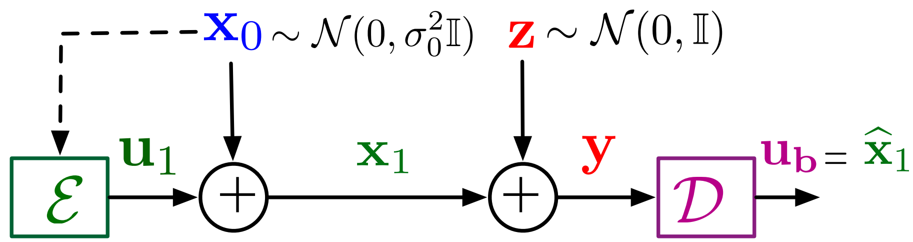
Asymptotically infinite-length extension of Witsenhausen's counterexample



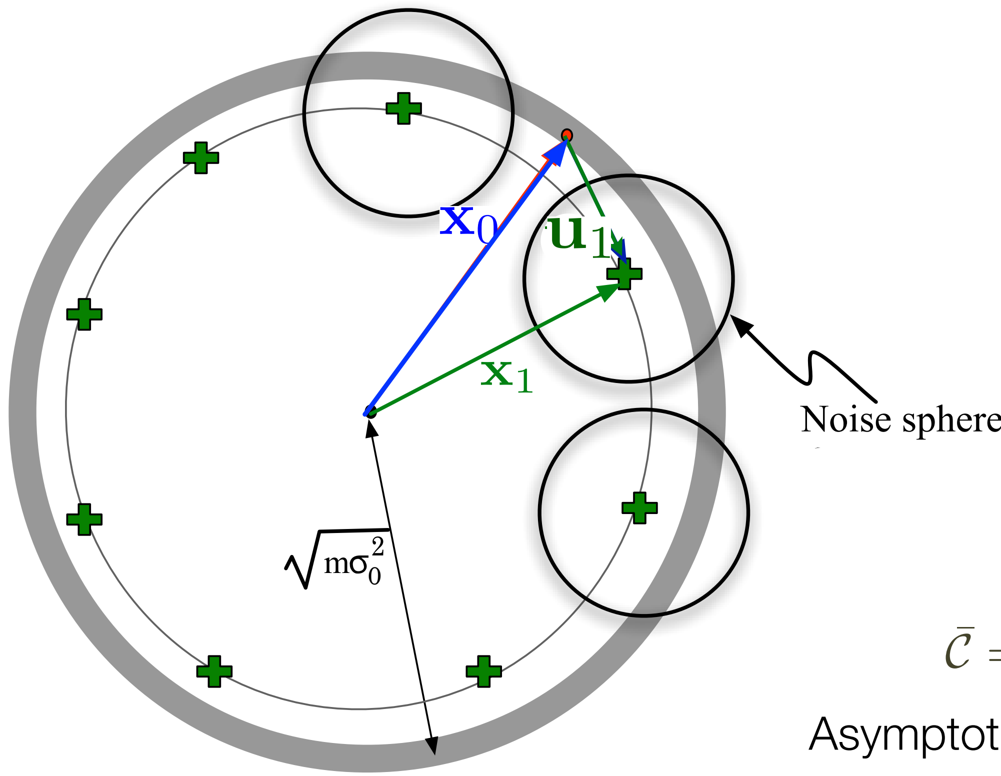
$$\min \left\{ \frac{k^2}{m} \mathbb{E}\left[\|\mathbf{u}_1^m\|^2\right] + \frac{1}{m} \mathbb{E}\left[\|\mathbf{x}_1^m - \widehat{\mathbf{x}}_1^m\|^2\right] \right\}$$

