2nd order symmetric eigenvalue perturbation theory

Equation (?) obtained from differentiating $S = Q\Lambda Q^T$ gives $Q^T\dot{S}Q = \dot{\Lambda} + [Q^T\dot{Q}, \Lambda]$. Thus

$\dot{\Lambda} = \mathrm{diag}(Q^T\dot{S}Q)$ and
$(Q^T\dot{Q})_{ij}(\lambda_j - \lambda_i) = (Q^T\dot{S}Q)_{ij}$ for $i \neq j$

Differentiating again
$\ddot{\Lambda} = 2\,\mathrm{diag}(Q^T\dot{S}\dot{Q}) + \mathrm{diag}(Q^T\ddot{S}Q)$.

We can specialize to the case where we linearly perturb a diagonal S by a symmetric matrix E:

$$S(\varepsilon) = \Lambda + \varepsilon E \qquad E \text{ symmetric}$$
$$\Lambda(\varepsilon) = \Lambda + \varepsilon\dot{\Lambda} + \frac{\varepsilon^2}{2}\ddot{\Lambda} + \ldots$$
$$Q(\varepsilon) = I + \varepsilon\dot{Q} \qquad \dot{Q} \text{ anti-symmetric}$$

Where $\dot{\Lambda} = \mathrm{diag}(E)$

$$\dot{Q}_{ij} = \frac{E_{ij}}{\lambda_j - \lambda_i} \quad (i \neq j)$$

$$\ddot{\Lambda} = 2\,\mathrm{diag}(E\dot{Q}) \quad \text{or}$$

$$\ddot{\lambda}_i = 2\sum_{k \neq i} \frac{E_{ik}^2}{\lambda_i - \lambda_k}$$

Thus $\lambda_i(\varepsilon) = \lambda_i + \varepsilon E_{ii} + \varepsilon^2 \sum_{k \neq i} \frac{E_{ik}^2}{\lambda_i - \lambda_k}$