Q.S growth

1

$\lambda = 1^{r_1} 2^{r_2} 3^{r_3} \dots$

$r_1 = \# \lambda_i = 1$

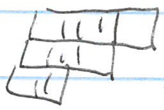
$r_2 = \# \lambda_i = 2$

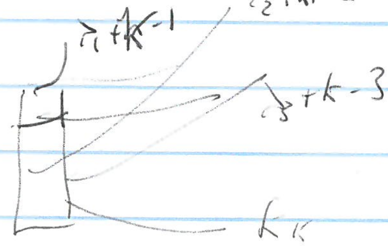
$r = \text{hist}(\lambda)$

Schur Polynomials

$\lambda \vdash n \qquad \lambda = (\lambda_1, \lambda_2, \dots \lambda_k) \qquad \lambda_1 \geq \lambda_2 \geq \dots \geq 0 \qquad k = |\lambda|$

$\lambda_1 + \lambda_2 + \dots = n$

Ferrer's Diagram 

Hook length: 

$|\lambda| = \#$ rows

$$\text{vandermond}(x_1, \dots x_k) = \begin{vmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_k \\ x_1^2 & & x_k^2 \\ \vdots & & \vdots \\ x_1^{k-1} & \dots & x_k^{k-1} \end{vmatrix} = \begin{vmatrix} x_1^{k-1} & \dots & x_k^{k-1} \\ \vdots & & \vdots \\ x_1 & & \vdots \\ 1 & & 1 \end{vmatrix}$$

$$\text{Gen-Vander}(x_1, \dots, x_k; \lambda) = \begin{vmatrix} x_1^{\lambda_k} \\ x_1^{\lambda_{k-1}+1} \\ x_1^{\lambda_{k-2}+2} \\ \vdots \\ x_1^{\lambda_1+k-1} \end{vmatrix} = \begin{vmatrix} x_1^{\lambda_1+k-1} & & x_k^{\lambda_1+k-1} \\ x_1^{\lambda_2+k-2} & & x_k^{\lambda_2+k-2} \\ \vdots & & \vdots \\ \vdots & & \vdots \end{vmatrix}$$

$$S_\lambda(x) = \frac{\text{Gen-Vander}(x_1, \dots, x_k; \lambda)}{\text{vander}(x_1, \dots, x_k)}$$

$$S_{(2,1,1)}(x_1, x_2, x_3) = x_1 x_2 x_3 (x_1 + x_2 + x_3)$$

Character Table : (# of partitions of n) $\times$ (# of partitions of n)

S_3

χ $\qquad\qquad$ $\chi^T \chi = n!\, I$

$$\begin{matrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \\ (1) & (3) & (2) \\ 1/6 & 1/2 & 1/3 \end{matrix}$$

$\uparrow$

Squares count standard Young Tableaux

λ		z_λ	$\frac{n!}{z_\lambda}$
(3)	$1^0 2^0 3^1$	3	2
$(2,1)$	$1^1 2^1$	2	3
$(1,1,1)$	1^3	$3!$	1

$$\lambda = 1^{r_1} 2^{r_2} 3^{r_3}$$

$$z_\lambda = r_1!\, r_2!\, r_3!\, 1^{r_1} 2^{r_2} 3^{r_3} \cdots$$

$$\chi \left(\operatorname{diag}(1/z_\lambda) \right) \chi^T = I$$

$$\cancel{\tilde{\chi} = \chi \operatorname{diag}(1/\sqrt{z_\lambda})}$$

$$S_\lambda = \sum \frac{\chi^\lambda_\nu}{z_\nu} p_\nu$$

$$p_\nu = \sum \chi^\nu_\lambda S_\lambda \qquad p_\nu = \prod_{i=1,\ldots} \sum x_i^{\nu_i}$$

③

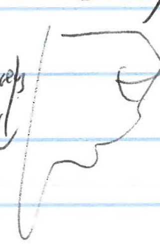
$$S_\lambda(1,1,\dots,1) = \prod_{i<j} \frac{(\lambda_i+k-i) - (\lambda_j+k-j)}{(k-i) - (k-j)} = \prod \frac{(h_i - h_j)}{(j-i)}$$

$h_i = \lambda_i + k - i$

Vandermond like with h's

$S_\lambda(1,\dots,1)$ counts semi-standard Young tableaux

Standard Tableau: increase → increase ↓



#s $1,\dots,n$ each appear once

Semi-standard: weakly increase → increase ↓

#s from $1,\dots,n$ can appear more than once or not at all

$\lambda = (3,2)$

1 3 3
2 4

1 2 3
3 4

1 2 4
3 3

T = semi-standard tableau

hist(T) = t_1 = # 1's
t_2 = # 2's
t_5 = # 5's

Sym polynomial

$$S_\lambda(x) = \sum_{\substack{T \\ \text{shape}(T)=\lambda}} x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}$$

④

~~S_λ (Matrix)~~

$S_\lambda(A :: \text{Matrix})$ is by defn $S_\lambda(\text{eig}(A) \ldots)$

$$\underset{U \text{ unitary}}{E_U}\left(S_\lambda(U)\overline{S_\mu(U)}\right) = \delta_{\lambda\mu}$$

$\ell \times \ell$

U ~~has to~~ $\ell \geq |\lambda|$

$$\underset{\text{unitr}}{E_U}\left(P_\lambda(U)\overline{P_\mu(U)}\right) = \underset{\text{unt}}{E}\left(\sum_{\kappa} \chi^\lambda_\kappa S_\kappa\right)\left(\sum_{\kappa'} \chi^\mu_{\kappa'} \bar{S}_{\kappa'}\right)$$

$\lambda = \mu = (1, \ldots, 1)$

$$\sum_{\lambda,\mu} \chi^\lambda_\kappa \chi^\mu_\kappa S_\lambda \bar{S}_\mu = \sum_{|\kappa| \leq \ell} \chi^{(1,\ldots,1)}_\kappa$$

~~$P(\text{lis} \leq \ell) = \frac{1}{n!} \sum_{|\lambda| \leq \ell} \chi$~~

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$$\cancel{P_{1\ldots1}(U) = \sum \mathrm{eig}(U) = \mathrm{Trace}(U)}$$

$$\cancel{P_{n\ldots n}(U) = \mathrm{Tr}\, U^n}$$

$$P_1(U) = \mathrm{Tr}(U)$$

$$P_{1,\ldots,1}(U) = [\mathrm{Tr}(U)]^n$$

$$P_{1,\ldots,1}(U)\, \overline{P_{1,\ldots,1}(U)} = |\mathrm{Tr}(U)|^{2n}$$

$$\left[\chi_K^{(1,1,\ldots,1)}\right]^2 = \# \text{ of stan Young tableau of shape } K$$

$$\sum_{\substack{|K| = \ell \\ K \vdash n}} \chi_K^{(1,\ldots,1)^2} = \# \text{ of permutations with lis} = \ell$$

$$\sum_{\substack{|K| \le \ell \\ K \vdash n}} \chi_K^{(1,\ldots,1)^2} = \# \text{ of perm with lis} \le \ell$$