hope : can use laws of large numbers to simplify

A strategy : vector quantization



$$\bar{\mathcal{C}} = k^2 P + MMSE$$

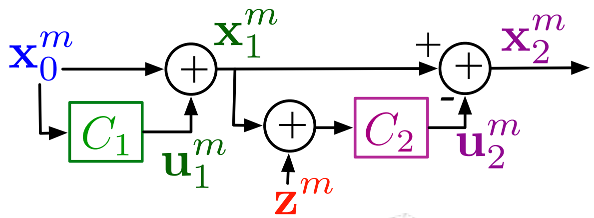


$$\bar{\mathcal{C}} = k^2 P + MMSE$$

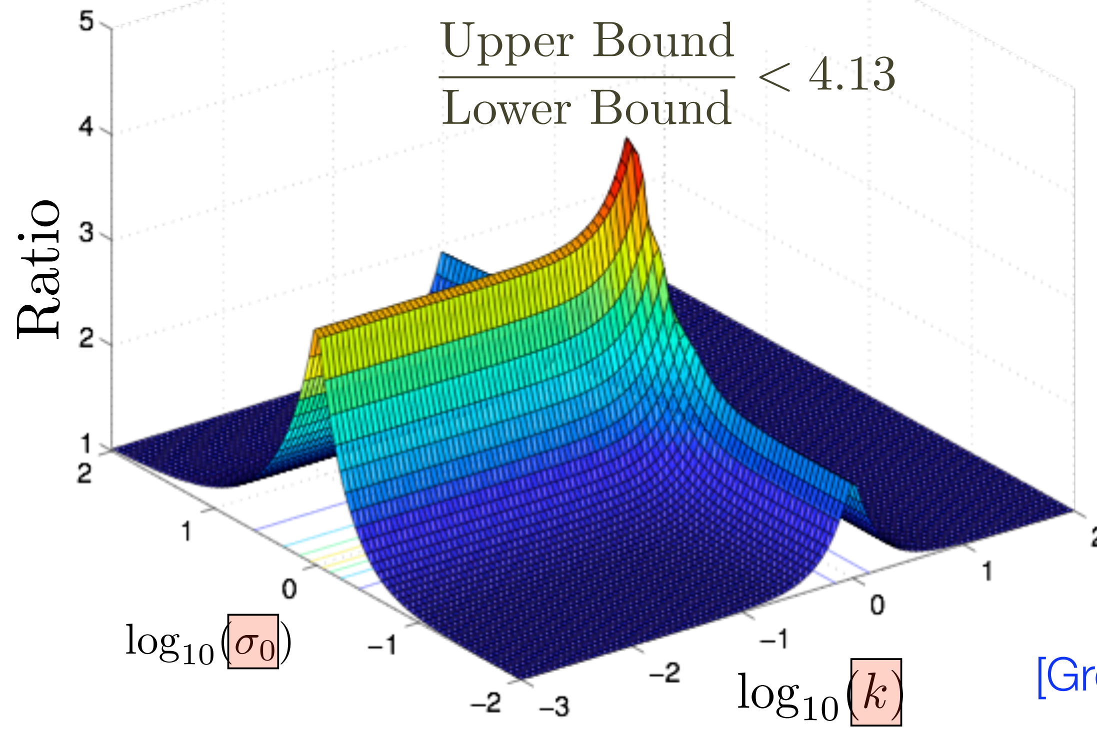
Asymptotic upper bound : $\bar{\mathcal{C}} = k^2 + 0$

