Numerical Evaluation of Standard Distributions in Random Matrix Theory

A Review of Folkmar Bornemann's MATLAB Package and Paper

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Level Spacing Function

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Definition (Gaussian Ensemble Spacing Function)

Let $J \subset \mathbb{R}$ be an open interval.

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Let $J = (0, s)$. Then $E_2(0; J) =$ probability no eigenvalues lie in $(0, s)$.

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Theorem (Gaudin 1961)

Given $K_{\sin}(x, y) = sinc(\pi(x - y))$,

$$E_2(0, J) = \det\left(I - K_{\sin} \restriction_{L^2_J}\right)$$

Note the operator's restriction to square integrable functions over J. In general we will choose $J = (0, s)$, and will notate $E_2(0, (0, s))$ as $E_2(0, s)$ as per Bornemann's conventions.

Integral Formulation

Theorem (Jimbo, Miwa, Mori, Sato 1980)

$$E_2(0; s) = \exp\left(-\int_0^{\pi s} \frac{\sigma(x)}{x}\, dx\right)$$

where $\sigma(x)$ solves a particular form of the Painleve V equation:

$$(x\sigma'')^2 = 4(\sigma - x\sigma')(x\sigma - \sigma - (\sigma')^2), \quad \sigma(x) \approx \frac{x}{\pi} + \frac{x^2}{\pi^2} \quad (x \to 0)$$

Tracy-Widom Distribution

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Definition (Tracy-Widom Distribution)

Let $F_2(s) \equiv \mathbb{P}($ no eigenvalues of large-matrix limit GUE lie in $(s, \infty))$

Determinantal Representation

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Theorem (Bronk 1964)

Given

$$K_{Ai}(x,y) = \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x-y}$$

we have

$$F_2(s) = \det\left(I - K_{Ai} \restriction_{L^2_{(s,\infty)}}\right)$$

Integral Formulation

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Theorem (Tracy, Widom 1993)

$$F_2(s) = \exp\left(-\int_s^\infty (x-s)u(x)^2 dx\right)$$

where $u(x)$ is the Hastings-McLeod (1980) solution to the Painleve II equation

$$u'' = 2u^3 + xu, \quad u(x) \approx Ai(x) \quad (x \to \infty)$$

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Bornemann's view:

- Numerical evaluation of Painleve transcendents is actually fairly involved. Stability is a major concern.
- There exists a simple, fast, accurate numerical method for evaluating Fredholm determinants
- Many multivariate functions (joint prob. dists.) have a nice representation as a Fredholm determinant, but no representation in terms of a nonlinear PDE.

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Given an interval (a, b), we seek $u(x)$ that solves

$$u''(x) = f(x, u(x), u'(x))$$

subject to either of the asymptotic one-sided conditions

$$u(x) \approx u_a(x) \quad (x \to a)$$

or

$$u(x) \approx u_b(x) \quad (x \to b)$$

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$$v''(x) = f(x, v(x), v'(x))$$

$$v(a_+) = u_a(a_+), \quad v'(a_+) = u_a'(a_+)$$

or

$$v(b_-) = u_b(b_-), \quad v'(b_-) = u_b'(b_-)$$

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Example: Computing $F_2(s) = \exp\left(-\int_s^\infty (x-s)u(x)^2 dx\right)$:

$$v(x)'' = 2v(x)^3 + xv(x), \quad v(b_-) = \text{Ai}(b_-), \quad v'(b_-) = \text{Ai'}(b_-)$$

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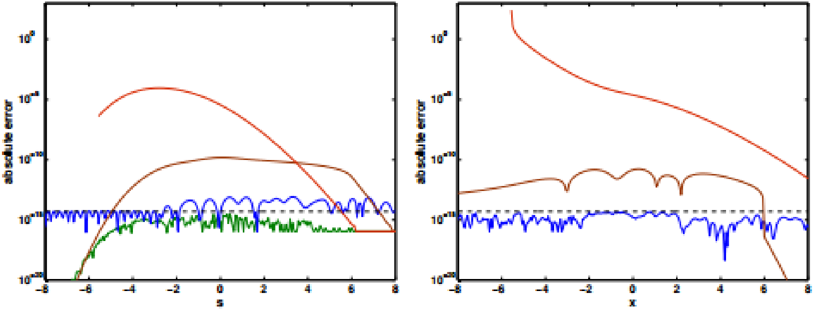
Choosing $b_- \geq 8$ gives initial values accurate to machine precision (about 10^{-16} for IEEE doubles). Choose $b_- = 12$ yields these results:

Stability Issues



a. error in evaluating $F_2(s)$ b. error in evaluating $u(x)$

method	reference	max. error	run time
IVP/Matlab's ode45	Edelman and Persson (2005)	$9.0 \cdot 10^{-5}$	11 sec
BVP/Matlab's bvp4c	Dieng (2005)	$1.5 \cdot 10^{-10}$	3.7 sec
BVP/spectral colloc.	Driscoll et al. (2008)	$8.1 \cdot 10^{-14}$	1.3 sec
Fredholm determinant	Bornemann (2010a)	$2.0 \cdot 10^{-15}$	0.69 sec

