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Orthogonal Polynomials

$w(x) \geq 0$ on I

$\pi_i(x)$ $i=0,1,\ldots$ degree i

$p_i(x)$ = monic orthog. polyn.

$$\langle \pi_i, \pi_j \rangle = \int_I \pi_i(x)\pi_j(x)\,dx = \delta_{ij}$$

$$\pi_i(x) = k_i x^i + \ldots$$

$$p_i(x) = \pi_i(x)/k_i$$

$$\langle p_i, p_i \rangle = 1/k_i^2 \equiv h_i$$

$$A = \begin{bmatrix} \pi_0(x) & \pi_1(x) & \cdots & \pi_{n-1}(x) \\ & & & \\ & & & \end{bmatrix} \qquad D = \begin{bmatrix} \ddots & & \\ & w(x) & \\ & & \ddots \end{bmatrix}$$

Cauchy Binet:

$$\mathrm{Det}(A^T D A) = \sum_{i_1 < i_2 \ldots} d_{i_1} d_{i_2} \ldots A\begin{pmatrix} i_1 & i_2 & i_3 & \ldots \\ 1 & 2 & 3 & \ldots \end{pmatrix}^2$$

$$\longrightarrow \int_{x_1 < \ldots < x_n} \prod w(x_i) \begin{vmatrix} \pi_0(x_1) & \cdots & \pi_{n-1}(x_1) \\ \pi_0(x_n) & & \pi_{n-1}(x_n) \end{vmatrix}^2 dx_1 \ldots dx_n$$

$$= \frac{1}{n!} \int_{x_1, \ldots, x_n} \prod w(x_i) \left| \qquad \right|^2 dx_1 \ldots dx_n$$

$$\det\left[\int_I \pi_i(x)\pi_j(x) w(x)\,dx \right]_{0 \leq i,j \leq n-1} = \frac{1}{n!} k_0^2 k_1^2 k_2^2 \ldots k_{n-1}^2 \int_{x_1 \ldots x_n} \prod w(x_i) \prod (x_i - x_j)^2 \, dx_1 \ldots dx_n$$

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$$\int_{x_1,\dots,x_n} \prod w(x_i) \prod |x_i - x_j|^2 \, dx_1 \dots dx_n = n! \, h_0 h_1 h_2 \dots h_{n-1}$$

Let $w(x) = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2}$ (normal with $\sigma^2 = \cancel{1}/(2\alpha)$)

$$\frac{1}{n! \, h_0 \dots h_{n-1}} \left(\frac{\alpha}{\pi}\right)^{n/2} e^{-\alpha(\sum x_i^2)} \prod |x_i - x_j|^2$$ is a prob density on $\mathbb{R}^n$

$$\left[\frac{1}{h_0 \dots h_{n-1}}\right] = \frac{2^{n(n-1)/2}}{\prod_{i=1} \Gamma(i)}$$

$$\Pr(\text{all } x_i \leq y) = \frac{2^{n(n-1)/2}}{n! \prod \Gamma(i)} \int \dots \int_{(-\infty, y)^n} \left(\frac{\alpha}{\pi}\right)^{n/2} e^{-\alpha \sum x_i^2} \prod (x_i - x_j)^2 \, dx_1 \dots dx_n$$

$$\frac{1}{n! \, h_0(\alpha, \infty) \dots h_{n-1}(\alpha, \dots, \alpha)} \int \dots \int \dots$$

$$\Pr(\text{all } x_i \leq y) = \frac{\prod_{i=0}^{n-1} h_i(\alpha, y)}{\prod_{i=0}^{n-1} h_i(\alpha, \infty)}$$

$\leftarrow w(x) = e^{-\alpha x^2}$ on $(-\infty, y)$

$\leftarrow w(x) = e^{-\alpha x^2}$ on $\mathbb{R}$

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Nothing really depended on $w(x)$

One can always define $w_y(x) = \begin{cases} w(x) & x \le y \\ 0 & x > y \end{cases}$

Obtain $\pi_i(x;y)$ with $h_i(;y)$

$$p(x_1,\dots,x_n) = \frac{1}{\frac{1}{n!}\prod_{i=0}^{n-1} h_i(\cancel{x;\infty})} \prod |x_i - x_j|^2 \prod w(x_i)$$ is a prob den on $\mathbb{R}^n$

$$pr(\text{all } x_i \le y) = \cancel{\prod_{i=0}^{n-1} h_i(x;\infty)} \quad \frac{\prod_{i=0}^{n-1} h_i(x;y)}{\prod_{i=0}^{n-1} h_i(x;\infty)}$$

Given $w(x)$ how do you compute h_i? Generally?