Ratio of upper and lower bounds

$\mathbf{x}_0^m \sim \mathcal{N}(0, \sigma_0^2 \mathbb{I})$

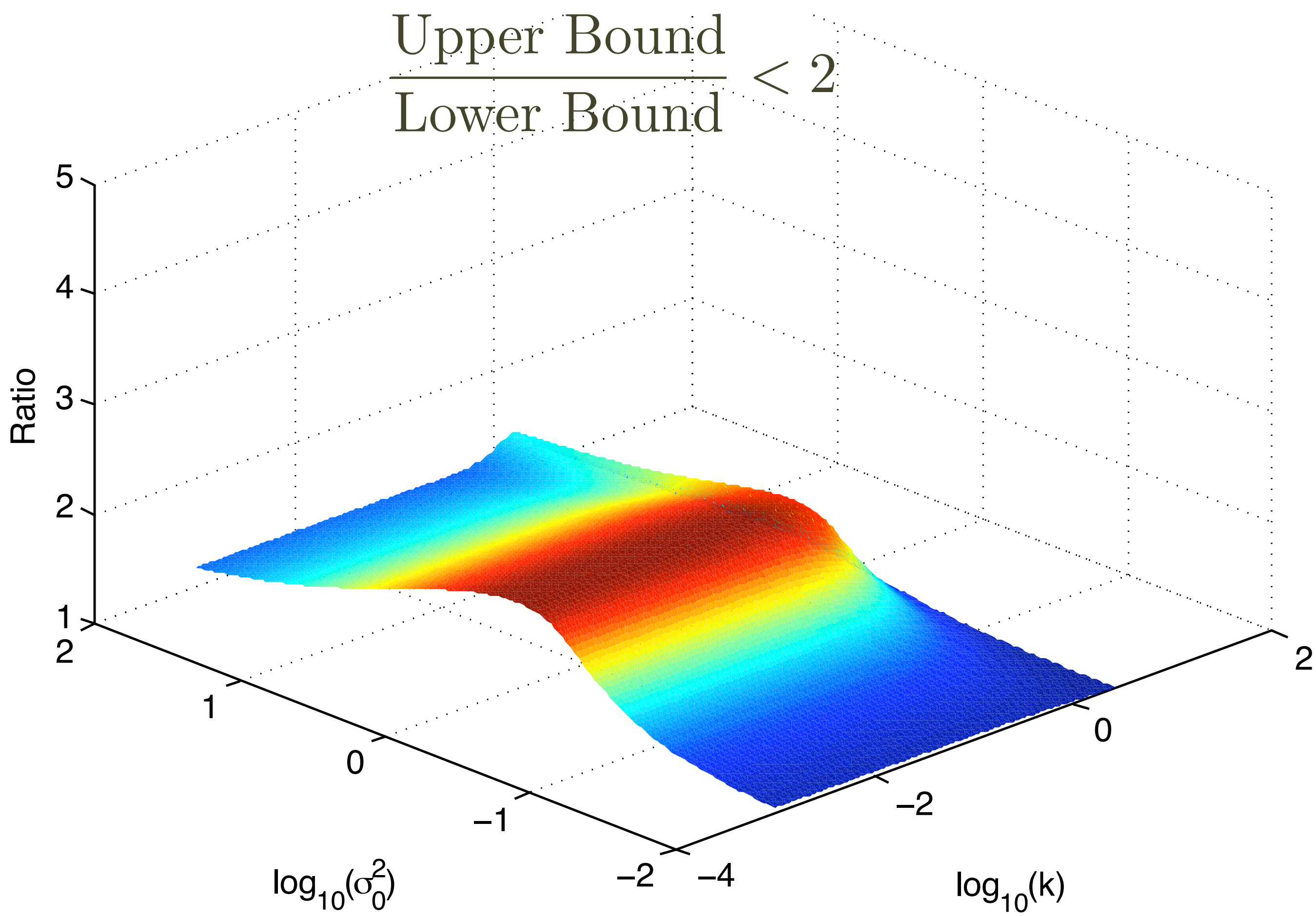


$$\min \left\{ \frac{k^2}{m} \mathbb{E}\left[\|\mathbf{u}_1^m\|^2\right] + \frac{1}{m} \mathbb{E}\left[\|\mathbf{x}_1^m - \widehat{\mathbf{x}}_1^m\|^2\right] \right\}$$