Less Straightforward Approach: Solving the BVP for Painleve

Stability issues described in depth in Bornemann's paper lead to a BVP approach.
We use asymptotic expression $u_a(x)$ at $(x \to a)$ to infer asymptotic expression $u_b(x)$ at $(x \to b)$, or vice versa.
Approximate $u(x)$ by solving BVP:

$$v''(x) = f(x, v(x), v'(x)), \quad v(a_+) = u_a(a_+), \quad v(b_-) = u_b(b_-)$$

Requires four choices: values of a_+, b_-, and order of asymptotic accuracy for $u_a(x)$ and $u_b(x)$

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By definition, $u(x) \approx \text{Ai}(x) \quad (x \to \infty)$ so we take $u_b(x) = \text{Ai}(x)$.
Choose $a_+ = -10$, $b_- = 6$ (Dieng, 2005).

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By definition, $u(x) \approx \mathrm{Ai}(x) \quad (x \to \infty)$ so we take $u_b(x) = \mathrm{Ai}(x)$.
Choose $a_+ = -10$, $b_- = 6$ (Dieng, 2005).
We need to choose a sufficiently accurate asymptotic expansion for $u_a(x)$. Tracy and Widom show

$$u(x) = \sqrt{-\frac{x}{2}} \left(1 + \frac{1}{8} x^{-3} - \frac{73}{128} x^{-6} + \frac{10657}{1024} x^{-9} + O(x^{-12}) \right), \quad (x \to -\infty)$$

so we'll use that for $u_a(x)$.

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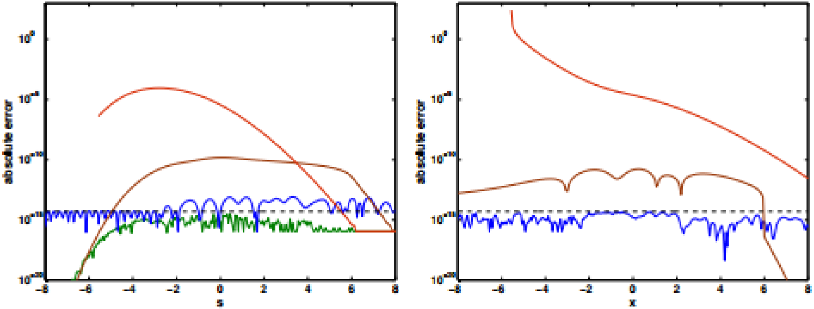

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Punchline: BVP approach is insufficiently "black-box" for us.

Better Approach: Numerical Evaluation of Fredholm Determinants

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Choose your favorite quadrature rule (Clenshaw-Curtis is good) over nodes $x_j \in (a, b)$ and positive weights w_j: $\sum_{j=1}^{m} w_j f(x_j) \approx \int_a^b f(x)dx$

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The Fredholm determinant

$$d(z) = \det\left(I - zK \restriction_{L^2_{(a,b)}}\right)$$

has the approximation

$$A_m = K(x_i, y_j)_{i,j=1}^{m}$$

$$d_m(z) = \det\left(\delta_{ij} - z \cdot w_i^{1/2} A_m w_j^{1/2}\right)$$

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  Compute LU of $(I - zA_m)$, get determinant from $\prod_{j=1}^{m} U_{jj}$
  Computing $d_m(z)$ for a single z takes $O(m^3)$ time.
- We need the value at many points, want $d_m(z)$ as polynomial.
  Compute eigenvalues λ_j of A_m via QR (one-time cost of $O(m^3)$ time, but worse constant factor than LU in practice), then form

$$d_m(z) = \prod_{j=1}^{m} (1 - z\lambda_j)$$

  Computing $d_m(z)$ takes $O(m)$ time.

Sample Matlab Code

The following code computes $F_2(0)$ to one unit of precision in the last decimal place:

```
>> m = 64; [w, x] = ClenshawCurtis(0, inf, m); w2 = sqrt(w);
>> [xi, xj] = ndgrid(x, x);
>> KAi = @AiryKernel;
>> F20 = det(eye(m) - (w2' * w2).*KAi(x, x))
F20 = 0.969372828355262
```

Wrapup

- Computing Fredholm Determinants is faster, easier, and more stable than integrating Painleve IVP or BVP.
- Being able to handle things that are expressed in non-PDE form is useful.
- Bornemann uses the toolset to identify (and subsequently prove) several new results (omitted here for brevity) about distributions of the k-th largest eigenvalue in the soft-edge scaling limit of the GOE and GSE – the numerical code generates immediate insights!