Angular Sychronization by Spectral Methods

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An expository presentation of using eigenvalue/eigenvector methods to solve synchronization problems

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Outline

1. Spiking Covariance Matrices
2. Angular Synchronization via Spectral Methods
3. Further Directions

Phase Transition of Spiked Covariance Matrices

Problem

Suppose we are given a Wigner $n \times n$ matrix M. Consider the matrix

$$M' = \frac{\lambda}{n} zz^* + \frac{1}{\sqrt{n}} M$$

How large does λ have to be so that the largest eigenvalue of M' lies outside the support of the semicircle law with high probabiliy?

Transition at $\lambda > 1$

- Baik, Ben Arous, Peche (2004)- transition when M is Gaussian
- Feral, Peche (2006)- transition for M Wigner under some conditions (sub-Gaussian moments, uniformly bounded second moments)

The Angular Synchronization Problem

Problem

Suppose we want to find n unknown angles $\theta_1, \ldots, \theta_n \in [0, 2\pi)$. We are given $m \leq \binom{n}{2}$ "noisy" measurements δ_{ij} which are $\theta_i - \theta_j$ with probability p and uniformly chosen from $[0, 2\pi)$ with probability $1 - p$. Our goal is to devise a method that with high probability recovers the angles, under some conditions we impose.

Approaches that Don't Quite Work

- Method of Least Squares
- Maximum Likelihood

Eigenvector Method

1. Define a matrix H with complex entries

$$H_{ij} = \begin{cases} 1 & \text{if } i = j \\ e^{i\delta_{ij}} & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E \end{cases}.$$

2. Compute the top eigenvector v_1 of H.

3. Set $e^{i\theta_j} = \frac{v_1(j)}{|v_1(j)|}$

n=200
p=0.5
<v_1,z>=0.992756349558468

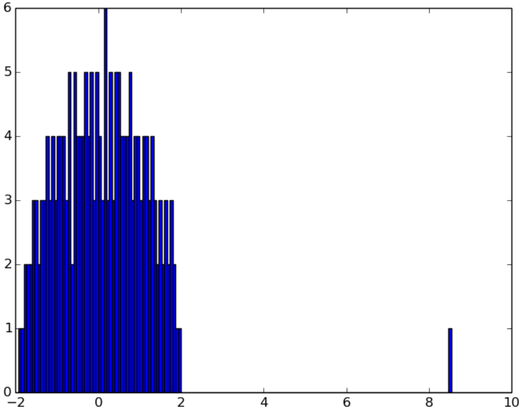

Figure: Plot of Eigenvalues

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n=200
p=0.1
<v_1,z>=0.7251118230422289
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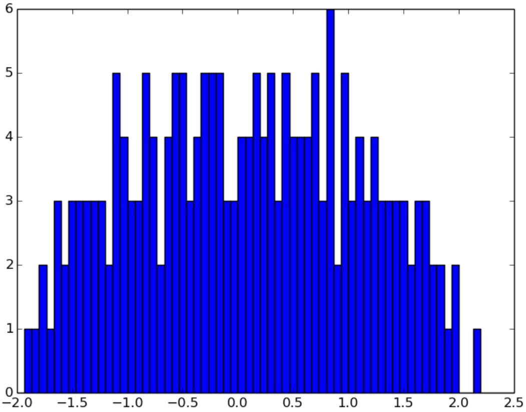

Figure: Plot of Eigenvalues

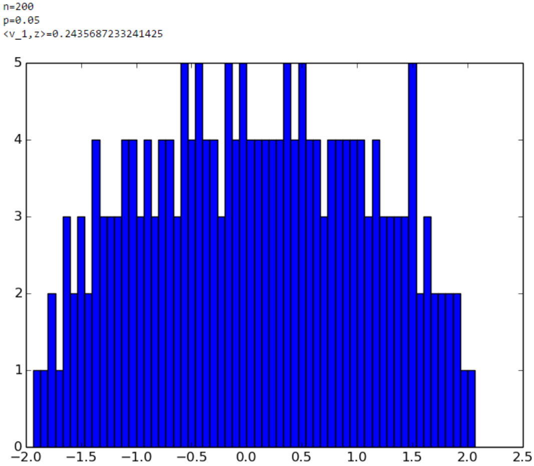

Figure: Plot of Eigenvalues

Analysis when $m = \binom{n}{2}$

- Take an Erdos-Renyi model $G(n,p)$ for the "good" edges.
- Set $z_j = \frac{1}{\sqrt{n}} e^{i\theta_j}$, and set $H = npzz^* + R$.
-

$$R_{ij} = \begin{cases} (1-p)e^{i(\theta_i - \theta_j)} & \text{with probability } p \\ e^{i\phi} - pe^{i(\theta_i - \theta_j)} & \text{with probability } 1-p \text{ and } \phi \text{ uniform} \end{cases}$$

- R_{ij} is mean 0 and variance $1 - p^2$.
- Use spiked covariance result! As long as $np > \sqrt{n(1-p^2)}$

Above Random Correlation of Top Eigenvector

1. $\lambda_1(H)v_1 = (npzz^* + R)v_1 \Rightarrow \lambda_1(H) = np| < z, v_1 > |^2 + v_1^* R v_1$
2. $\Rightarrow | < z, v_1 > |^2 \geq \frac{\lambda_1(H) - \lambda_1(R)}{np} > \frac{1}{n}$
3. Plug in some formulas, get $p > \frac{1}{\sqrt{n}}$

General Case

- The idea is to generalize the decomposition of H from the complete graph case.
- Let $A = \sum \lambda_i \psi_i \psi_i^T$ be the adjacency matrix for the good edges, and Z be the diagonal matrix with entries $e^{i\theta_i}$
- $B = ZAZ^*$, so $B_{ij} = e^{i(\theta_i - \theta_j)}$ for good edges, with eigenvectors $\phi_i = Z\psi_i$.
- Then $H = B + R$, where R has an $e^{i\delta_{ij}}$ for bad edges.

General Case

- R is a sparse matrix whose nonzero entries have zero mean and unit variance

$$\limsup_{n\to\infty} \frac{\sqrt{n}}{\sqrt{2m_{bad}}}\lambda_1(R) \leq 2$$

- Recall eigenvectors of B can be written as Z times eigenvectors of A.
- Perron Frobenius says top eigenvector of A is all positive, so top eigenvector of B is the true angles.
- Correlation as long as spectral gap is bigger than $\frac{1}{2}\lambda_1(R)$.

Semidefinite Program

- We can write our problem as trying to find

$$
\begin{aligned}
\max_{\Theta \in \mathbb{C}^{n \times n}} \ & trace(H^* \Theta) \\
\Theta & \succeq 0 \\
\Theta_{ii} & = 1 \;\; i = 1, \ldots, n \\
\operatorname{rank}(\Theta) & \leq 1
\end{aligned}
$$

Other Synchronization Problems

- More general problem- given pairwise multiplications $g_i g_j^{-1}$ of a group G, recover g_i
- AMP- approximate message passing techniques
- Cryo-EM

Bibliography

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