On the Minimal Eigenpair of Erdos-Renyi Graphs

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Erdos-Renyi Graph

Let $G(n, p)$ be a random graph for some fixed $n \in \mathbf{N}$ and $0 \leq p \leq 1$, where each edge is included with probability p, and each edge is independent.

Let us introduce the graph Laplacian, namely, the billinear form $L(G)$ satisfying

$$(Lu, v) = \sum_{i \sim j} (u_i - u_j)(v_i - v_j).$$

Graph Laplacian

The graph Laplacian of $G(n, p)$ is a random matrix with

$$L(i,i) = \sum_{j \neq i} \xi_{i,j},$$

$$L(i,j) = -\xi_{i,j},$$

where $\{\xi_{i,j}\}_{i<j}$ are independent Bernoulli random variables with parameter p, and $\xi_{i,j} := \xi_{j,i}$ for $i > j$.

Algebraic Connectivity

It is known that L is positive semidefinite, and contains $\{c(1, ..., 1)^T | c \in \mathbb{R}\}$ in the kernel. Therefore $\lambda_1 = 0$ is the minimal eigenvalue.

We call the second minimal eigenvalue $a(G)$ the algebraic connectivity of the graph. If G is connected, then $a(G) > 0$. The corresponding eigenvector is called the Fiedler vector.

Sparse and Connected Graph Laplacian

It was shown by Erdos and Renyi that if $p > \frac{\log n}{n}$, then the graph is connected with high probability.

We will investigate the region of p for which the graph is connected, namely, $a(G)$ is non-zero, and the resulting matrix is still very sparse. Namely, let

$$p = p_0 \frac{\log n}{n}$$

for some constant $p_0 > 1$ and also $p_0 = O(1)$.

Minimum Degree

It is known that the minimum degree of L is

$$d_{\min} \sim a(p_0)np + O(\sqrt{np}),$$

where $a = a(p_0)$ is in $(0, 1)$ and the solution to

$$a(1 - \ln a) = \frac{p_0 - 1}{p_0}.$$

By "$\sim$" we mean it is both greater than or equal to and less than or equal to the given quantity with high probability.

Algebraic Connectivity

Using techniques of Kahn and Szemeredi, one can show that

Theorem

Suppose there exists a $p_0 > 0$ *so that* $np \geq p_0 \log n$ *and* $C, c_1 > 0$ *so that*

$$|d_{min} - c_1 np| \leq C\sqrt{np}$$

with probability at least $1 - O(e^{-\Omega(\sqrt{np})})$. *Then there exists a* $\tilde{C} > 0$ *so that*

$$|\lambda_2 - c_1 np| \leq \tilde{C}\sqrt{np}$$

with probability at least $1 - O(e^{-\Omega(\sqrt{np})})$.

Dense and Connected Graph Laplacian

If p is a constant independent of n, then it is well known that

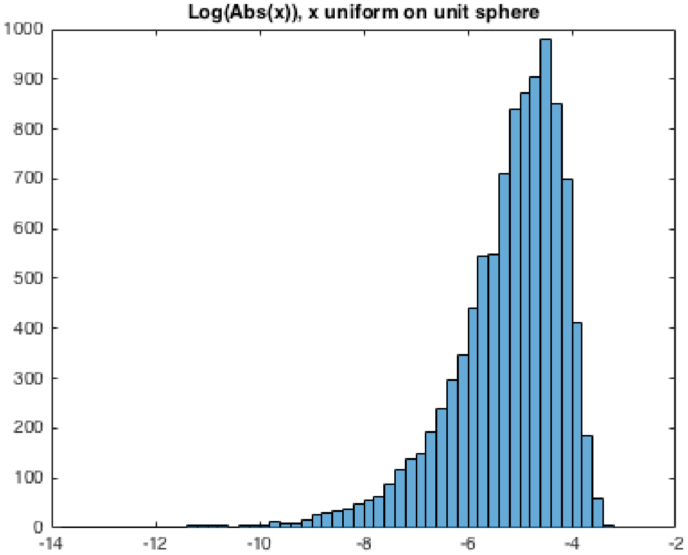
Theorem

$$\lambda_2 = np + o(n^{1/2+\epsilon})$$

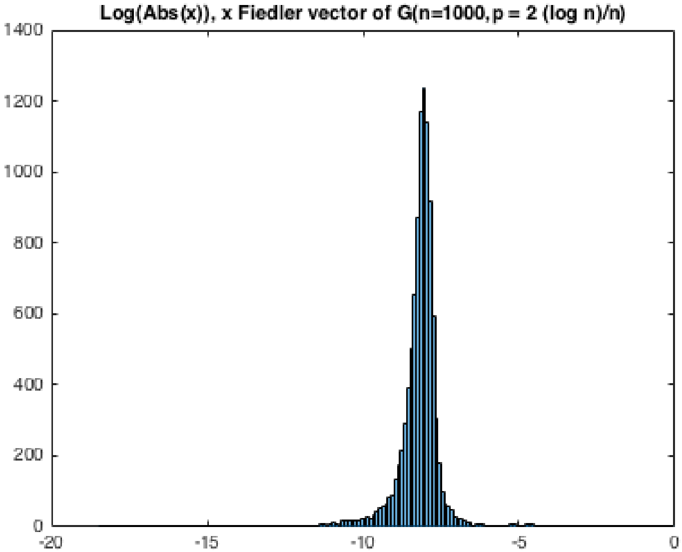
But What About the Fiedler Vector?

While there are results for the algebraic connectivity, there is very little in the way of the Fiedler vector. This is for both near the connectivity threshold and for constant p.

Histogram of Entries



Histogram of Entries



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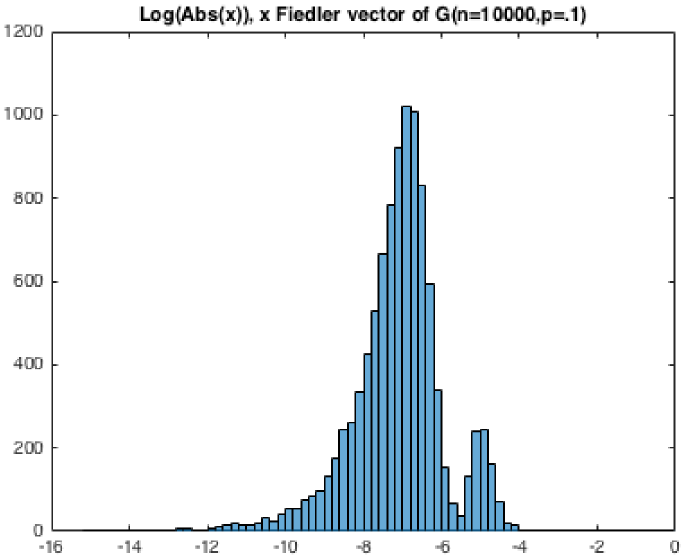

Diffusion Mapping