# Probabilistic Logics and the Synthesis of Reliable Organisms from Unreliable Components

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## Outline

- Automata Theory

- Error in Automata

- Controlling Error

- Extensions

## Motivation

Goals

- Provide a framework for exploring automata
- Give performance guarantees from unreliable automata
- Relate theory to electronic or neural circuitry

Historical Context

- First presented at Caltech by von Neumann
  - Manuscript based on lecture notes taken by R.S. Pierce
  - von Neumann added text to prepare for publication
- Five versions of the manuscript exist with different typesets, figures and content
- This presentation is based on the first manuscript

## Automata Theory

### Definition

A **single-output automaton with time delay** $\tau$ is a finite set of inputs, one output and a set of preferred subsets of inputs. The automaton stimulates its output at time $t + \tau$ if a preferred state appears at time $t$.
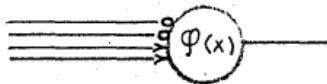
### Comments

- Automata differs from logic in that there is a time dimension
- We treat automata as "black boxes"
- Each input or output is allowed two states:
    - "unstimulated"(0)
    - "stimulated" (1)
- For $n$ inputs, there exist $2^{2n}$ automata of a certain $\tau$

## Example

- **Excitatory** inputs
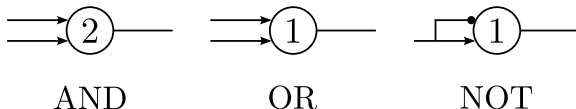- **Inhibitory** inputs
- Threshold function

$$\varphi(x) = \begin{cases} 0, & x < h; \\ 1, & x \geq h, \end{cases}$$

where $h = \#_{\mathrm{ex}} + \#_{\mathrm{in}}$.

AND          OR          NOT

## Definition

Two single-output automata are **equivalent in the wider sense** if they differ only in their time delays, not input-output behavior.

## Theorem

*Any single-output automaton is equivalent in the wider sense to a network of basic organs. There exists a unique $\tau^\star$ such that the latter network exists iff its time delay is $\tau > \tau^\star$*

Comment: Actually, any two organs above form a basis.

# Single Basic Organs

Scheffer Stroke (NAND)

- $\bar{A} = S(A, A)$
- $A \cdot B = S(S(A, B), S(A, B))$
- $A + B = S(S(A, A), S(B, B))$
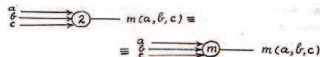


Majority Organ ("best out of three")

- $A \cdot B = M(A, B, 0)$
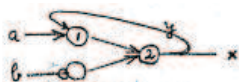- $A + B = M(A, B, \infty)$
- $\bar{A}$ can be derived from AND and OR

## Examples

Memory Machine: $X$ is stimulated $\tau$ after the first time $A$ is stimulated (i.e. memory tape)



Fig. 16

Memory Organ: $X$ is stimulated iff $A$ was stimulated earlier such that no stimulation of $b$ occurred since

# Error in Automata

Error considerations

- Mechanical and electric units are subject to failure
- Assume: for every operation, the organ will fail to function correctly with precise probability $\epsilon$
- Failures $\varepsilon$ are assumed statistically independent of time and state of network
- More generally, failures can be dependent but are upper-bounded by $\epsilon$

Goal: Find the limits of the smallness of $\epsilon$ such that performance of automata can be reliable ($\Pr(\varepsilon) < \delta$)

## Example: The Memory Organ

Scenario
- Memory machine
- Stimulation only at time $t$
- Error probability $\epsilon$

After $s$ cycles after $t$, probability organ is still stimulated is

$$\rho_s \approx \frac{1}{2} + \frac{1}{2}e^{-2\epsilon s}$$

Conclusion: $\rho_s \to 1/2$ as $s \to \infty$

"... is not so much that incorrect information will be obtained, but rather that irrelevant results will be produced."