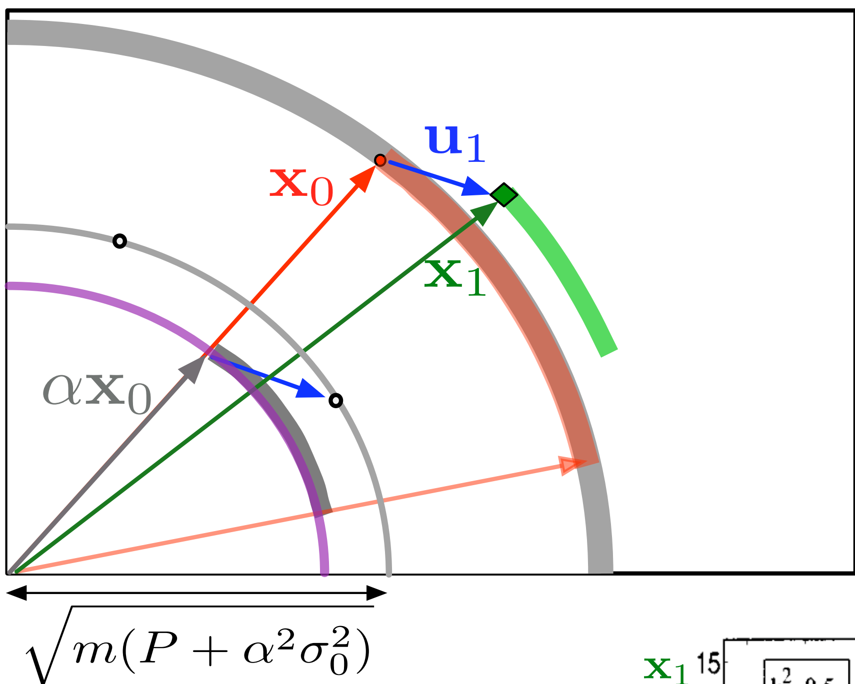


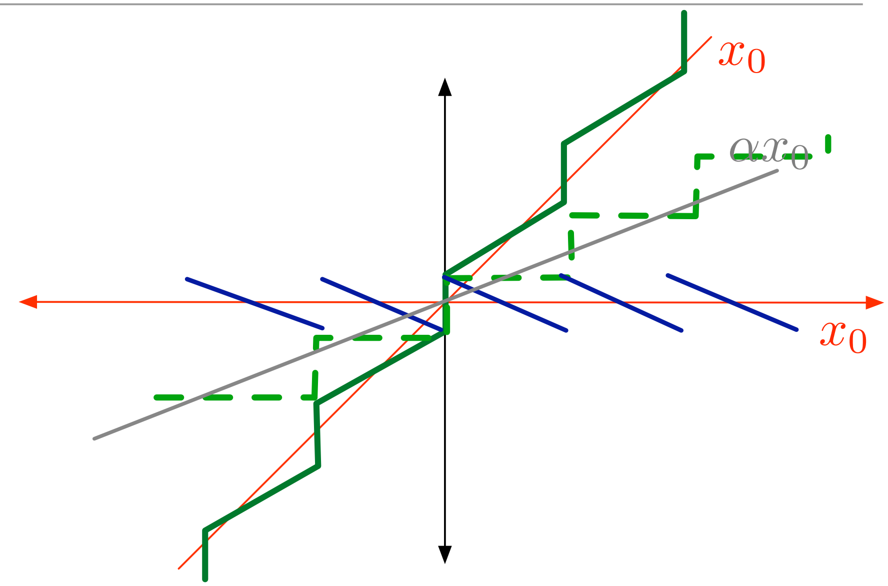
[Grover, Sahai '08]

