Jacobi Ensemble (also known as MANOVA Matrix, Multivariate Beta)

Constantine 1963

$A = \text{randn}(m1, n)$

$B = \text{randn}(m2, n)$

MANOVA: $(A^TA) / (A^TA + B^TB)$ or $(A^TA + B^TB) \backslash (A^TA)$

or $(A^TA + B^TB)^{-1/2} (A^TA) (A^TA + B^TB)^{-1/2}$

(similar matrices!)

$Y = qr([A;B])[1]$

Julia: $Y = \text{qrfact}([A;B])[:Q]$

MATLAB $[Y, \sim] = qr([A;B], 0)$

QR Factor Formulation

$$\begin{array}{c c} & n \\ m_1 & \\ m_2 & \end{array}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} R$$

$Y_1^TY_1 + Y_2^TY_2 = I_n$ $\leftarrow$ orthogonality

$A^TA = R^T(Y_1^TY_1)R$

$A^TA + B^TB = R^T(Y_1^TY_1 + Y_2^TY_2)R = R^TR$

$(A^TA)/(A^TA + B^TB) = R^T(Y_1^TY_1)R^{-T} \sim Y_1^TY_1$

Joint Eigenvalue Density is known:

$$\text{constant} \times \prod_{i<j} |\lambda_i - \lambda_j| \prod_{i=1}^{n} \left\{ \lambda_i^{(m_1-n-1)/2} (1-\lambda_i)^{(m_2-n-1)/2} \right\}$$

$$\text{Constant} = \frac{V_{\text{Stiefel}}(m_1, n)\, V_{\text{Stiefel}}(m_2, n)}{V_{\text{grassmann}}(m, n)\, V_{\text{phases}}(n)}$$

The Generalized Singular Value Decomposition

MATLAB: gsvd(A,B) returns certain format

Julia: $\mathrm{svd}(A,B) \longrightarrow U, V, Q, C, S, R$

MATLAB: $[U, V, X, C, S] = \mathrm{gsvd}(A,B)$

$$\begin{array}{c} \\ m_1 \\ m_2 \end{array}\begin{array}{c} n \\ \begin{bmatrix} A \\ B \end{bmatrix} \end{array} = \begin{bmatrix} U & \\ & V \end{bmatrix}\begin{bmatrix} C \\ S \end{bmatrix} X^T$$

U, V orthogonal

C, S diagonal cosines + sines

X non-singular

Julia: $\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} U & \\ & V \end{bmatrix}\begin{bmatrix} C \\ S \end{bmatrix}(QR)^T$

$(A^TA)/(A^TA+B^TB) = XC^2X^TX^{-T}X^{-1} \sim C^2$

$\lambda_i = c_i^2 \quad 1-\lambda_i = s_i^2$

Special case $\mathrm{gsvd}(A, B) = \mathrm{svd}(A/B)$ if $A + B$ square + invertible

Geometric Meaning not really found in numerical linear algebra books,

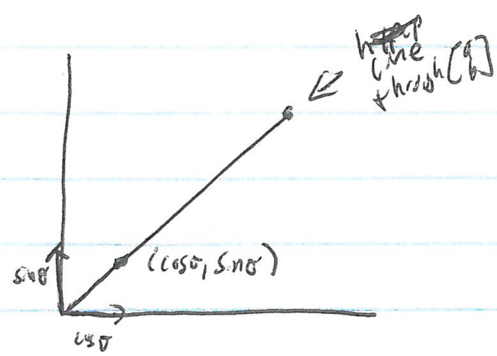
Equivalence Class $\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \equiv \begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$ if $\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} M$ for M invertible

~~Alge~~' GSVD: U, V, C, S depends on equivalence class only

X depends on more

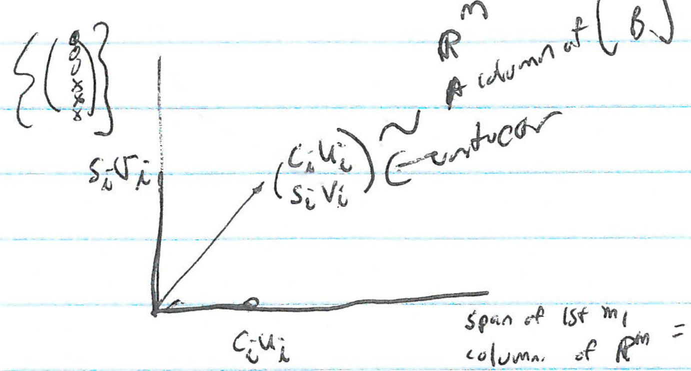
Geometrically: $\begin{bmatrix} A \\ B \end{bmatrix}$'s equivalence class is the hyperplane spanned by the columns of $\begin{bmatrix} A \\ B \end{bmatrix}$

e.g. a + b are scalars



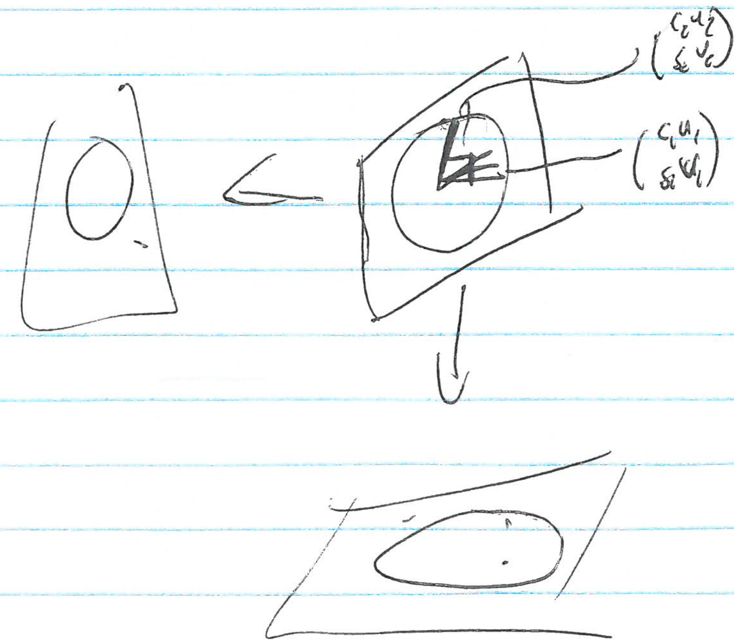
$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} r$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} r$$



$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} VC \\ VS \end{bmatrix}$$

span of 1st m_1 column of R^m = $\left\{\begin{array}{c} \begin{pmatrix} x \\ x \\ x \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{array}\begin{array}{l} m_1 \\ \\ m_2 \end{array}\right\}$



Characterizat

as rand hyperplane