18.338 Project: log-Determinant of a Wigner matrix

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Abstract

The *log-determinant* of a Wigner matrix is a linear statistic of its singular values, the sum of the logarithm of the singular values:

$$\Sigma_{k=1}^{n} \log \sigma_k$$

. We describe some structural properties of the log-determinant of the Gaussian Unitary Ensemble and the Gaussian Orthogonal Ensemble, and prove a central limit theorem about general Wigner matrices as $n \to \infty$. We follow the computations of [**?**] and [**?**] and the universality proof of [**?**]. Finally, we present numerical results on the log-determinant.

1 Introduction

The goal of this expository paper is to study the fluctuations of the log-determinant

$$\log(|\det(H_n)|) = 2\sum_{k=1}^{n} \log(\sigma_k) \tag{1}$$

where $\sigma_1, \ldots, \sigma_n$ are the singular values of a Wigner matrix H_n.

Definition 1.1. *A Wigner matrix is an $n \times n$ Hermitian random matrix H_n whose off-diagonal entries $H_n(i,j) = \overline{H_n(j,i)}$ are centered, independent identically distributed complex random variables satisfying $\mathbb{E}|H_n(i,j)|^2 = 1$ and $\mathbb{E}H_n(i,j)^2 = 0$ for all $i \neq j$ and whose diagonal entries $H_n(i,i)$ are centered, independent identically distributed* *real random variables satisfying $\mathbb{E}H_n(i,i)^2 < \infty$. We assume that the common distribution μ of $H_n(i,j)$ matches a Gaussian up to the fourth moment. In case μ is a Gaussian, the ensemble is known as the Gaussian Unitary Ensemble (GUE).*

Remark 1.1. *As the name suggests, the distribution of the GUE is invariant under conjugation by a unitary matrix. A symmetric counterpart to the GUE is known: the Gaussian Orthogonal Ensemble (GOE).*

2 Factorization of the Determinant

In [**?**], the La Croix and Bornemann discovered a beautiful structure in what were at the time purely asymptotic results about the log-determinant of GOE. Tao and Vu, in [**?**], had proved a central limit theorem but speculated that a factorization would be out of reach. La Croix and Bornemann found it (for GUE see [**?**]), and it included the term:

$$\chi_{2m}^{\circ} = (\xi_1^4 + 2\xi_1^2\xi_{2m}^2)^{1/4},$$

(notation mine) where ξ_1 is distributed according to χ_1 and ξ_{2m} is distributed according to χ_{2m}. I have computed the density of this random variable in the Appendix in Theorem 5.1.

3 Universality

Since the work of [**?**], there has been a great interest in understanding the extent to which the properties of the GUE and GOE are observed in more general random matrix ensembles.

For ensembles which are invariant under symmetries (unitary transformation, orthogonal transformation), see [**?**]. Both the heuristics and the techniques of invariant ensembles work were difficult to extend to the class of Wigner matrices. Recently this project has been amenable to the tool-kits of analysts ([**?**]).

In this section, we give an exposition to some of the core ideas in that thread of research. In particular, we explain the concept of microscopic universality, a kind of behavior of the spectrum which is more rich in allowing a central limit theorem for the log-determinant to be established than, say, the semi-circle theorem.

We also explain how some of the more central ideas of linear algebra, such as the Cauchy Interpolation Theorem, can be approached to create a powerful result in random matrix theory, the Four Moment Theorem ([**?**]).

4 Illustration of the theory

In [**?**], an effort was made toward an understanding of the accuracy of local laws for Wigner matrices for matrices of small dimension, i.e. non-asymptotic random matrix theory for Wigner matrices.

In that vein, we present some Julia code and plots that demonstrates both the Gaussian behavior toward classical universality (i.e. the comparison of the log-determinant of the GUE and GOE with a standard normal appropriately rescaled) and the histogram comparing the "insensitivity" of the log-determinant (i.e. the comparison of the GUE/GOE log-determinant with a Wigner matrix log-determinant).

5 Appendix: Density of χ_{2m}°

Theorem 5.1. *Let $f_{\chi_{2m}^{\circ}}(t)$ be the density of χ_{2m}°. Then*

$$f_{\chi_{2m}^{\circ}}(t) = \frac{2^{1-m}t^3}{\Gamma(m)\sqrt{2\pi}} \int_0^t \left(\sqrt{\frac{1}{2}\left(\frac{t^4}{s^2} - s^2\right)}\right)^{m-2} \cdot \frac{1}{s^2} \cdot e^{-s^2/2} e^{-\sqrt{\frac{1}{2}(\frac{t^4}{s^2}-s^2)}/2} ds. \tag{2}$$

Proof. Note that

$$f_{\chi_{2m}^{\circ}}(t) = \frac{d}{dt}\Pr((\xi_1^4 + 2\xi_1^2\xi_{2m}^2)^{1/4} \leq t).$$

Now observe that

$$\Pr((\xi_1^4 + 2\xi_1^2\xi_{2m}^2)^{1/4} \leq t) = \Pr(\xi_1^2 \leq \sqrt{t^4 + \xi_{2m}^4} - \xi_{2m}^2),$$

and now using the density of ξ_1, we have:

$$\Pr(\xi_1^2 \leq \sqrt{t^4 + \xi_{2m}^4} - \xi_{2m}^2) = \frac{2}{\sqrt{2\pi}} \int_0^t \Pr(s^2 \leq \sqrt{t^4 + \xi_{2m}^4} - \xi_{2m}^2) e^{-s^2/2} ds,$$

with the upper bound on the integral due to the fact that $h_t(x) = \sqrt{t^4 + x^4} - x^2$ is bounded by t^2. Let the inverse of $h_t(x)$ in the variable

x be $g_t(y)$. Note that $h_t(x)$ is monotone decreasing in x, and therefore $g_t(y)$ is monotone decreasing in y. Therefore

$$\Pr((\xi_1^4 + 2\xi_1^2\xi_{2m}^2)^{1/4} \leq t) = \frac{2}{\sqrt{2\pi}} \int_0^t F_{\chi_{2m}}(g_t(s^2))e^{-s^2/2}ds,$$

where $F_{\chi_{2m}}$ is the characteristic distribution function of χ_{2m}. Note that $F_{\chi_{2m}}(g_t(t^2)) = 0$. Therefore, we may simply pass the derivative into the integral to obtain $f_{\chi_{2m}^\circ}(t)$. Hence,

$$f_{\chi_{2m}^\circ}(t) = \frac{2}{\sqrt{2\pi}} \int_0^t f_{\chi_{2m}}(g_t(s^2))\partial_t(g_t(s^2))e^{-s^2/2}ds.$$

Here we use $f_{\chi_{2m}}(\cdot)$ to denote the probability density function of χ_{2m}. Note that

$$g_t(s^2) = \sqrt{\frac{1}{2}\left(\frac{t^4}{s^2} - s^2\right)}.$$

To complete, we simply evaluate the derivative and finally substitute the expression for $f_{\chi_{2m}}$ into the integral.

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