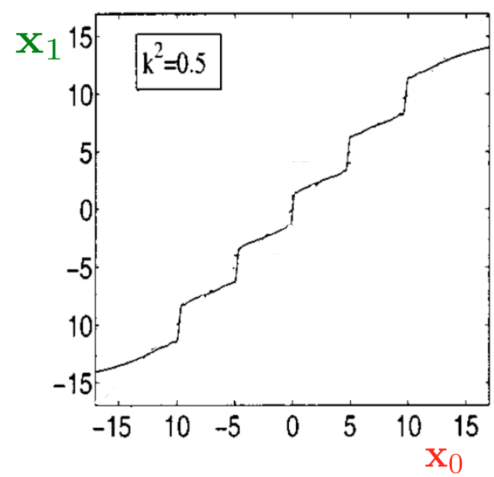
. . . with dirty-paper coding strategy



Conjectured optimal strategy: Dirty-paper coding



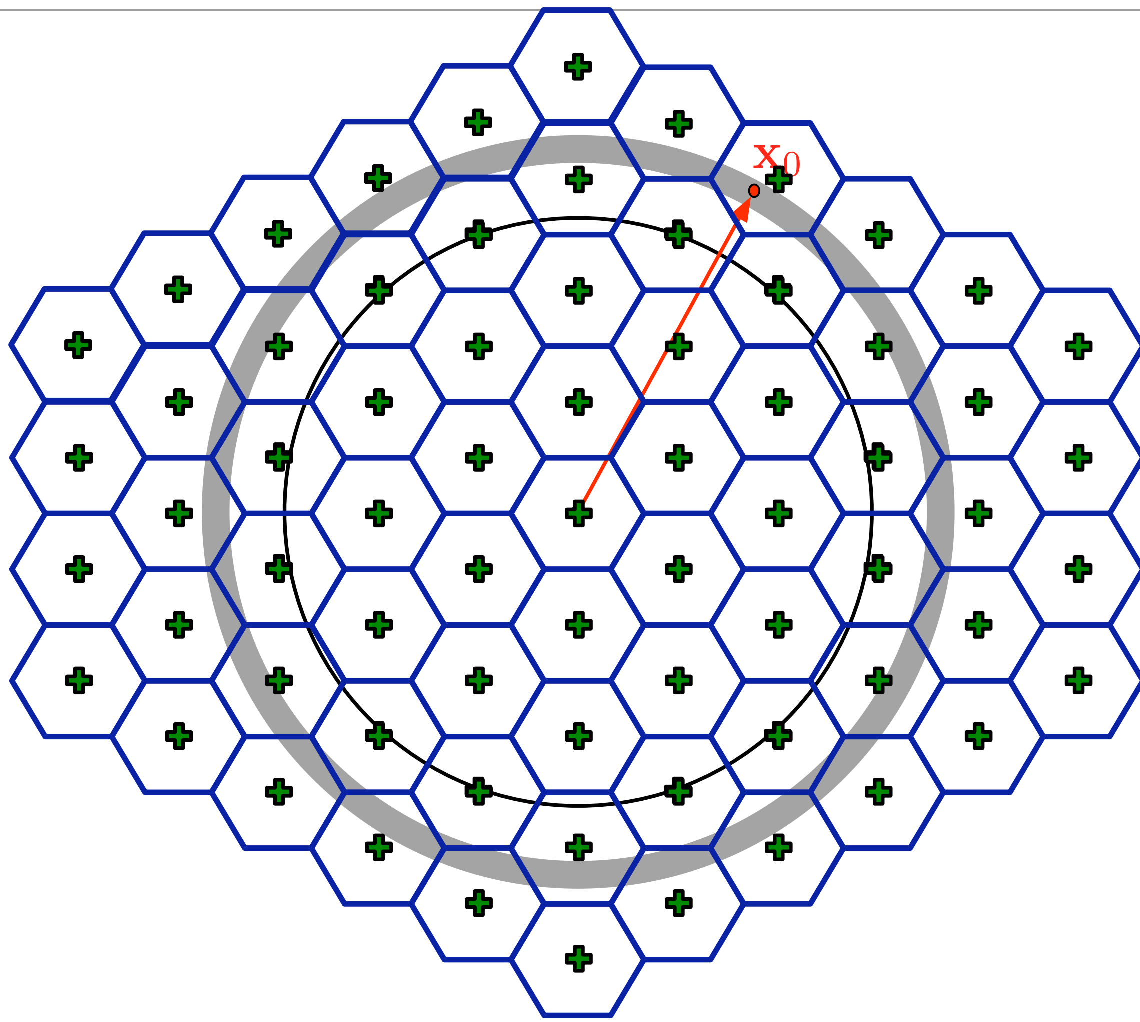


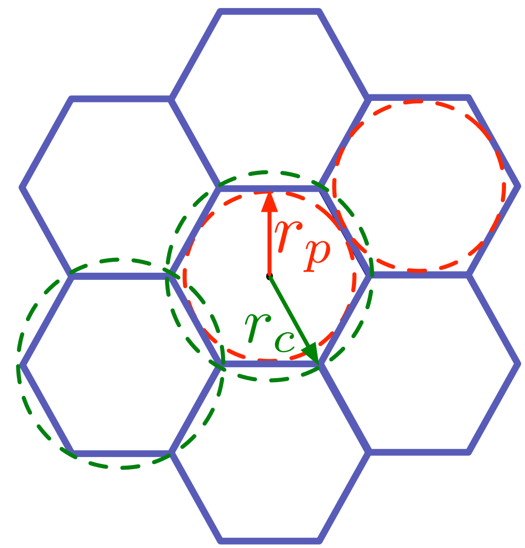


[Baglietto, Parisini, Zoppoli ’01]
[Lee, Lau and Ho ’01]

Finite-vector lengths

Quantization upper bounds





$$\zeta = \frac{r_c}{r_p}$$

Lower bound

