An Exact Formula For Integrating Polynomials Over $U(d)$

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B. Collins, P. Sniady, *Integration With Respect to the Haar Measure on Unitary, Orthogonal and Symplectic Group*, Commun. Math. Phys. 264 (2006) 773-795.
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Question

We can use symmetries of Haar measure and index permutation tricks to compute integrals over $U(d)$ like:

- $\int U_{ij} dU = 0$
- $\int |U_{11}|^2 dU = 1/d$
- $\int U_{i_1 j_1} \cdots U_{i_n j_n} \overline{U}_{i'_1 j'_1} \cdots \overline{U}_{i'_m j'_m} dU = 0$ if $m \neq n$ or if there is are no permutations $\sigma, \tau \in S_n$ such that $\sigma(i) = i'$ and $\tau(j) = j'$

Is there a general formula for the *moments* of $U(d)$?

(this would let us compute polynomials in U_{ij} and $\overline{U}_{ij}$)

Answer

- Yes, and it still involves symmetries of Haar measure and index permutations.

- Based on the “classic” Schur-Weyl duality but only discovered in 2004

- I hope you like Representation Theory!

Representation Theory

Partition $\boldsymbol{\lambda \vdash n}$: non-increasing sequence of non-neg. integers that sum to n.

$\boldsymbol{P_{n,d}}$: set of $\lambda \vdash n$ with $\leq d$ non-zero entries

Facts:

- Each $\lambda \in P_{n,d}$ gives a distinct irred. rep'n of $U(d)$ denoted $\boldsymbol{\rho^{\lambda}_{U(d)}: U(d) \to V^{\lambda}}$
- Each $\lambda \vdash n$ gives a distinct irred. rep'n of S_n denoted $\boldsymbol{\rho^{\lambda}_{S_n}: S_n \to W^{\lambda}}$. Denote the character by $\boldsymbol{\chi^{\lambda}}$.

Schur-Weyl Duality

$U \in U(d)$ acts $\mathbb{C}$-linearly on $\left(\mathbb{C}^d\right)^{\otimes n}$

$$\boldsymbol{v_1 \otimes \cdots \otimes v_n \mapsto (Uv_1) \otimes \cdots \otimes (Uv_n)}$$

$\sigma \in S_n$ acts $\mathbb{C}$-linearly on $\left(\mathbb{C}^d\right)^{\otimes n}$

$$\boldsymbol{v_1 \otimes \cdots \otimes v_n \mapsto v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}}$$

Schur-Weyl Duality characterizes joint representation:

$$\left(\mathbb{C}^d\right)^{\otimes n} \cong \bigoplus \left(V^\lambda \otimes W^\lambda\right)$$

($\bigoplus$ is over $P_{n,d}$)

Main Theorem

$$\int_{U(d)} U_{i_1 j_1} \overline{U}_{i'_1 j'_1} \cdots U_{i_n j_n} \overline{U}_{i'_n j'_n} dU = \sum_{\{\sigma(i)=i'\}} \sum_{\{\tau(j)=j'\}} Wg(\tau\sigma^{-1})$$

Where the **Weingarten function** Wg is defined by:

$$Wg(\sigma) = \frac{1}{(n!)^2} \sum_{\lambda \in P_{n,d}} \frac{\left(d_{S_n}^{\lambda}\right)^2}{d_{U(d)}^{\lambda}} \chi^{\lambda}(\sigma)$$

Proof Sketch

For $A \in End\left(\mathbb{C}^d\right)^{\otimes n}$ define **conditional expectation:**

$$E(A) = \int U^{\otimes n} A (U^*)^{\otimes n} dU$$

Properties: (use Haar invariance to prove)

- $E(A)$ commutes with all unitary actions; unitary piece "integrated out", result lives in S_n piece
- $Tr\big(E(A)\big) = Tr(A)$; $E(A)$ is "trace on $U(d)$ piece"
- $E\left(A\rho^d_{S_n}(\sigma)\right) = E(A)\rho^d_{S_n}(\sigma)$; leaves alone S_n actions

Sketch cont'd

Let $A_{(i)}(e_{i_1} \otimes \cdots \otimes e_{i_n}) = e_{i'_1} \otimes \cdots \otimes e_{i'_n}$

and $B_{(j)}\left(e_{j'_1} \otimes \cdots \otimes e_{j'_n}\right) = e_{j_1} \otimes \cdots \otimes e_{j_n}$ and define both to be zero on other std basis vectors. Then:

$$Tr\big(A_{(i)}E\big(B_{(j)}\big)\big) = \int_{U(d)} U_{i_1 j_1} \cdots U_{i_n j_n} \overline{U}_{i'_1 j'_1} \cdots \overline{U}_{i'_n j'_n} dU$$

Which is the LHS of the theorem. For RHS need some algebraic properties…

Sketch cont'd

Define Φ: $End\ (\mathbb{C})^{\otimes n} \to \mathbb{C}_d[S_n] \subset \mathbb{C}[S_n]$ by:

$$\Phi(A) = \sum_{\sigma \in S_n} \left[Tr\left(A\rho_{S_n}^d(\sigma^{-1})\right)\right]\sigma$$

Properties:

- $\Phi(A)$ compatible with left and right multiplication
- $\Phi(A) = E(A)\Phi(\mathrm{id})$
- $\Phi(id) = \chi_{S_n}^d = \sum d_{U(d)}^{\lambda}\chi^{\lambda}$
- $\Phi(id)^{-1} = Wg$ (use Schur ortho relations, etc)
- $\Phi\left(AE(B)\right) = \Phi(A)\Phi(\mathrm{B})Wg$

Sketch Conclusion

- Two slides ago: $\Phi\left(A_{(i)}E\left(B_{(j)}\right)\right)_e$ = LHS of main thm
- Previous slide: $\Phi\left(A_{(i)}E\left(B_{(j)}\right)\right)_e = \left[\Phi\left(A_{(i)}\right)\Phi\left(B_{(j)}\right)Wg\right]_e$ too
- $\left[\Phi\left(A_{(i)}\right)\right]_\sigma = 1$ if $\sigma(i) = i'$, zero otherwise
- $\left[\Phi\left(B_{(j)}\right)\right]_{\tau^{-1}} = 1$ if $\tau(j) = j'$, zero otherwise
- Products are convolutions
- $\int_{U(d)} U_{i_1 j_1} \cdots U_{i_n j_n} \overline{U}_{i_1' j_1'} \cdots \overline{U}_{i_n' j_n'} dU = \sum_{\sigma:\sigma(i)=i'} \sum_{\tau:\tau(j)=j'} Wg(\tau\sigma^{-1})$ (QED)

Bonus!

- I coded up some MATLAB routines to compute arbitrary moments for $n \leq 5$ and any d

- If someone feels like coding up the **Monaghan-Nakayama rule** algorithm we can make it compute for arbitrary n although performance might be bad for large n…
  - D. Bernstein, *The computational complexity of rules for the character table of* S_n, Journal of Symbolic Computation 37 (6) (2004) 727-748.