Run $m$ versions of network $\mathcal{O}$ in parallel, use majority organ to determine output values

Analysis

- Assume $\eta$ is upper bound on error of $\mathcal{O}$
- Error at majority organ output is upper bounded by
  $$\eta^\star = \epsilon + (1 - 2\epsilon)(3\eta^2 - 2\eta^3)$$
- Roots at
  $$\frac{1}{2}, \eta_0 = \frac{1}{2}\left(1 - \sqrt{\frac{1 - 6\epsilon}{1 - 2\epsilon}}\right), 1 - \eta_0$$
- Latter two roots are only real if $\epsilon < 1/6$

Consider successive occurences of the network ($\eta$ becomes $\eta^\star$)

- If $\epsilon > 1/6$, $\eta \to 1/2$
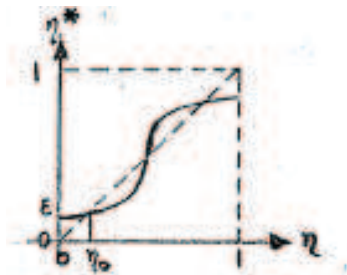- If $\epsilon < 1/6$, $\eta \to \eta_0$



Fig. 27

End Result: Reliable computation is possible, with necessary error level satisfying $\eta_0 \approx \epsilon + 3\epsilon^2$
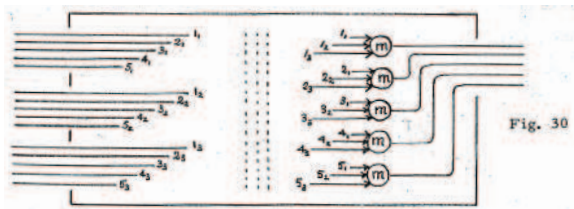
# Controlling Error 1: Multiple Machines

More general argument (Sections 8.3.2 – 8.4 )

- Network $\mathcal{P}$ made up of arbitrary basic organs
- Each output's error probability is bounded by $\eta_1$
- Use induction to reduce $\mathcal{P}$ into networks with smaller serial chains with maximum length denoted $\mu$
- Necessary error level satisfies $\eta_1 = 4\epsilon + 152\epsilon^2$
- Procedure is impractical because new network needs $3^{\mu(\mathcal{P})}$ as many organs as the original network $\mathcal{P}$

Each message is carried on a "bundle" of $N$ wires

- For $\Delta_H < 1/2$, stimulation of at least $(1 - \Delta_H)N$ lines is interpreted as 1
- For $\Delta_L < 1/2$, stimulation of at most $\Delta_L N$ lines is 0
- Everything else is malfunction



Fig. 30

**Executive organ**

Issue: Output wires may have different values
Fix: Allow for a **restoring organ** to ensure bundle wires have the same value

Error Types

- Organ malfunction (with probability $\epsilon$)
- Bundles not entirely stimulated or unstimulated (errors in wires)
    - $\Pr(0 \rightarrow 1) = \eta$
    - $\Pr(1 \rightarrow 0) = \xi$

Goal: For a computation, find $N$ such that probability of malfunction is $\eta$

Assume: Scheffer (NAND) for executive organs, majority for restoring organ

# Controlling Error 2: Multiplexing

Proof Sketch

- Use combinatorial argument to determine probability of error propagated by wire errors
- Use Stirling's formula to get Gaussian approximation of error
- Variance increases with executive and restoring organ failures

Constructive scheme

1. Design network $\mathcal{R}$ for function to be computed assuming error-free parts
2. Determine number of basic organs needed (denoted $m$) and let $\delta = \eta/m$
3. Find $\mathcal{N}$ so that error probability of each organ is $\delta$
4. Create multiplexed system with bundles of $\mathcal{N}$ wires

Comments

- For fixed $\mathcal{N}$ and $\epsilon$, $\delta_H = \delta_L = 0.07$ is best, according to back-of-the-envelope calculation
- Maximum allowable $\epsilon$ is 0.0107
- For two practical examples, $\mathcal{N} = 20,000$ is a good approximation

Comparison to multiple machines

- Requires $\mathcal{N}$ as many wires and $3\mathcal{N}$ as many basic organs versus exponentially more

# Extensions

Memory considerations

- Model assumes random permutations of wires in a bundle to simplify calculations
- This is hard to maintain when network contains feedback

Multiplexing as analog computation

- For large $\mathcal{N}$, can model bundle as an analog signal
- This is similar to modern logic gates

Multiplexing to explain neuroscience

- von Neumann discusses implications in neural circuits
- At the time of the presentation, neural systems are the only ones dense and reliable enough to match his theory

## A Final Note

In the introduction, von Neumann writes:

"Our present treatment of error is unsatisfactory and ad hoc. It is the author's conviction, voiced over many years, that error should be treated by thermodynamical methods and be the subject of a thermodynamical theory, as information has been by the work of L. Szilard and C.E. Shannon."