I believe best is to get T as fast as possible:

$$T = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \ddots & \ddots & \\ & \ddots & \ddots & \beta_{n-1} \\ & & \beta_{n-1} & \alpha_n \end{pmatrix}$$

$x\pi_K = \alpha_{K+1}\pi_K + \beta_K \pi_{K-1} + \beta_{K+1}\pi_{K+1}$

If $\int_I w(x) = 1$ then $\pi_0(x) = 1$

$$T\begin{pmatrix} \pi_0(x) \\ \pi_1(x) \\ \vdots \end{pmatrix} = x\begin{pmatrix} \pi_0(x) \\ \pi_1(x) \\ \vdots \\ \pi_{n-1} \end{pmatrix}$$

$\beta_{K-1}\pi_{K-2}(x)$

$\alpha_K \pi_{K-1}(x)$

$+ \beta_K \pi_K(x)$

$= x\,\pi_{K-1}(x)$

$$K_i = \beta_1 \cdots \beta_{i+1} = 1/\sqrt{h_i}$$

$$h_i = \frac{1}{\beta_1^2 \cdots \beta_{i+1}^2}$$

How to compute T:

Moment Method (Ill conditioned)

$$m_K = \int_I x^K dx$$

$$H = \begin{bmatrix} m_0 & m_1 & m_2 \\ m_1 & m_2 & \vdots \\ m_2 & \vdots & \vdots \\ \vdots & \vdots & \end{bmatrix}$$ Hankel

Cholesky $R^T R = H$ $R =$

$$\beta_K = \frac{R_{K+1,K+1}}{R_{K,K}}$$

$$\alpha_K = \frac{R_{K+1,K+2}}{R_{K+1,K+1}} - \frac{R_{K,K+1}}{R_{K,K}}$$

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$$x\pi_K = \alpha_{K+1}\pi_K + \beta_K \pi_{K-1} + \beta_{K+1}\pi_{K+1}$$

$$x\pi_i = \alpha_{i+1}\pi_i + \beta_i \pi_{i-1} + \beta_{i+1}\pi_{i+1}$$

$$\pi_i = K_i \rho_i$$

$$x K_i \rho_i = \alpha_{i+1} K_i \rho_i + \beta_i K_{i-1}\rho_{i-1} + \beta_{i+1} K_{i+1}\rho_{i+1}$$

$$\rho_{i+1}(x) + \frac{\alpha_{i+1}\cancel{K_i}}{\cancel{\beta_{i+1}K_{i+1}}}\rho_i(x) + \frac{\beta_i K_{i-1}}{\beta_{i+1}K_{i+1}}\rho_{i-1}(x)$$

$$= \frac{x\cancel{K_i}}{\cancel{\beta_{i+1}K_{i+1}}}\rho_i(x)$$

$$\boxed{\frac{K_i}{\beta_{i+1}K_{i+1}} = 1}$$

$$\cancel{S_i = \alpha_{i+1}}$$

$$\rho_{i+1} + \underbrace{\alpha_{i+1}}_{S_i}\rho_i + \underbrace{\beta_i\frac{K_{i-1}}{K_i}}_{R_i}\rho_{i-1} = x\rho_i$$

$$\rho_{i+1} + \alpha_{i+1}\rho_i + \beta_i\beta_{i+1}\rho_{i-1} = x\rho_i$$

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$$R_T = \frac{h_i}{h_{i-1}} = \frac{1}{\beta_{i+1}^2}$$

$\langle \pi_{n}, x\pi_n \rangle = \alpha_{n+1}$

$$\int_{-\infty}^{y} x\, p_n^2(x)\, e^{-\alpha x^2}\, dx = y p_n e^{-\alpha y^2} - \int p_n e^{-\alpha y^2}$$

$$\int_{-\infty}^{y} p_n^2(x)\, e^{-\alpha x^2}\, dx$$

$$\frac{\partial}{\partial y}\left[\quad\uparrow\quad\right] = p_n^2(y)\, e^{-\alpha y^2}$$

$$+ \cancel{\int 2\pi_n(x)\, \frac{\partial p_n(x)}{\partial y}\, e^{-\alpha x^2}\, dx}$$

$dw \; p_n^2 e^{-\alpha x^2}$

$u = x$

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$$dv = y e^{-\alpha y^2} \qquad v = -\frac{e^{-\alpha y^2}}{2\alpha}$$

$$u = P_n^2(y) \qquad du = 2P_n P_n'$$

$$\int u\,dv = \cancel{uv} - \int$$

$$-\frac{P_n^2(y)}{2\alpha} e^{-\alpha y^2} + \int \cancel{2P_n P_n'} \ldots$$