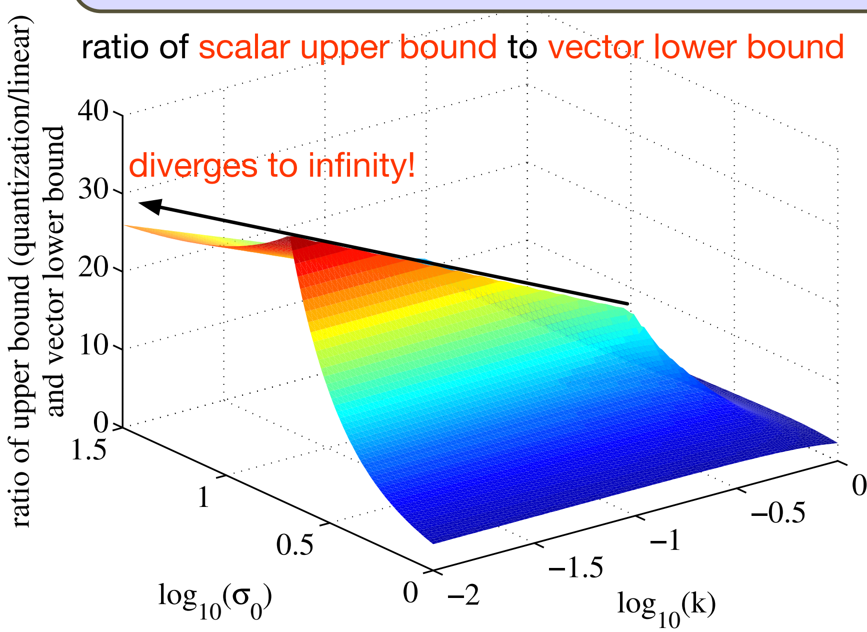
Use old lower bound

Theorem [Grover, Sahai '08]

$$\bar{\mathcal{C}}_{\min} \geq \inf_{P \geq 0} k^2 P + \left(\left(\sqrt{\kappa(P)} - \sqrt{P}\right)^+\right)^2$$

$$\kappa(P) = \frac{\sigma_0^2}{(\sigma_0 + \sqrt{P})^2 + 1}$$

ratio of scalar upper bound to vector lower bound



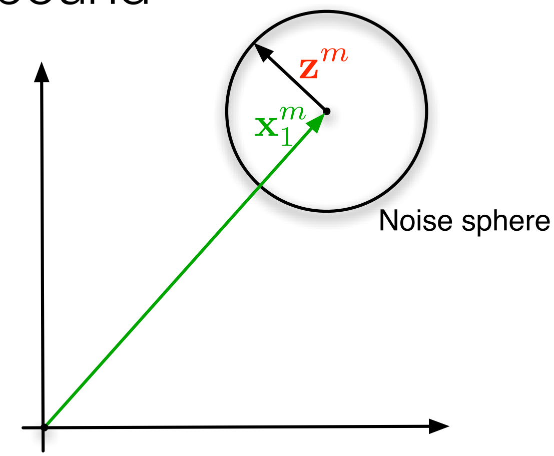
Need tighter lower bounds for tiny blocklengths

[Shannon '59][Shannon, Gallager, Berlekamp '67][Blahut '74][Pinsker '67][Sahai '06][Sahai, Grover '07][Polyanskiy, Poor, Verdu '08]

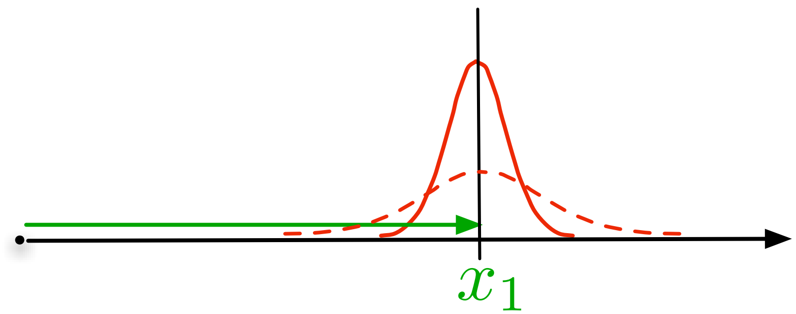
Need bounds that work in a distortion setting!

Large-deviation technique to tighten the lower bound

Lower bound



reality



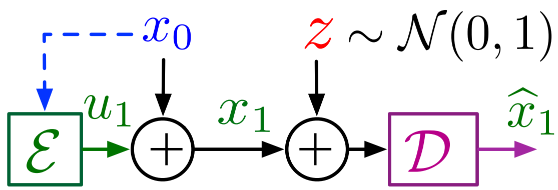
Noise can behave atypically!

Atypical noise behaviors are typical under a different distribution

$$\bar{\mathcal{C}}_{\min} \geq \inf_{P \geq 0} k^2 P + \left(\left(\sqrt{\kappa(P)} - \sqrt{P} \right)^+ \right)^2$$

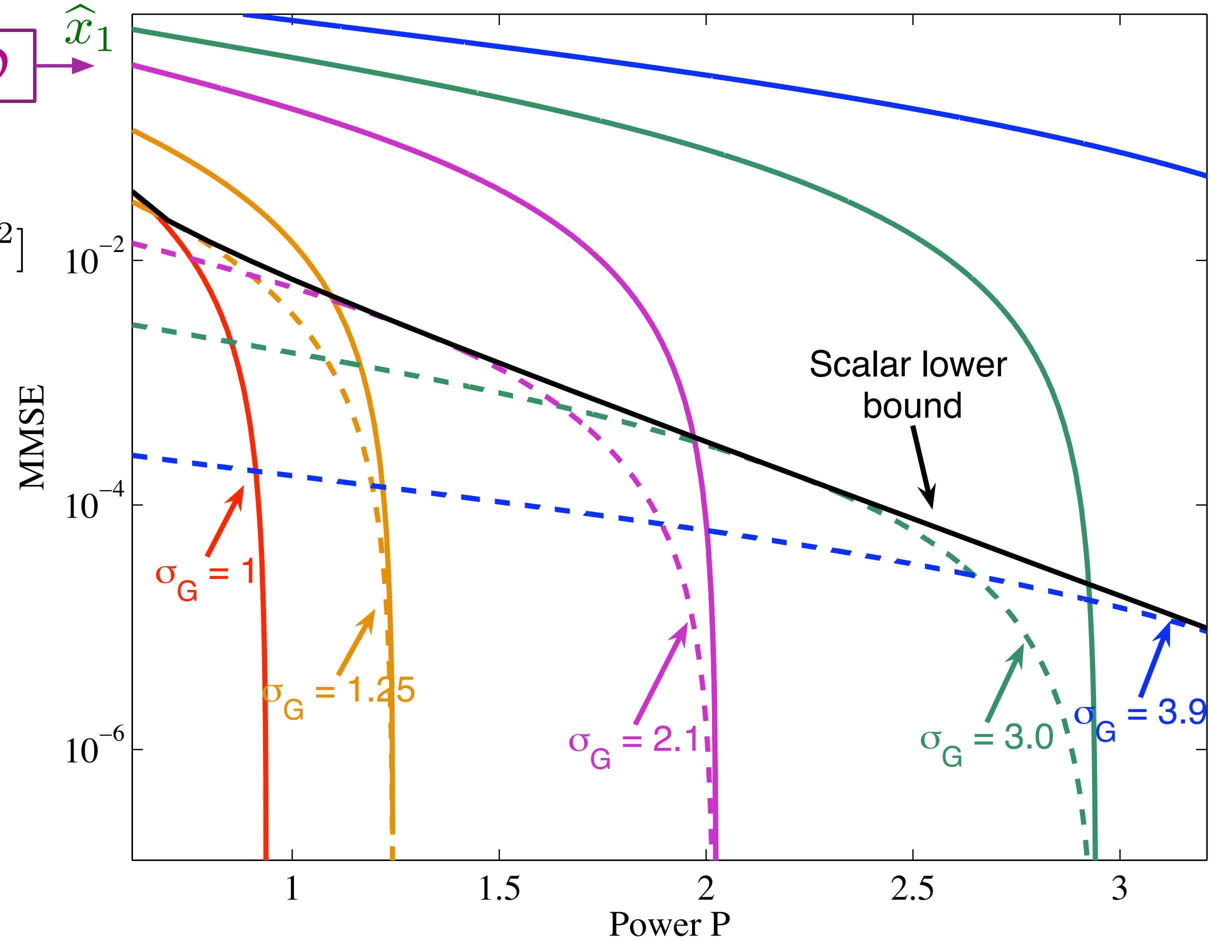
$$\kappa(P) = \frac{\sigma_0^2 {\sigma_G}^2}{(\sigma_0 + \sqrt{P})^2 + {\sigma_G}^2}$$

“Sphere-packing” extension of lower bound



$$P = \mathbb{E}[u_1^2]$$

$$MMSE = \mathbb{E}[(x_1 - \widehat{x}_1)^2]$$



Summary

Implicit communication promises substantial gains

. . . can be understood using information theory

Deterministic abstractions yield useful insights

Large-deviation techniques are needed to obtain finite-length results

The finite-length lower bound

For $\sigma_G^2 \geq 1$ and $L > 0$

$$\bar{J}_{\min}(m, k^2, \sigma_0^2) \geq \inf_{P \geq 0} k^2 P + \eta(P, \sigma_0^2, \sigma_G^2, L),$$

$$\eta(P, \sigma_0^2, \sigma_G^2, L) = \frac{\sigma_G^m}{c_m(L)} \exp\left(-\frac{mL^2(\sigma_G^2 - 1)}{2}\right)\left(\left(\sqrt{\kappa_2(P, \sigma_0^2, \sigma_G^2, L)} - \sqrt{P}\right)^+\right)^2,$$

$$\kappa_2(P, \sigma_0^2, \sigma_G^2, L) := \frac{\sigma_0^2 \sigma_G^2}{c_m^{\frac{2}{m}}(L) e^{1-d_m(L)}\left((\sigma_0 + \sqrt{P})^2 + d_m(L)\sigma_G^2\right)}$$

$$c_m(L) := \frac{1}{\Pr(\|\mathbf{Z}^m\|^2 \leq mL^2)} = (1 - \psi(m, L\sqrt{m}))^{-1}$$

$$d_m(L) := \frac{\Pr(\|\mathbf{Z}^{m+2}\|^2 \leq mL^2)}{\Pr(\|\mathbf{Z}^m\|^2 \leq mL^2)} = \frac{1 - \psi(m+2, L\sqrt{m})}{1 - \psi(m, L\sqrt{m})}$$